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SPLITTINGS FOR THE BRAID-PERMUTATION GROUP

Chan-Seok Jeong* and Yongjin Song**

Abstract. The braid-permutation group is a group of welded braids which is the extension of Artin's braid groups by the symmetric groups. It is also described as a subgroup of the automorphism group of a free group. We also show that the plus-construction of the classifying space of the infinite braid-permutation group has the following two types of splittings

$$BBP_\infty^+ \simeq B\Sigma_\infty^+ \times X,$$
$$BBP_\infty^+ \simeq B\mathbb{Z}^+ \times Y = S^1 \times Y,$$

where X, Y are some spaces.

1. Introduction

Braids arise as isotopy classes of a collection of n connected strings in three-dimensional space. A braid diagram may be thought of as a composite of two types of *crossings* of strings (Figure 2.1). A welded braid diagram is obtained from the composite of these crossings and the *welded crossings* (Figure 2.2). The set of welded braids forms a group, called the braid-permutation group BP_n(cf. [3], [4]). It was shown by R. Fenn, Rimányi and Rourke([3], [4]) that BP_n is also given by the set of generators $\{\xi_i, \sigma_i \mid i = 1, 2, \ldots, n-1\}$ and three types of relations: braid group relations, symmetric group relations, mixed relations. This expression of BP_n is analogous to the classical group presentation of the braid group given by Artin([1], [2]).

The group BP_n may also be regarded as a subgroup of the automorphism group $\mathrm{Aut}F_n$ of a free group on $\{x_1, \ldots, x_n\}$. The generators σ_i of BP_n, called the braid group generator, is given by

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$$\begin{cases} x_i & \mapsto x_{i+1} \\ x_{i+1} & \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j & \mapsto x_j, j \neq i, i+1. \end{cases}$$

The generator ξ_i, called the symmetric group generator, is given by

$$\begin{cases} x_i & \mapsto x_{i+1} \\ x_{i+1} & \mapsto x_i \\ x_j & \mapsto x_j, j \neq i, i+1. \end{cases}$$

The braid group B_n and the symmetric group Σ_n are naturally embedded in BP_n.

One of the interesting properties of BP_n is that the subgroup PC_n of $\mathrm{Aut}F_n$ of the automorphisms of permutation-conjugacy type is isomorphic to BP_n. Moreover, BP_n is isomorphic to the automorphism group $\mathrm{Aut}FQ_n$ of the free quandle rank n, and is closely related to the automorphism group $\mathrm{Aut}FR_n$ of the free rack of rank n(cf. [4]) and these groups have relations to invariants of classical knots and links in the 3-sphere.

We, in this paper, show that the plus-construction of the classifying space of infinite braid-permutation group, up to homotopy, has the following two types of splittings:

$$BBP_\infty^+ \simeq B\Sigma_\infty^+ \times X,$$
$$BBP_\infty^+ \simeq B\mathbb{Z}^+ \times Y = S^1 \times Y$$

for some topological spaces X and Y. In the proof of this theorem we use the classical splitting theorem (Corollary 3.3). The key part of the proof is to find the elements c and d satisfying the conditions of the splitting theorem. We have found these elements in an explicit expression in terms of the generators of the braid-permutation group, which may attract an independent interest.

2. The braid-permutation group

Let F_n be the free group of rank n with the set of generators $\{x_1, \ldots, x_n\}$, and let $\mathrm{Aut}F_n$ be the group of automorphisms of F_n. There are the standard inclusions of the symmetric group Σ_n and the braid group B_n into $\mathrm{Aut}F_n$. They can be described as follows:

Let $\xi_i \in \mathrm{Aut}F_n$, $i = 1, 2, \ldots, n-1$, be given by the following formula

$$\begin{cases} x_i & \mapsto x_{i+1} \\ x_{i+1} & \mapsto x_i \\ x_j & \mapsto x_j, j \neq i, i+1. \end{cases} \tag{2.1}$$

Let $\sigma_i \in \mathrm{Aut} F_n$, $i = 1, 2, \ldots, n-1$, be given by the following formula

$$(2.2) \qquad \begin{cases} x_i & \mapsto x_{i+1} \\ x_{i+1} & \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j & \mapsto x_j, j \neq i, i+1. \end{cases}$$

Let BP_n be the subgroup of $\mathrm{Aut} F_n$ generated by ξ_i's and σ_i's of (2.1) and (2.2). It is called the *braid-permutation group.* It was proved by R. Fenn, R. Rimányi and C. Rourke in [3], [4] that this group is given by the set of generators $\{\xi_i, \sigma_i \mid i = 1, 2, \ldots, n-1\}$ and the following relations:

The symmetric group relations,

$$\begin{cases} \xi_i^2 & = 1, \\ \xi_i \xi_j & = \xi_j \xi_i, \quad \text{if } |i-j| > 1, \\ \xi_i \xi_{i+1} \xi_i & = \xi_{i+1} \xi_i \xi_{i+1}. \end{cases}$$

The braid group relations,

$$\begin{cases} \sigma_i \sigma_j & = \sigma_j \sigma_i, \text{ if } |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i & = \sigma_{i+1} \sigma_i \sigma_{i+1}. \end{cases}$$

The mixed relations,

$$\begin{cases} \sigma_i \xi_j & = \xi_j \sigma_i, \text{ if } |i-j| > 1, \\ \xi_i \xi_{i+1} \sigma_i & = \sigma_{i+1} \xi_i \xi_{i+1}, \\ \sigma_i \sigma_{i+1} \xi_i & = \xi_{i+1} \sigma_i \sigma_{i+1}. \end{cases}$$

Fenn, Rimányi and Rourke also gave the geometrical interpretation of BP_n as a group of *welded braids.* First they defined a *welded braid diagram* on n strings as a collection of n monotone arcs starting from n points on a horizontal line of a plane (the top of the diagram) and going down to n points on another horizontal line (the bottom of the diagram). The diagrams can have crossings of two types: (A) ordinary braids in Figure 2.1; (B) welds in Figure 2.2.

FIGURE 2.1.

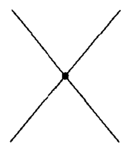


FIGURE 2.2.

An example of a welded braid diagram is shown in Figure 2.3.

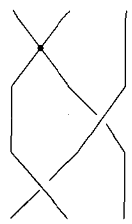


FIGURE 2.3.

Fenn, Rimányi and Rourke defined the following types of allowable transformations on welded braid diagrams. They are described in Figures 2.6, 2.7, 2.8. The transformations in Figure 2.7 are Reidemeister transformations of knot theory. The first transformation in Figure 2.8 corresponds to the relation

$$\xi_i^2 = 1.$$

The transformation in Figure 2.9 is the geometric form of the commutativity from the mixed relations. There are also analogous transformations corresponding to the commutativity from the symmetric group and the braid group relations.

A *welded braid* is defined as an equivalence class of welded braid diagrams under allowable transformations. It was proved by Fenn, Rimányi and Rourke that welded braids form a group, and this group is isomorphic to the braid-permutation group BP_n. The generator σ_i corresponds to the canonical generator of the braid group B_n and is shown in Figure 2.4.

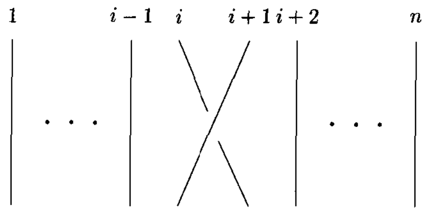


FIGURE 2.4.

The generators ξ_i correspond to the welded braids shown in the Figure 2.5.

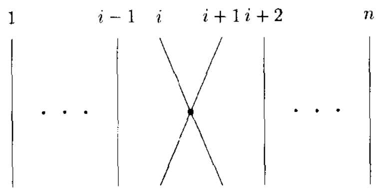


FIGURE 2.5.

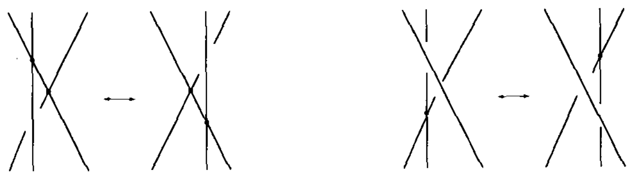


FIGURE 2.6.

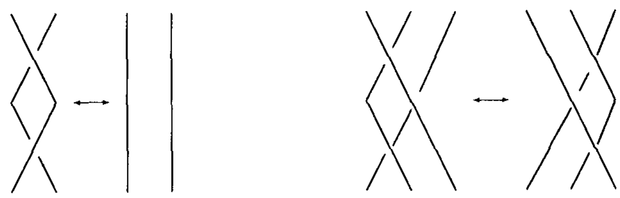


FIGURE 2.7.

FIGURE 2.8.

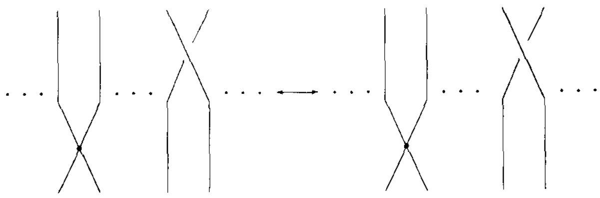


FIGURE 2.9.

3. Splittings for the braid-permutation group

Let j_n be the inclusion of the group $\mathbb{Z}$ into B_n:

$$j_n : \mathbb{Z} \to B_n,$$

where the generator of the cyclic group is mapped to one of the generators say, $j_n(1) = \sigma_1$.

There are epimorphisms

$$\alpha_n : BP_n \to \mathbb{Z},$$

$$\beta_n : BP_n \to \Sigma_n,$$

which are given by the following formulas:

$$\alpha_n(\xi_i) = 0, \quad \alpha_n(\sigma_i) = 1 \quad \text{and} \quad \beta_n(\xi_i) = \xi_i, \quad \beta_n(\sigma_i) = \xi_i \quad \text{for all } i.$$

Its composition with the canonical inclusions i_∞ of Σ_∞ and j_∞ of $\mathbb{Z}$ in BP_∞ are equal to the identity maps of Σ_∞ and $\mathbb{Z}$, respectively. These homomorphisms generate maps of classifying spaces Bi_∞, $B\beta_\infty$ and Bj_∞, $B\alpha_\infty$ such that their compositions

$$B\Sigma_\infty \xrightarrow{Bi_\infty} BBP_\infty \xrightarrow{B\beta_\infty} B\Sigma_\infty$$

$$B\mathbb{Z} \xrightarrow{Bj_\infty} BBP_\infty \xrightarrow{B\alpha_\infty} B\mathbb{Z}$$

are equal to the identity maps.

The homomorphisms α_n induce maps of classifying spaces

$$B\alpha_n : BBP_n \longrightarrow S^1.$$

Similarly, the homomorphisms β_n induce maps of classifying spaces

$$B\beta_n : BBP_n \longrightarrow B\Sigma_n.$$

Splitting theorem

We describe a splitting theorem(cf. [5]) which plays a key role in the proof of main theorems.

A group G is *perfect* if every element can be written as a product of commutators, that is, $[G, G] = G$. Any group G has a unique maximal perfect subgroup which we will denote by $P(G)$. Recall that a group G is called a *direct sum group* if there is a homomorphism $\oplus : G \times G \to G$.

Consider the more general case.

DEFINITION 3.1. G and H form a *direct sum pair* if H is a subgroup of G and there is a homomorphism $\oplus : H \times G \to G$ such that for any $g_1, \ldots, g_s \in G$ and $h_1, \ldots, h_s \in H$ there exist elements $c \in P(G)$ and $d \in P(H)$ satisfying the following;

$$(**) \qquad 1 \oplus g_i = c g_i c^{-1} \text{ and } h_i \oplus 1 = d h_i d^{-1} \text{ for all } i = 1, \ldots, s.$$

THEOREM 3.2. *BG^+ admits a left H-action by BH^+.*

This means there is a map $\mu : BH^+ \times BG^+ \to BG^+$ such that $\mu|_{BH^+}$ is homotopic to the map induced by the inclusion $i : H \hookrightarrow G$ and $\mu|_{BG^+}$ is homotopic to the identity.

Proof. Note that $(BH \times BG)^+ = BH^+ \times BG^+$. Thus the direct sum homomorphism $\oplus$ induces a map

$$m : BH^+ \times BG^+ \to BG^+.$$

Let $*$ denote the basepoint of BG^+ and BH^+. The map $m(\_, *) : BH^+ \to BG^+$ is induced by $\_ \oplus 1$. By $(**)$, $\_ \oplus 1$ factors through H. We show that the induced map $f : BH^+ \to BH^+$ is a homotopy equivalence. Since $P(H) \triangleleft H$, $BP(H)$ is a regular cover of BH, and hence $BP(H)^+$ is the universal cover of BH^+. By $(**)$, the map $BP(H)^+ \to BP(H)^+$ induced by f is the identity on homology ([5], Lemma 1.3). Hence, by the Whitehead theorem, it is a homotopy equivalence. Also, f is a homotopy equivalence. Similarly, $m(*, \_)$ is a homotopy equivalence of BG^+. Choose homotopy inverses r and l for these two maps. Then $\mu = m \circ (r \times l) : BH^+ \times BG^+ \to BG^+$ defines an H-action. □

COROLLARY 3.3. *If there is a splitting homomorphism $\phi : G \to H$, then $BG^+ \simeq BH^+ \times F$, where F is the homotopy fiber of the map $BG^+ \to BH^+$.*

Proof. Let F be the homotopy fiber of the map $B\phi^+ : BG^+ \to BH^+$, and let $s : F \to BG^+$ denote the inclusion of the fiber. Define $BH^+ \times F \to BG^+$ by mapping (x, y) to $\mu(x, s(y))$. Because μ defines an H-action, this induces an isomorphism on homotopy groups and hence is a homotopy equivalence. □

We have the following main theorem.

THEOREM 3.4. *There exist maps*

$$Bi_\infty^+ : B\Sigma_\infty^+ \to BBP_\infty^+$$

and

$$B\beta_\infty^+ : BBP_\infty^+ \to B\Sigma_\infty^+$$

such that $B\beta^+_\infty$ splits by the map Bi^+_∞.

If a space X is a fiber of the map $B\beta^+_\infty$, then we have the following splitting:

$$BBP^+_\infty \simeq B\Sigma^+_\infty \times X.$$

Proof. Note that Σ_∞ is a subgroup of BP_∞, and $\Sigma_\infty \hookrightarrow BP_\infty \overset{\beta_\infty}{\to} \Sigma_\infty$ is equal to the identity map of Σ_∞.

Define a map $\oplus_1 : \Sigma_\infty \times BP_\infty \to BP_\infty$ as follows:

Let $(\xi, \sigma) \in \Sigma_\infty \times BP_\infty$. Take $n = \max\{k, l\}$. Then we can consider ξ, σ as elements of Σ_n and BP_n, respectively, by inserting trivial strings in each ξ and σ. Now define $\xi \oplus_1 \sigma$ as an element of BP_{2n} by putting σ on odd strings and ξ on even strings. If strings of ξ and σ are crossed, then we think that they generate welded crossings; for example, choose an element $(\xi_1, \sigma_2) \in \Sigma_3 \times BP_4$. Then we may regard ξ_1 as an element of Σ_4. Hence, $\xi_1 \oplus_1 \sigma_2$ is just like in the following Figure.

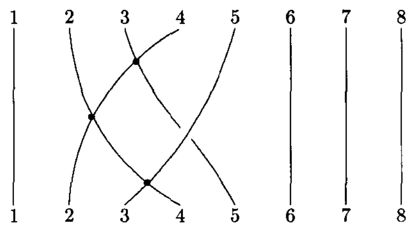


FIGURE 3.1.

It is clear that $\oplus_1$ is a homomorphism by definition.

Let $\xi_1, \ldots, \xi_{n-1}, \sigma_1, \ldots, \sigma_{n-1} \in BP_n$ for some n. By the splitting theorem it suffices to find the elements c and d satisfying the conditions of the splitting theorem. Put

$$\begin{aligned} c =& (\xi_{2n-2}\xi_{2n-1})(\xi_{2n-4}\xi_{2n-3}\xi_{2n-2}\xi_{2n-1}) \\ & \cdots (\xi_4\xi_5 \cdots \xi_{2n-1})(\xi_2\xi_3\xi_4 \cdots \xi_{2n-1}), \end{aligned}$$

where $n = 2, 3, 4, \ldots$ and

$$d = d_n d_{n-1} d_{n-2} d_{n-3} \cdots d_2 d_1,$$

where

$$d_n = \xi_{2n-1},\ d_{n-1} = \xi_{2n-3}\xi_{2n-2} \cdots \xi_{2n-1},$$
$$d_{n-2} = \xi_{2n-5}\xi_{2n-4} \cdots \xi_{2n-2}\xi_{2n-1},\ d_{n-3} = \xi_{2n-7}\xi_{2n-6} \cdots \xi_{2n-1}, \ldots,$$
$$d_2 = \xi_3\xi_4\xi_5 \cdots \xi_{2n-2}\xi_{2n-1},\ d_1 = \xi_1\xi_2 \cdots \xi_{2n-1} \quad \text{for } n = 2, 4, 6, \ldots.$$

Note that c and d belong to the maximal perfect subgroup of BP_m because they are even words. By the definition of $\oplus_1$ we have

$$id \oplus_1 \xi_i = \xi_{2i}\xi_{2i-1}\xi_{2i}, \quad id \oplus_1 \sigma_i = \xi_{2i}\sigma_{2i-1}\xi_{2i},$$
$$\xi_i \oplus_1 id = \xi_{2i+1}\xi_{2i}\xi_{2i+1} \quad \text{for } i = 1, 2, 3, \dots, n-1.$$

In general, we have

$$id \oplus_1 \xi_i = c\xi_i c^{-1} \text{ for all } i = 1, \dots, n-1, \tag{3.1}$$

$$id \oplus_1 \sigma_i = c\sigma_i c^{-1} \text{ for all } i = 1, \dots, n-1, \tag{3.2}$$

and

$$\xi_i \oplus_1 id = d\xi_i d^{-1} \quad \text{for all } i = 1, \dots, n-1. \tag{3.3}$$

Proof of (3.1). We prove this by induction. Let

$$c = c_{n-1}c_{n-2}\cdots c_i \cdots c_3c_2c_1,$$

where $c_i = (\xi_{2i}\xi_{2i+1}\cdots\xi_{2n-2}\xi_{2n-1})$ for $i = 1, \dots, n-1$.
For $i = 1$,

$$\begin{aligned} c\xi_1c^{-1} &= c_{n-1}c_{n-2}\cdots c_2(\xi_2\xi_3\xi_4\cdots\xi_{2n-1})\xi_1 \\ &\quad (\xi_{2n-1}\xi_{2n-2}\cdots\xi_4\xi_3\xi_2)\cdots c_{n-1} \\ &= c_{n-1}c_{n-2}\cdots c_2(\xi_2\xi_1\xi_2)\cdots c_{n-1} \\ &= \xi_2\xi_1\xi_2. \end{aligned}$$

Suppose (3.1) is true for $i-1$.
Then we have

$$\begin{aligned} c\xi_ic^{-1} &= (c\xi_{i-1}c^{-1})(c\xi_{i-1}\xi_i\xi_{i-1}c^{-1})(c\xi_{i-1}c^{-1}) \\ &= (c\xi_{i-1}c^{-1})c_{n-1}\cdots c_2(\xi_2\xi_3\cdots\xi_{i-1}\xi_{i+1}\xi_i\xi_{i+1}\xi_{i-1}\cdots\xi_3\xi_2) \\ &\quad c_2\cdots c_{n-1}(c\xi_{i-1}c^{-1}) \\ &= (c\xi_{i-1}c^{-1})c_{n-1}\cdots c_2(\xi_i\xi_{i+1}\xi_i)c_2\cdots c_{n-1}(c\xi_{i-1}c^{-1}) \\ &= (c\xi_{i-1}c^{-1})c_{n-1}\cdots c_i(\xi_{2i-2}\xi_{2i-1}\xi_{2i-3}\xi_{2i-2}\xi_{2i-3}\xi_{2i-1}\xi_{2i-2}) \\ &\quad c_i\dots c_{n-1}(c\xi_{i-1}c^{-1}) \\ &= (c\xi_{i-1}c^{-1})(\xi_{2i-2}\xi_{2i-3})c_{n-1}\cdots c_i(\xi_{2i-2}\xi_{2i-1}\xi_{2i-2})c_i\cdots c_{n-1} \\ &\quad (\xi_{2i-3}\xi_{2i-2})(c\xi_{i-1}c^{-1}) \\ &= (c\xi_{i-1}c^{-1})(\xi_{2i-2}\xi_{2i-3})(\xi_{2i-2})c_{n-1}\cdots c_{i+1}(\xi_{2i-1}\xi_{2i}\xi_{2i-1})c_{i+1} \\ &\quad c_{i+1}\cdots c_{n-1}(\xi_{2i-2})(\xi_{2i-3}\xi_{2i-2})(c\xi_{i-1}c^{-1}) \\ &= (\xi_{2i-2}\xi_{2i-3}\xi_{2i-2})(\xi_{2i-2}\xi_{2i-3})(\xi_{2i-2})(\xi_{2i-1}\xi_{2i}\xi_{2i-1}) \\ &\quad (\xi_{2i-2})(\xi_{2i-3}\xi_{2i-2})(\xi_{2i-2}\xi_{2i-3}\xi_{2i-2}) \\ &= \xi_{2i-1}\xi_{2i}\xi_{2i-1} \\ &= \xi_{2i}\xi_{2i-1}\xi_{2i}. \end{aligned}$$

By the similar calculations to the above, we can prove (3.2).

Proof of (3.3). Let

$$d = d_n d_{n-1} \cdots d_i \cdots d_3 d_2 d_1,$$

where $d_i = (\xi_{2i-1}\xi_{2i}\xi_{2i-1} \cdots \xi_{2n-2}\xi_{2n-1})$ for $i = 1, \ldots, n-1$.
For $i = 1$, we have

$$\begin{aligned} d\xi_1 d^{-1} &= d_n d_{n-1} \cdots d_2 (\xi_1\xi_2\xi_3 \cdots \xi_{2n-2}\xi_{2n-1})\xi_1 \\ &\quad (\xi_{2n-1}\xi_{2n-2} \cdots \xi_3\xi_2\xi_1) \\ &\quad d_2 \cdots d_{n-1} d_n \\ &= d_n d_{n-1} \cdots d_2 (\xi_1\xi_2\xi_1\xi_2\xi_1) d_2 \cdots d_{n-1} d_n \\ &= d_n d_{n-1} \cdots d_2 (\xi_2) d_2 \cdots d_{n-1} d_n \\ &= d_n d_{n-1} \cdots d_3 (\xi_3\xi_2\xi_3) d_3 \cdots d_{n-1} d_n \\ &= \xi_3\xi_2\xi_3. \end{aligned}$$

Suppose (3.3) is true for $i-1$. Then we have

$$\begin{aligned} d\xi_i d^{-1} &= (d\xi_{i-1}d^{-1})(d\xi_{i-1}\xi_i\xi_{i-1}d^{-1})(d\xi_{i-1}d^{-1}) \\ &= (d\xi_{i-1}d^{-1}) d_n \cdots d_2 (\xi_1\xi_2\xi_3 \cdots \xi_{2n-1})(\xi_{i-1}\xi_i\xi_{i-1}) \\ &\quad (\xi_{2n-1} \cdots \xi_3\xi_2\xi_1) d_2 \cdots d_n (d\xi_{i-1}d^{-1}) \\ &= (d\xi_{i-1}d^{-1}) d_n \cdots d_2 (\xi_1\xi_2\xi_3 \cdots \xi_{i-1}\xi_{i+1}\xi_i\xi_{i+1}\xi_{i-1} \cdots \xi_3\xi_2\xi_1) \\ &\quad d_2 \cdots d_n (d\xi_{i-1}d^{-1}) \\ &= (d\xi_{i-1}d^{-1}) d_n \cdots d_2 (\xi_i\xi_{i+1}\xi_i) d_2 \cdots d_{n-1} (d\xi_{i-1}d^{-1}) \\ &= (d\xi_{i-1}d^{-1}) d_n \cdots d_i (\xi_{2i-2}\xi_{2i-1}\xi_{2i-2}) d_i \cdots d_n (d\xi_{i-1}d^{-1}) \\ &= (d\xi_{i-1}d^{-1})(\xi_{2i-1}\xi_{2i-2}) d_n \cdots d_{i+1} (\xi_{2i-1}\xi_{2i}\xi_{2i-1}) \\ &\quad d_{i+1} \cdots d_n (\xi_{2i-2}\xi_{2i-1})(d\xi_{i-1}d^{-1}) \\ &= (d\xi_{i-1}d^{-1})(\xi_{2i-1}\xi_{2i-2})(\xi_{2i-1}) \\ &\quad d_n \cdots d_{i+2} (\xi_{2i+1}\xi_{2i+2}\xi_{2i}\xi_{2i+2}\xi_{2i+1}) \\ &\quad d_{i+2} \cdots d_n (\xi_{2i-1})(\xi_{2i-2}\xi_{2i-1})(d\xi_{i-1}d^{-1}) \\ &= (\xi_{2i-1}\xi_{2i-2}\xi_{2i-1})(\xi_{2i-1}\xi_{2i-2})(\xi_{2i-1}) \\ &\quad d_n \cdots d_{i+2} (\xi_{2i+1}\xi_{2i+2}\xi_{2i}\xi_{2i+2}\xi_{2i+1}) \\ &\quad d_{i+2} \cdots d_n (\xi_{2i-1})(\xi_{2i-2}\xi_{2i-1})(\xi_{2i-1}\xi_{2i-2}\xi_{2i-1}) \\ &= (\xi_{2i-1}\xi_{2i-2}\xi_{2i-1})(\xi_{2i-1}\xi_{2i-2})(\xi_{2i-1})(\xi_{2i+1}\xi_{2i+2}\xi_{2i}\xi_{2i+2}\xi_{2i+1}) \\ &\quad (\xi_{2i-1})(\xi_{2i-2}\xi_{2i-1})(\xi_{2i-1}\xi_{2i-2}\xi_{2i-1}) \\ &= \xi_{2i+1}\xi_{2i}\xi_{2i+1}. \end{aligned}$$

□

The following Figures illustrate the calculations in the proof of the theorem.

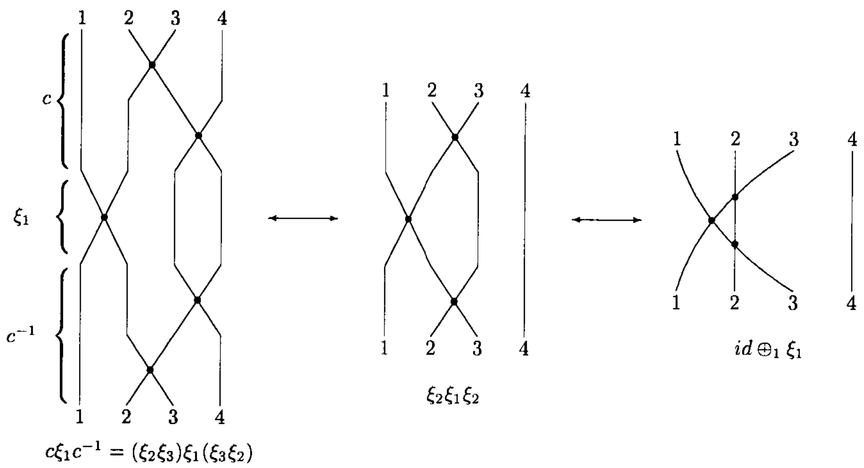


Figure 3.2.

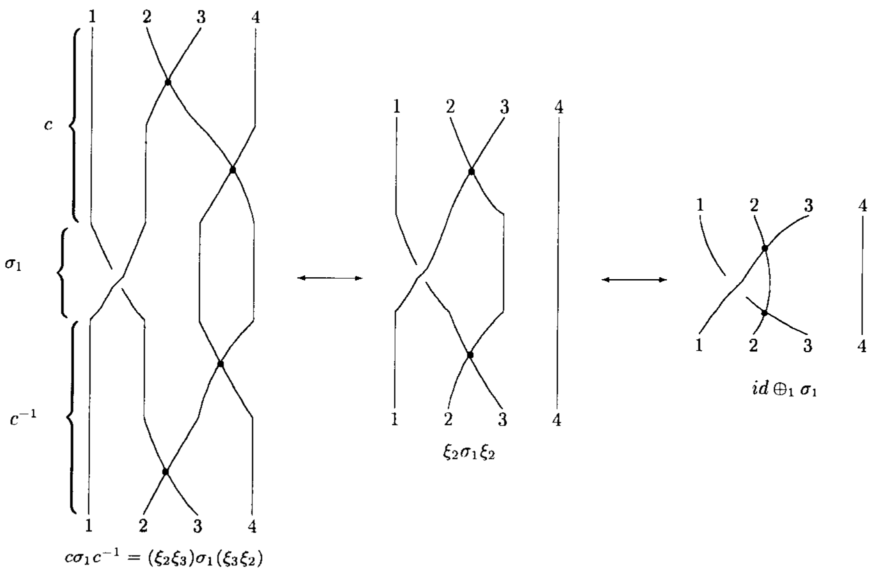


Figure 3.3.

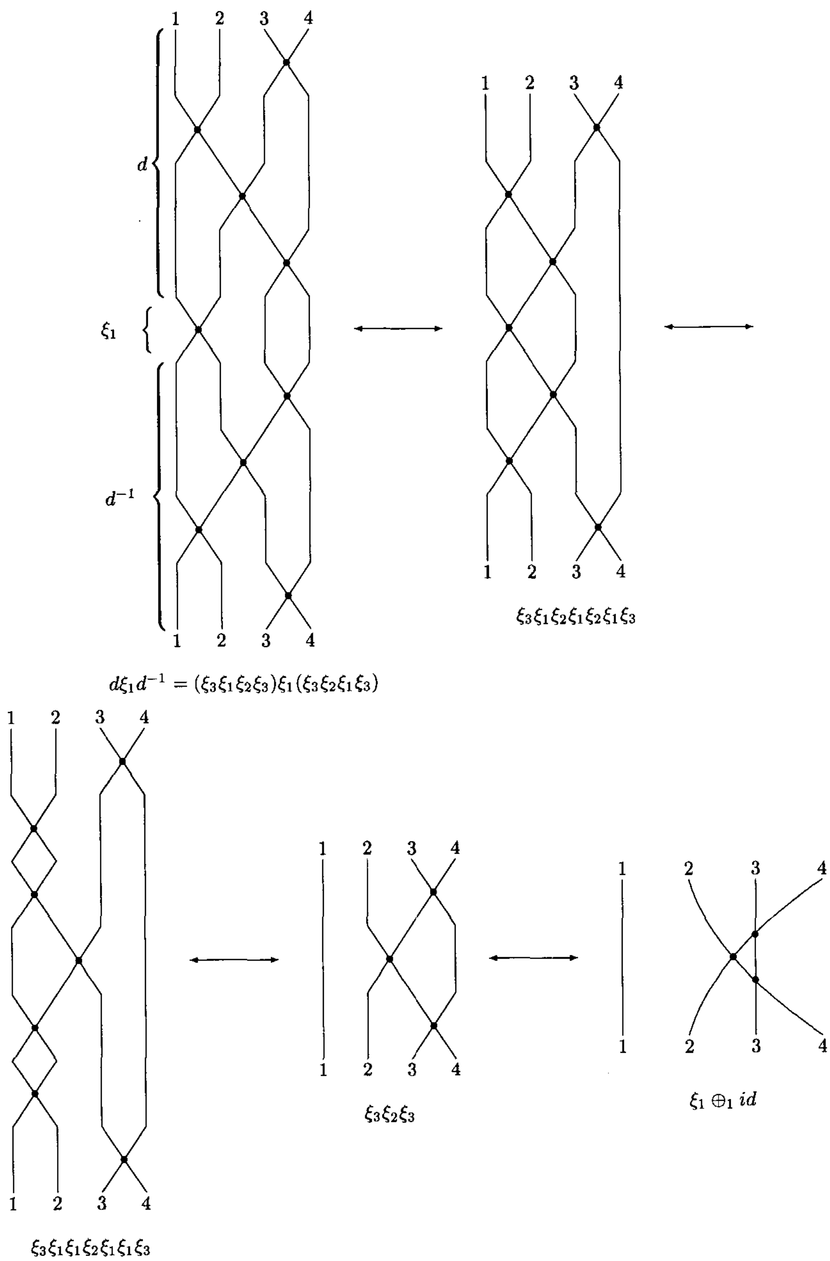

1 2 3 4
d
ξ_1
d^{-1}
$d\xi_1 d^{-1} = (\xi_3\xi_1\xi_2\xi_3)\xi_1(\xi_3\xi_2\xi_1\xi_3)$
$\xi_3\xi_1\xi_2\xi_1\xi_2\xi_1\xi_3$
$\xi_3\xi_1\xi_1\xi_2\xi_1\xi_1\xi_3$
$\xi_3\xi_2\xi_3$
$\xi_1 \oplus_1 id$

Figure 3.4.

THEOREM 3.5. *There exist maps*

$$Bj_{\infty}^{+} : B\mathbb{Z}^{+} \to BBP_{\infty}^{+}$$

and

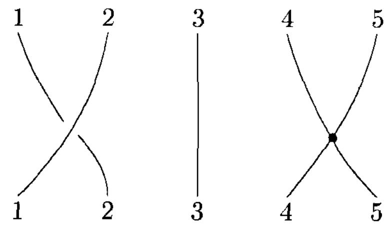
$$B\alpha_{\infty}^{+} : BBP_{\infty}^{+} \to B\mathbb{Z}^{+}$$

such that $B\alpha_{\infty}^{+}$ *splits by the map* Bj_{∞}^{+}.

If a space Y *is a fiber of the map* $B\alpha_{\infty}^{+}$, *then we have the following splitting:*

$$BBP_{\infty}^{+} \simeq B\mathbb{Z}^{+} \times Y = S^1 \times Y.$$

Proof. We may regard $\mathbb{Z}$ as the infinite cyclic subgroup of B_n generated by σ_1, the first generator of B_n. Define a map $\oplus_2 : \mathbb{Z} \times BP_{\infty} \to BP_{\infty}$ by $n \oplus_2 \sigma = \sigma_1^n \amalg \sigma$ for $n \in \mathbb{Z}$, $\sigma \in BP_{\infty}$. Here $\sigma_1^n \amalg \sigma$ means the juxtaposition of σ_1^n and σ. We may think that σ_1^n lies on the left-hand side of σ. For example, for $(1, \xi_2) \in \mathbb{Z} \times BP_{\infty}$, $1 \oplus_2 \xi_2$ looks as follows:



$1 \oplus_2 \xi_2$

Figure 3.5.

It is clear that $\oplus_2$ is a homomorphism.

For $\xi_1, \ldots, \xi_{n-1}, \sigma_1, \ldots, \sigma_{n-1} \in BP_n$, let

$$c = (\xi_2\xi_3 \cdots \xi_{n+1})(\xi_1\xi_2 \cdots \xi_n), \quad d = id.$$

Note that c belongs to the maximal perfect subgroup of BP_m, and by the definition of $\oplus_2$ we have

$$0 \oplus_2 \xi_i = \xi_{i+2}, \quad 0 \oplus_2 \sigma_i = \sigma_{i+2},$$

$$n \oplus_2 id = \sigma_1^n \quad \text{for } i = 1, 2, 3, \ldots, n-1.$$

It suffices to show that the following equations:

$$n \oplus_2 id = d\sigma_1^n d^{-1} \quad \text{for } n \in \mathbb{Z}, \tag{3.4}$$

$$0 \oplus_2 \xi_i = c\xi_i c^{-1} \quad \text{for } i = 1, \ldots, n-1, \tag{3.5}$$

and

$$(3.6) \qquad 0 \oplus_2 \sigma_i = c\sigma_i c^{-1} \quad \text{for } i = 1, \ldots, n-1;$$

Proof of (3.5).

$$\begin{aligned}
c\xi_i c^{-1} &= (\xi_2\xi_3\cdots\xi_i\xi_{i+1}\cdots\xi_{n+1})(\xi_1\xi_2\cdots\xi_i\xi_{i+1}\cdots\xi_n)\xi_i \\
&\quad (\xi_n\cdots\xi_{i+1}\xi_i\cdots\xi_2\xi_1)(\xi_{n+1}\cdots\xi_{i+1}\xi_i\cdots\xi_3\xi_2) \\
&= (\xi_2\xi_3\cdots\xi_{n+1})(\xi_1\xi_2\cdots\xi_i\xi_{i+1}\xi_i\xi_{i+1}\xi_i\cdots\xi_2\xi_1) \\
&\quad (\xi_{n+1}\cdots\xi_3\xi_2) \\
&= (\xi_2\xi_3\cdots\xi_{n+1})(\xi_1\xi_2\cdots\xi_i\xi_i\xi_{i+1}\xi_i\xi_i\cdots\xi_2\xi_1) \\
&\quad (\xi_{n+1}\cdots\xi_3\xi_2) \\
&= (\xi_2\xi_3\cdots\xi_{n+1})\xi_{i+1}(\xi_{n+1}\cdots\xi_3\xi_2) \\
&= (\xi_2\xi_3\cdots\xi_i\xi_{i+1}\xi_{i+2}\xi_{i+1}\xi_{i+2}\xi_{i+1}\xi_i\cdots\xi_3\xi_2) \\
&= (\xi_2\xi_3\cdots\xi_i\xi_{i+1}\xi_{i+1}\xi_{i+2}\xi_{i+1}\xi_{i+1}\xi_i\cdots\xi_3\xi_2) \\
&= \xi_{i+2}.
\end{aligned}$$

Proof of (3.6).

$$\begin{aligned}
c\sigma_i c^{-1} &= (\xi_2\xi_3\cdots\xi_i\xi_{i+1}\cdots\xi_{n+1})(\xi_1\xi_2\cdots\xi_i\xi_{i+1}\cdots\xi_n)\sigma_i \\
&\quad (\xi_n\cdots\xi_{i+1}\xi_i\cdots\xi_2\xi_1)(\xi_{n+1}\cdots\xi_{i+1}\xi_i\cdots\xi_3\xi_2) \\
&= (\xi_2\xi_3\cdots\xi_{n+1})(\xi_1\xi_2\cdots\xi_i\xi_{i+1}\sigma_i\xi_{i+1}\xi_i\cdots\xi_2\xi_1) \\
&\quad (\xi_{n+1}\cdots\xi_3\xi_2) \\
&= (\xi_2\xi_3\cdots\xi_{n+1})(\xi_1\xi_2\cdots\xi_i\xi_i\sigma_{i+1}\xi_i\xi_i\cdots\xi_2\xi_1) \\
&\quad (\xi_{n+1}\cdots\xi_3\xi_2) \\
&= (\xi_2\xi_3\cdots\xi_{n+1})\sigma_{i+1}(\xi_{n+1}\cdots\xi_3\xi_2) \\
&= (\xi_2\xi_3\cdots\xi_i\xi_{i+1}\xi_{i+2}\sigma_{i+1}\xi_{i+2}\xi_{i+1}\xi_i\cdots\xi_3\xi_2) \\
&= (\xi_2\xi_3\cdots\xi_i\xi_{i+1}\xi_{i+1}\sigma_{i+2}\xi_{i+1}\xi_{i+1}\xi_i\cdots\xi_3\xi_2) \\
&= \sigma_{i+2}.
\end{aligned}$$

□

There is a geometric interpretation of the above calculations as we have had in the Figures 3.2, 3.3, 3.4. We leave the finding of this to the readers.

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