Burn-in When Repair Costs Vary With Time

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ABSTRACT

Burn-in is a widely used method to eliminate the initial failures. Preventive maintenance policy such as block replacement with minimal repair at failure is often used in field operation. In this paper burn-in and maintenance policy are taken into consideration at the same time. The cost of a minimal repair is assumed to be a non-decreasing function of its age. The problems of determining optimal burn-in times and optimal maintenance policy are considered.

1. Introduction

Let $F(t)$ be a distribution function of a lifetime X. If X has density $f(t)$ on $[0,\infty)$, then its failure rate function $h(t)$ is defined by $h(t)=f(t)/\overline{F}(t)$ where $\overline{F}(t)=1-F(t)$ is the survival function of X.

Based on the behavior of failure rate, various nonparametric classes of life distributions have been defined in the literature. The following is one definition of a bathtub-shaped failure rate function which we shall use.

Definition. A real-valued failure rate function $h(t)$ is said to be bathtub-shaped failure rate (BTR) with change points t_1 and t_2, if there exist change points $0\le t_1\le t_2<\infty$ such that $h(t)$ is strictly decreasing in $[0,t_1)$, constant in $[t_1, t_2)$ and then strictly increasing in $[t_2,\infty)$.

The time interval $[0,t_1]$ is called the infant mortality period; the interval $[t_1,t_2]$, where $h(t)$ is flat and attains its minimum value, is called the normal operating life or the useful life; the interval $[t_2,\infty)$ is called the wear-out period.

The most common popular maintenance policy might be the block replacement policy with minimal repair at failure. Under this policy the component is replaced at planned time $kT(k=1,2,\cdots)$, where T is a fixed number, and is minimally repaired at failure between planned replacements. Once a cost structure is established to model the total cost related to the maintenance policy adopted, an optimal T is determined (denoted by T^* and called the optimal maintenance policy) such that the cost will be minimized. Under the assumption that the underlying distribition F has increasing failure rate function, Barlow and Proschan (1965) have shown that an optimal block replacement policy exists, but may be infinite.

The optimal maintenance policy, however, clearly depends on the distribution function of the component used in operation. It is thus natural to take both burn-in and preventive maintenance into consideration. The burn-in procedure is stopped when a preassigned reliability goal is achieved, e.g. when the mean residual life is long enough. Since burn-in is usually costly, one of the major problem is to decide how long the procedure should continue. The best time to stop the burn-in process for a given criterion to be optimized is called the optimal burn-in time. An introduction to this important area of reliability can be found in Jensen and Petersen (1982). In the literature, certain cost structures have been proposed and the corresponding problem of finding the optimal burn-in time has been considered. See, for example, Clarotti and Spizzichino (1991) and Mi (1994). A survey of recent research in burn-in can be found in Block and Savits (1997).

Mi (1994) considers the following procedure. Consider a fixed burn-in time b and begin to burn-in a new component. If the component fails before burn-in time b, then repair it completely with shop repair cost, then burn-in the repaired component again and so on. If the component survives the burn-in time b, then it is put into field operation. For a burned-in component he considers the block replacement policy with minimal repair. He assumes that the cost of minimal repair at failure is constant.

In this paper, it is assumed that the cost of a minimal repair to the component which fails at age t is $C_m(t)$, where $C_m(t)$ is a continuous nondecreasing function of t. Hence, as the component ages it becomes more expensive to perform minimal repair. It is shown that the optimal burn-in time b^* must occur before the change point t_1 of h(t) under the assumption of a bathtub-shaped failure rate function.

2. Expected Minimal Repair Costs

Consider a fixed burn-in time b and begin to burn-in a new component. If the component fails before burn-in time b, then repair it with shop repair cost, c_s, then burn-in the repaired component again and so on. Here it is assumed that the repair is complete, i.e. the repaired component is as good as new. If the component survives the burn-in time b, then it is put into field operation. The cost for burn-in is assumed to be proportional to the total burn-in time with proportionality constant c_0.

Now denote the distribution function of a new device by F and let $\{X_i, i\geq 1\}$ be an i.i.d. sequence of random variables distributed according to F. Let $\eta-1$ be the random variable which is the number of shop repairs until the first device surviving burn-in is obtained. Thus the cost incurred by this burn-in procedure is given by

$$g(b)=c_0\left(\sum_{i=1}^{\eta-1}X_i+b\right)+c_s(\eta-1).$$

It is evident that

$$\begin{aligned}P(\eta=k) &= P(X_1\leq b,\cdots,X_{k-1}\leq b,X_k>b)\\ &= F^{k-1}(b)\overline{F}(b);\end{aligned}$$

that is, η has a geometric distribution with mean $E(\eta)=1/\overline{F}(b)$. Further, the lifetime of that device which has survived to burn-in time b is $X_\eta-b$ and is distributed according to

$$P(X_\eta-b>s)=\frac{\overline{F}(b+s)}{\overline{F}(b)},\quad \text{for all } s\geq 0.$$

As in Mi (1994), the mean burn-in cost, $C_1(b)$ is given by

$$\begin{aligned}C_1(b) &= E(g(b))\\ &= c_0\frac{\int_0^b\overline{F}(t)dt}{\overline{F}(b)}+c_s\frac{F(b)}{\overline{F}(b)}.\end{aligned}\qquad(1)$$

Let $N_b(T)$ be the random variable denoting the number of minimal rapairs performed on the component in $[b, b+T]$. We know that $N_b(T)$ has a Poisson distribution with parameter $H_b(t)=H(b+T)-H(b)$ where $H(t)=\int_0^t h(s)ds$. Now if $N_b(T)=k$, and $t_1,\cdots,t_k$ are the times of the minimal repairs, then the total minimal repair cost in the interval $[b, b+T]$ is $\sum_{i=1}^{k}C_m(t_i)$. Given $N_b(T)=k$, we know that $\tau_1=H(t_1),\cdots,\tau_k=H(t_k)$ are distributed as the order statistics of a random sample of size k from the uniform distribution on $[H(b), H(b+T)]$. Hence, the expected minimal repair cost given $N_b(T)=k$ is

$$\begin{aligned}&E(C_m(t_1)+\cdots+C_m(t_k)|N_b(T)=k)\\ &=E(C_m(H^{-1}(\tau_1))+\cdots+C_m(H^{-1}(\tau_k))|\\ &\qquad N_b(T)=k)\\ &=kE(C_m(H^{-1}(\tau_1))|N_b(T)=k)\\ &=\frac{k}{H(b+T)-H(b)}\\ &\qquad\int_{H(b)}^{H(b+T)}C_m(H^{-1}(t))\,dt.\end{aligned}$$

Therefore, the expected minimal repair cost in the interval $[b, b+T]$ is

$$E(E(C_m(t_1)+\cdots+C_m(t_{N_b(T)})|N_b(T)))$$

$$= E\left(\frac{N_b(T)}{H(b+T)-H(b)} \int_{H(b)}^{H(b+T)} C_m(H^{-1}(t))\,dt\right)$$

$$= \left(\frac{1}{H(b+T)-H(b)} \int_{H(b)}^{H(b+T)} C_m(H^{-1}(t))\,dt\right) E(N_b(T))$$

$$= \int_{H(b)}^{H(b+T)} C_m(H^{-1}(t))\,dt$$

$$= \int_b^{b+T} C_m(t)h(t)\,dt.$$

Thus, the total cost incurred by maintenance is

$$C_b(T) = c_r + \int_b^{b+T} C_m(t)h(t)dt. \qquad (2)$$

where c_r the cost of a replacement.

3. Optimal Burn-in

Let $N_r(t)$ be the random variable denoting the number of replacements during field operating time interval $[b, b+t]$. Then $N_r(t) = [t/T]$ where $[x]$ is the largest integer which is not greater than x. Since the cost of each burned-in component is $g(b)$, the associated cost is $(N_r(t)+1)g(b)$. Hence the corresponding long-run average cost is

$$\lim_{t\to\infty} \frac{E(N_r(t)+1)g(b)}{t}$$

$$= \lim_{t\to\infty} \frac{(N_r(t)+1)Eg(b)}{t} \qquad (3)$$

$$= \frac{1}{T}\left(c_0 \frac{\int_0^b \overline{F}(t)dt}{\overline{F}(b)} + c_s \frac{F(b)}{\overline{F}(b)}\right)$$

since $N_r(t) = [t/T]$ is not random. Combining (1), (2) and (3) the long-run average cost per unit time $C(b, T)$ is given by

$$C(b, T) = \frac{1}{T}\left(c_0 \frac{\int_0^b \overline{F}(t)dt}{\overline{F}(b)} + c_s \frac{F(b)}{\overline{F}(b)} + c_r + \int_b^{b+T} C_m(t)h(t)dt\right) \qquad (4)$$

The results regarding the optimal burn-in time b^* and the optimal age T^* which satisfy

$$C(b^*, T^*) = \min_{b\geq 0,\, T>0} C(b, T)$$

are given in the following theorem.

THEOREM 1 Suppose the failure rate function $h(t)$ is differentiable and BTR with change points t_1 and t_2. If $C_m(t)h(t)$ is not eventually constant, then the optimal burn-in time b^* and the corresponding optimal age $T^* = T^*_{b^*}$ satisfy

$$0 \leq b^* \leq t_1 \quad \text{and} \quad T^* = T^*_{b^*} > 0.$$

Proof. For any fixed $b \geq 0$

$$\frac{\partial}{\partial T} C(b, T) = \frac{1}{T^2}\{\eta_b(T) - [c_r + C_1(b)]\},$$

where $C_1(b)$ is as in (1) and $\eta_b(T) =$

$$TC_m(b+T)h(b+T) - \int_b^{b+T} C_m(t)h(t)dt.$$

Hence, $\partial C(b, T)/\partial T=0$ if and only if

$$\eta_b(T) = c_r + C_1(b). \qquad (5)$$

Note that $\eta_b(0)=0$, $\eta_b'(T) = T[C_m(b+T)h(b+T)]'$ and $\eta_b(\infty)=\infty$ since $C_m(t)h(t)$ is not eventually constant.

For $b \geq t_1$, $\eta_b(T)$ is nondecreasing for all $T \geq 0$ and the equation (5) has a unique solution which we denote by T_b^*.

For $0 \leq b \leq t_1$, $\eta_b(T)$ is nondecreasing for all $T \geq t_1 - b$. Hence the equation (5) has at least one solutions and $C(b, T)$ has at least one local minimum points. We take the largest point having minimum value of $C(b, T)$ among the local minimum points as T_b^*. This shows that the solution T_b^* must satisfy $T_b^* > 0$ for any given $b \geq 0$.

The fact that T_b^* satisfies equation (5) gives

$$T_b^* C_m(b+T_b^*)h(b+T_b^*) - \int_b^{b+T_b^*} C_m(t)h(t)dt = c_r + C_1(b). \qquad (6)$$

Combining (4) and (6), we obtain $C(b, T_b^*) = C_m(b+T_b^*)h(b+T_b^*)$. Thus minimizing $C(b, T_b^*)$ is equivalent to minimizing $C_m(b+T_b^*)h(b+T_b^*)$ for $b \geq 0$. If $b \geq t_1$, then $b+T_b^* \geq t_1$ and the problem of minimizing $C(b, T_b^*)$ is actually equivalent to minimizing $(b+T_b^*)$. Taking the derivative with respect to b on both sides of (6), we obtain

$$\begin{aligned} &T_b^*(1+T_b^{*\prime})[C_m(b+T_b^*)h(b+T_b^*)]' \\ &= C_m(b+T_b^*)h(b+T_b^*) - C_m(b)h(b) \qquad (7) \\ &\quad + C_1'(b), \end{aligned}$$

where $C_1'(b) =$

$$\frac{c_0 \overline{F}^2(b) + f(b)(c_s + c_0 \int_0^b \overline{F}(t)dt)}{\overline{F}^2(b)}.$$

If $b \geq t_1$, then $C_m(b+T_b^*)h(b+T_b^*) - C_m(b)h(b) > 0$. Since $C_1'(b) > 0$ for all $b \geq 0$, consequently, from (7)

$$T_b^*(1+T_b^{*\prime})\,[C_m(b+T_b^*)h(b+T_b^*)]' > 0$$

This yield $1+T_b^{*\prime} > 0$ since $b+T_b^* > t_1$ and thus $[C_m(b+T_b^*)\,h(b+T_b^*)]' > 0$. Hence

$$\frac{d}{db}(b+T_b^*) > 0, \text{ for all } b \geq t_1.$$

Therefore the minimum value of $b+T_b^*$ can be only be achieved on the interval $[0, t_1]$, i.e. $0 \leq b^* \leq t_1$. This completes the proof.

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