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ON STARLIKENESS AND CLOSE-TO-CONVEXITY OF CERTAIN MEROMORPHIC FUNCTIONS

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ABSTRACT. In this paper we derive some sufficient conditions for starlikeness and close-to-convexity of order α of meromorphic functions in the punctured unit disk.

1. INTRODUCTION

Let Σ_n $(n \in \mathbb{N} = \{1, 2, 3, \cdots\})$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=n-1}^{\infty} a_m z^m \tag{1}$$

which are analytic in the punctured unit disk $E_0 = \{z : 0 < |z| < 1\}$. A function $f(z) \in \Sigma_n$ is said to *be meromorphic starlike of order* α if it satisfies

$$-\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha \quad (z \in E = E_0 \cup \{0\}) \tag{2}$$

for some α $(0 \leq \alpha < 1)$. We denote by $\Sigma_n^*(\alpha)$ $(0 \leq \alpha < 1)$ the class of all meromorphic starlike functions of order α. Also we write $\Sigma_1 = \Sigma$ and $\Sigma_1^*(\alpha) = \Sigma^*(\alpha)$.

Let $MC_n(\alpha)$ be the subclass of Σ_n consisting of functions $f(z)$ which satisfy

$$-\operatorname{Re}\{z^2 f'(z)\} > \alpha \quad (z \in E) \tag{3}$$

for some α $(0 \leq \alpha < 1)$. A function $f(z)$ in $MC_n(\alpha)$ $(0 \leq \alpha < 1)$ is meromorphic close-to-convex of order α (see, *e. g.*, Ganigi & Uralegaddi [2]).

Meromorphic starlike functions and meromorphic close-to-convex functions have been studied by several authors.

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2. PRELIMINARIES

Let $f(z)$ and $g(z)$ be analytic in E. Then the function $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1 (z \in E)$ such that $f(z) = g(w(z))$ for $z \in E$. If $g(z)$ is univalent in E, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

To derive our results, we need the following lemmas.

Lemma 1. *Let $g(z) = b_0 + b_n z^n + b_{n+1}z^{n+1} + \cdots$ $(n \in \mathbb{N})$ be analytic in E and let $h(z)$ be analytic and starlike (with respect to the origin) univalent in E with $h(0) = 0$. If $zg'(z) \prec h(z)$, then*

$$g(z) \prec b_0 + \frac{1}{n}\int_0^z \frac{h(t)}{t}\,dt.$$

It is clear that this lemma is equivalent to of Yang [5, lemma 2].

Lemma 2 (Miller & Mocanu [4]). *Let $g(z)$ be analytic and univalent in E and let $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $g(E)$, with $\varphi(w) \neq 0$ when $w \in g(E)$. Set*

$$Q(z) = zg'(z)\varphi(g(z)), \quad h(z) = \theta(g(z)) + Q(z)$$

and suppose that

(i) *$Q(z)$ is univalent and starlike in E, and*
(ii) $\operatorname{Re}\dfrac{zh'z)}{Q(z)} = \operatorname{Re}\left\{\dfrac{\theta'(g(z))}{\varphi(g(z))} + \dfrac{zQ'(z)}{Q(z)}\right\} > 0 \ (z \in E)$.

If $p(z)$ is analytic in E, with $p(0) = g(0), p(E) \subset D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z), \tag{4}$$

then $p(z) \prec g(z)$ and $g(z)$ is the best dominant of (4).

Lemma 3 (Miller & Mocanu [3]). *Let $g(z) = b_0 + b_n z^n + b_{n+1}z^{n+1} + \cdots$ $(n \in \mathbb{N})$ be analytic in E with $g(z) \not\equiv b_0$. If $0 < |z_0| < 1$ and $\operatorname{Re} g(z_0) = \min_{|z| \le |z_0|} \operatorname{Re} g(z)$, then*

$$z_0 g'(z_0) \le -\frac{n|b_0 - g(z_0)|^2}{2\operatorname{Re}\{b_0 - g(z_0)\}}.$$

3. Main results

Applying Lemma 1, we derive the following theorem.

Theorem 1. *Let $f(z) \in \Sigma_n$ satisfy $f(z)f'(z) \neq 0$ in E_0 and*

$$-\alpha\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 2 - \alpha \prec \frac{az}{1-bz}, \tag{5}$$

where α, a and b are real numbers with $a \neq 0$ and $0 \leq b \leq 1$.

(i) *If $0 < a \leq n$ and $0 < b \leq 1$, then*

$$-\operatorname{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\} > \left(\frac{1}{1+b}\right)^{\frac{a}{nb}} \quad (z \in E).$$

(ii) *If $0 < a \leq n$ and $b = 0$, then*

$$-\operatorname{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\} > e^{-\frac{a}{n}} \quad (z \in E).$$

(iii) *If $a \neq 0$ and $0 < b \leq 1$, then*

$$\left|\left(-\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right)^{-\frac{nb}{a}} - 1\right| < b \quad (z \in E).$$

(iv) *If $a > 0$ and $b = 0$, then*

$$\left|\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)} + 1\right| < e^{\frac{a}{n}} - 1 \quad (z \in E).$$

Proof. Let $f(z) \in \Sigma_n$ with $f(z)f'(z) \neq 0$ $(z \in E_0)$ and define

$$g(z) = -\alpha\left(\frac{zf'(z)}{f(z)} + 1\right) + \frac{zf''(z)}{f'(z)} + 2.$$

Then $g(z) = b_n z^n + b_{n+1}z^{n+1} + \cdots$ is analytic in E and (5) can be rewritten as

$$g(z) \prec h(z), \tag{6}$$

where $h(z) = az/(1-bz)$ is analytic and starlike univalent in E. Applying Lemma 1 to (6) we have

$$\int_0^z \frac{g(t)}{t}\,dt \prec \frac{1}{n}\int_0^z \frac{h(t)}{t}\,dt,$$

that is,

$$-\alpha\int_0^z \left(\frac{f'(t)}{f(t)} + \frac{1}{t}\right)dt + \int_0^z \left(\frac{f''(t)}{f'(t)} + \frac{2}{t}\right)dt \prec \frac{a}{n}\int_0^z \frac{dt}{1-bt}. \tag{7}$$

(i) If $0 < a \le n$ and $0 < b \le 1$, then from (7) we deduce that

$$-\frac{z^{2-\alpha}f'(z)}{f^\alpha(z)} \prec \left(\frac{1}{1-bz}\right)^{\frac{a}{nb}} \equiv h_1(z). \tag{8}$$

The function $h_1(z)$ is analytic and convex univalent in E because

$$\mathrm{Re}\left\{1+\frac{zh_1''(z)}{h_1'(z)}\right\} = \mathrm{Re}\,\frac{1+(a/n)z}{1-bz} > \frac{1-(a/n)}{1+b} \ge 0 \quad (z \in E).$$

Also $h_1(E)$ is symmetric with respect t_0 the real axis. Hence $\mathrm{Re}\, h_1(z) > h_1(-1)$ in E and it follows from (8) that

$$-\mathrm{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^\alpha(z)}\right\} > \left(\frac{1}{1+b}\right)^{\frac{a}{nb}} \quad (z \in E).$$

(ii) If $0 < a \le n$ and $b = 0$, then from (7) we obtain

$$-\frac{z^{2-\alpha}f'(z)}{f^\alpha(z)} \prec e^{\frac{a}{n}z} \equiv h_2(z). \tag{9}$$

Since $h_2(z)$ is analytic and convex univalent in E and $h_2(E)$ is symmetric with respect to the real axis, it follows from (9) that

$$-\mathrm{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^\alpha(z)}\right\} > e^{-\frac{a}{n}} \quad (z \in E).$$

(iii) If $a \ne 0$ and $0 < b \le 1$, then by (8) we have

$$-\frac{z^{2-\alpha}f'(z)}{f^\alpha(z)} = \left(\frac{1}{1-bw(z)}\right)^{\frac{a}{nb}} \quad (z \in E),$$

where $w(z)$ is analytic in E with $|w(z)| \le |z|$ $(z \in E)$. Therefore,

$$\left|\left(-\frac{z^{2-\alpha}f'(z)}{f^\alpha(z)}\right)^{-\frac{nb}{a}} - 1\right| = |-bw(z)| < b \quad (z \in E).$$

(iv) If $a > 0$ and $b = 0$, then from (9) we get

$$-\frac{z^{2-\alpha}f'(z)}{f^\alpha(z)} = e^{\frac{a}{n}w(z)} \ (z \in E),$$

where $w(z)$ is analytic in E with $|w(z)| \le |z|$ $(z \in E)$. Thus

$$\left|\frac{z^{2-\alpha}f'(z)}{f^\alpha(z)} + 1\right| = \left|e^{\frac{a}{n}w(z)} - 1\right| \le e^{\frac{a}{n}|w(z)|} - 1 < e^{\frac{a}{n}} - 1$$

for $z \in E$. The proof of the theorem is now complete. □

By specifying the values of the parameters appearing in Theorem 1, we can obtain several useful consequences.

Taking $n = 1$, $0 < a = 2(2 - \alpha - \beta) \leq 1$ and $b = 1$, Theorem 1(i) reduces to the following corollary.

Corollary 1. *If $f(z) \in \Sigma$ satisfies $f(z)f'(z) \neq 0$. In E_0 and*

$$\operatorname{Re}\left\{\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < 2(2-\alpha) - \beta \quad (z \in E), \tag{10}$$

where α is real and $3/2 - \alpha \leq \beta < 2 - \alpha$, then

$$-\operatorname{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\} > \frac{1}{4^{2-\alpha-\beta}} \quad (z \in E).$$

Remark 1. Cho & Owa [1] proved that if $f(z) \in \Sigma_2$ satisfies $f(z)f'(z) \neq 0$ in E_0 and (10) for $\alpha \leq 2$ and $3/2 - \alpha \leq \beta < 2 - \alpha$, then

$$-\operatorname{Re}\left\{\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)}\right\} > \frac{1}{1 + 2(2-\alpha-\beta)} \quad (z \in E).$$

In view of $\Sigma_2 \subset \Sigma$ and $2^a < 1 + a$ $(0 < a < 1)$, our Corollary 1 is a better result than the main theorem of Cho & Owa [1].

Corollary 2. *If $f(z) \in \Sigma_n$ satisfies $f(z)f'(z) \neq 0$ in E_0 and*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < 1 + \frac{a}{2} \quad (z \in E) \tag{11}$$

for some a $(0 < a \leq n)$, then $f(z) \in \Sigma_n^*(2^{-a/n})$ and the order $2^{-a/n}$ is sharp.

Proof. Letting $\alpha = b = 1$ in Theorem 1(i) and using (11), we see that

$$f(z) \in \Sigma_n^*(2^{-a/n}).$$

To show that the order $2^{-a/n}$ cannot be increased, we consider

$$f(z) = \frac{1}{z}\exp\int_0^z \frac{1 - (1+t^n)^{-a/n}}{t}\,dt \in \Sigma_n.$$

It is easy to verify that this function $f(z)$ satisfies (11) and

$$-\operatorname{Re}\frac{zf'(z)}{f(z)} = \operatorname{Re}\left\{\left(\frac{1}{1+z^n}\right)^{\frac{a}{n}}\right\} \to \left(\frac{1}{2}\right)^{\frac{a}{n}}$$

as $z \to 1$. The proof is complete. □

Putting $\alpha = 0$ and $b = 1$ in Theorem 1(i), we have the following corollary.

Corollary 3. *If $f(z) \in \Sigma_n$ satisfies $f'(z) \neq 0$ in E_0 and*

$$-\operatorname{Re}\frac{zf''(z)}{f'(z)} < 2 + \frac{a}{2} \quad (z \in E)$$

for some a $(0 < a \leq n)$, then $f(z) \in MC_n(2^{-a/n})$.

Remark 2. For $n = 1$, Corollary 2 (with $0 < a = 2(1-\beta) \leq 1$) and Corollary 3 (with $0 < a = 2(2-\beta) \leq 1$) are better than the corresponding results in Cho & Owa [1]. Setting $\alpha = 0, 1$ in Theorem 1(ii), we have the following two corollaries.

Corollary 4. *If $f(z) \in \Sigma_n$ satisfies $f'(z) \neq 0$ in E_0 and*

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| < a \quad (z \in E)$$

for some a $(0 < a \leq n)$, then $f(z) \in MC_n(e^{-a/n})$.

Corollary 5. *If $f(z) \in \Sigma_n$ satisfies $f(z)f'(z) \neq 0$ in E_0 and*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < a \quad (z \in E)$$

for some a $(0 < a \leq n)$, then $f(z) \in \Sigma_n^*(e^{-a/n})$ and the order $e^{-a/n}$ is sharp with the extremal function

$$f(z) = \frac{1}{z}\,\exp\int_0^z \frac{1 - e^{-(a/n)t^n}}{t}\,dt.$$

For $\alpha = 1$ and $a = -nb$ $(0 < b \leq 1)$, Theorem 1(iii) yields.

Corollary 6. *If $f(z) \in \Sigma_n$ satisfies $f(z)f'(z) \neq 0$ in E_0 and*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec -\frac{nbz}{1 - bz}$$

for some b $(0 < b \leq 1)$, then $f(z) \in \Sigma_n^*(1-b)$ and the order $1-b$ is sharp with the extremal function $f(z) = e^{(b/n)z^n}/z$.

Next, applying Lemma 2, we derive the following two results.

Theorem 2. *If $f(z) \in \Sigma$ satisfies $f(z) \neq 0$ in E_0 and*

$$-\frac{zf'(z)}{f(z)} - \frac{z^2f''(z)}{f(z)} + 2\left(\frac{zf'(z)}{f(z)}\right)^2 \prec h(z), \tag{12}$$

where

$$h(z) = \frac{(1-2\alpha)^2z^2 + 2(2-3\alpha)z + 1}{(1-z)^2}, \quad 0 \leq \alpha < 1, \tag{13}$$

then $f(z) \in \Sigma^*(\alpha)$ and the order α is sharp.

Proof. Let us put

$$-\frac{zf'(z)}{f(z)} = (1-\alpha)p(z) + \alpha \tag{14}$$

for $0 \le \alpha < 1$. Then $p(z)$ is analytic in E and $p(0) = 1$. Differentiating (14) logarithmically, we find that

$$\begin{aligned} -\frac{zf'(z)}{f(z)} - \frac{z^2 f''(z)}{f(z)} + 2\left(\frac{zf'(z)}{f(z)}\right)^2 & \\ = -\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right) & \\ = (1-\alpha)zp'(z) + ((1-\alpha)p(z)+\alpha)^2. & \end{aligned} \tag{15}$$

From (12) and (15) we have

$$(1-\alpha)zp'(z) + (1-\alpha)^2 p^2(z) + 2\alpha(1-\alpha)p(z) + \alpha^2 \prec h(z). \tag{16}$$

We choose

$$g(z) = \frac{1+z}{1-z}, \quad \theta(w) = (1-\alpha)^2 w^2 + 2\alpha(1-\alpha)w + \alpha^2, \quad \varphi(w) = 1-\alpha. \tag{17}$$

Then $g(z)$ is analytic and univalent in E, $\operatorname{Re} g(z) > 0$ $(z \in E)$, $\theta(w)$ and $\varphi(w)$ are analytic with $\varphi(w) \neq 0$ in the w-plane. The function

$$Q(z) = zg'(z)\varphi(g(z)) = 2(1-\alpha)\frac{z}{(1-z)^2} \tag{18}$$

is univalent and starlike in E. Further,

$$\begin{aligned} &\theta(g(z)) + Q(z) \\ &\quad = (1-\alpha)^2\left(\frac{1+z}{1-z}\right)^2 + 2\alpha(1-\alpha)\left(\frac{1+z}{1-z}\right) + \alpha^2 + 2(1-\alpha)\frac{z}{(1-z)^2} \\ &\quad = \frac{(1-2\alpha)^2 z^2 + 2(2-3\alpha)z + 1}{(1-z)^2} = h(z) \end{aligned} \tag{19}$$

and

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left\{2(1-\alpha)g(z) + 2\alpha + \frac{zQ'(z)}{Q(z)}\right\} = (3-2\alpha)\operatorname{Re}\frac{1+z}{1-z} + 2\alpha > 0$$

for $z \in E$. In view of (16)–(19), we see that

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z).$$

Therefore, Lemma 2 leads to $p(z) \prec g(z)$, which implies that $f(z) \in \Sigma^*(\alpha)$.

Next we consider

$$f(z) = \frac{(1-z)^{2(1-\alpha)}}{z} \in \Sigma. \tag{20}$$

It is easy to see that

$$-\frac{zf'(z)}{f(z)} - \frac{z^2 f''(z)}{f(z)} + 2\left(\frac{zf'(z)}{f(z)}\right)^2 = h(z)$$

and

$$-\operatorname{Re}\frac{zf'(z)}{f(z)} = \operatorname{Re}\frac{1+(1-2\alpha)z}{1-z} \longrightarrow \alpha$$

as $z \longrightarrow -1$. The proof of the theorem is complete. □

Theorem 3. *If $f(z) \in \Sigma$ satisfies $f(z) \neq 0$ in E_0 and*

$$-\frac{zf'(z)}{f(z)} - 2\alpha\frac{z^2 f''(z)}{f(z)} + 4\alpha\left(\frac{zf'(z)}{f(z)}\right)^2 \prec h(z), \tag{21}$$

where

$$h(z) = \frac{(2\alpha-1)^3 z^2 + 2\alpha(3-4\alpha)z + 1}{(1-z)^2}, \quad 0 \leq \alpha < 1,$$

then $f(z) \in \Sigma^*(\alpha)$ and the order α is sharp.

Proof. It suffices to prove the theorem for $0 < \alpha < 1$. We define the function $p(z)$ by (14). Then $p(z)$ is analytic in E and $p(0) = 1$. By a brief calculation, we find that

$$\begin{aligned} &-\frac{zf'(z)}{f(z)} - 2\alpha\frac{z^2 f''(z)}{f(z)} + 4\alpha\left(\frac{zf'(z)}{f(z)}\right)^2 \\ &= -\frac{zf'(z)}{f(z)}\left(1 + 2\alpha\frac{zf''(z)}{f'(z)} - 4\alpha\frac{zf'(z)}{f(z)}\right) \\ &= 2\alpha(1-\alpha)zp'(z) + 2\alpha(1-\alpha)^2 p^2(z) \\ &\qquad + (1-\alpha)(1-2\alpha+4\alpha^2)p(z) + \alpha(1-2\alpha+2\alpha^2). \end{aligned}$$

Thus the subordination (21) becomes

$$2\alpha(1-\alpha)zp'(z) + 2\alpha(1-\alpha)^2 p^2(z) + (1-\alpha)(1-2\alpha+4\alpha^2)p(z) + \alpha(1-2\alpha+2\alpha^2) \prec h(z).$$

Set $g(z) = (1+z)/(1-z)$, $\theta(w) = 2\alpha(1-\alpha)^2 w^2 + (1-\alpha)(1-2\alpha+4\alpha^2)w + \alpha(1-2\alpha+2\alpha^2)$ and $\varphi(w) = 2\alpha(1-\alpha)$. Then $g(z)$, $\theta(w)$ and $\varphi(w)$ satisfy the conditions of Lemma 2. The function

$$Q(z) = zg'(z)\varphi(g(z)) = 4\alpha(1-\alpha)\frac{z}{(1-z)^2}$$

is univalent and starlike in E. Further,

$$\begin{aligned}
&\theta(g(z)) + Q(z) \\
&\quad = 2\alpha(1-\alpha)^2\left(\frac{1+z}{1-z}\right)^2 + (1-\alpha)(1-2\alpha+4\alpha^2)\left(\frac{1+z}{1-z}\right) \\
&\qquad + \alpha(1-2\alpha+2\alpha^2) + 4\alpha(1-\alpha)\frac{z}{(1-z)^2} \\
&\quad = \frac{(2\alpha-1)^3 z^2 + 2\alpha(3-4\alpha)z + 1}{(1-z)^2} = h(z)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Re}\frac{zh'(z)}{Q(z)} &= \operatorname{Re}\left\{2(1-\alpha)g(z) + \frac{1-2\alpha+4\alpha^2}{2\alpha} + \frac{zQ'(z)}{Q(z)}\right\} \\
&= (3-2\alpha)\operatorname{Re}\frac{1+z}{1-z} + \frac{1-2\alpha+4\alpha^2}{2\alpha} > 0
\end{aligned}$$

for $z \in E$. Note that

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z).$$

Hence, an application of Lemma 2 yields that $p(z) \prec g(z)$, that is, $f(z) \in \Sigma^*(\alpha)$.

For the function $f(z)$ defined by (20), we have

$$-\frac{zf'(z)}{f(z)} - 2\alpha\frac{z^2 f''(z)}{f(z)} + 4\alpha\left(\frac{zf'(z)}{f(z)}\right)^2 = h(z)$$

and

$$-\operatorname{Re}\frac{zf'(z)}{f(z)} \to \alpha \quad \text{as} \quad z \to -1.$$

Hence the theorem is proved. □

Finally, by using Lemma 3, we prove have the following theorem.

Theorem 4. *Let $f(z) \in \Sigma_n$ satisfy $f(z) \neq 0$ in E_0 and*

$$\left|\arg\left\{-\lambda\left(\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)}\right) + (1+\lambda)\left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{n\lambda}{2}\right\}\right| < \pi \quad (z \in E) \tag{22}$$

for some λ ($\lambda > 0$). Then $f(z) \in \Sigma_n^*(0)$ and the order 0 is sharp.

Proof. The function $g(z)$ defined by

$$g(z) = -\frac{zf'(z)}{f(z)} = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$$

is analytic in E and it is easily verified that

$$-\lambda\left(\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)}\right) + (1+\lambda)\left(\frac{zf'(z)}{f(z)}\right)^2 = g^2(z) + \lambda zg'(z) \quad (z \in E). \tag{23}$$

suppose that there exists a point $z_0 \in E_0$ such that

$$\operatorname{Re} g(z) > 0 \quad (|z| < |z_0|), \quad g(z_0) = i\beta, \tag{24}$$

where β is a real number. Then, applying Lemma 3, we have

$$z_0 g'(z_0) \leq -\frac{n(1+\beta^2)}{2}. \tag{25}$$

Thus it follows from (23), (24) and (25) that

$$\begin{aligned}
&-\lambda\left(\frac{z_0 f'(z_0)}{f(z_0)} + \frac{z_0^2 f''(z_0)}{f(z_0)}\right) + (1+\lambda)\left(\frac{z_0 f'(z_0)}{f(z_0)}\right)^2 + \frac{n\lambda}{2} \\
&= g^2(z_0) + \lambda z_0 g'(z_0) + \frac{n\lambda}{2} \\
&\leq -\beta^2 - \frac{n\lambda(1+\beta^2)}{2} + \frac{n\lambda}{2} \\
&\leq 0
\end{aligned}$$

for $\lambda > 0$, which contradicts (22). Hence $\operatorname{Re} g(z) > 0$ $(z \in E)$, that is, $f(z) \in \Sigma_n^*(0)$.

If we let $f_n(z) = (1-z^n)^{2/n}/z \in \Sigma_n$, then

$$\begin{aligned}
&-\lambda\left(\frac{z f_n'(z)}{f_n(z)} + \frac{z^2 f_n''(z)}{f_n(z)}\right) + (1+\lambda)\left(\frac{z f_n'(z)}{fn(z)}\right)^2 + \frac{n\lambda}{2} \\
&\qquad = \left(1 + \frac{n\lambda}{2}\right)\left(\frac{1+z^n}{1-z^n}\right)^2 \quad (z \in E),
\end{aligned}$$

and so the function $f_n(z)$ satisfies (22). Noting that

$$-\operatorname{Re}\frac{z f_n'(z)}{f_n(z)} = -\operatorname{Re}\frac{1+z^n}{1-z^n} \to 0$$

as $z \to e^{i\pi/n}$, we conclude that the order 0 is best possible. □

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