INVARIANT MEASURE AND THE EULER CHARACTERISTIC OF PROJECTIVELY FLAT MANIFOLDS

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ABSTRACT. In this paper, we show that the Euler characteristic of an even dimensional closed projectively flat manifold is equal to the total measure which is induced from a probability Borel measure on $\mathbb{R}P^n$ invariant under the holonomy action, and then discuss its consequences and applications. As an application, we show that the Chern's conjecture is true for a closed affinely flat manifold whose holonomy group action permits an invariant probability Borel measure on $\mathbb{R}P^n$; that is, such a closed affinly flat manifold has a vanishing Euler characteristic.

1. Introduction

In this paper, we will show how the Euler characteristic of a projectively flat manifold M can be viewed as a total measure of M, where the measure is induced from a probability measure on $\mathbb{R}P^n$ invariant under the holonomy action, and then we will also discuss its consequences and applications.

A projectively flat manifold M is a manifold which is locally modelled on the projective space with its natural projective geometry, i.e., M admits a cover of coordinate charts into the projective space $\mathbb{R}P^n$ whose coordinate transitions are projective transformations. By an analytic continuation of coordinate maps from its universal covering $\widetilde{M}$,

we obtain a developing map from $\tilde{M}$ into $\mathbb{R}P^n$ and this map is rigid in the sense that it is determined only by a local data. Therefore the deck transformation action on $\tilde{M}$ induces the holonomy action via the developing map by the rigidity.(See for example [4, 14, 15] for more details.) Suppose there is a probability Borel measure λ on $\mathbb{R}P^n$ which is invariant under this holonomy action. Then we will first show that a Borel measure μ on M is induced from λ by the invariance property of λ, and then show the following Main Theorem.

THEOREM 1.1 (The Main Theorem). *Let M be an even dimensional closed projectively flat manifold and λ be a holonomy invariant finitely additive probability Borel measure on $\mathbb{R}P^n$.*
Then

$$\chi(M) = \mu(M),$$

where μ is the Borel measure on M induced from λ.

This result and its consequences are significantly refined and evolved versions of our earlier results [8, 9] in its perspective. Our investigations have been motivated from the effort to resolve the Chern's conjecture (or also known as Sullivan's conjecture): "A closed affinely flat manifold has vanishing Euler characteristic."

An (X, G)-manifold is a manifold which is locally modelled on X with the geometry determined by the Lie group G acting on X analytically. For example, projectively flat manifold is a special case of (X, G)-manifold with $X = \mathbb{R}P^n$ and $G=\mathrm{PGL}(n+1, \mathbb{R})$ and so is an affinely flat manifold with $X = \mathbb{E}^n$, the standard Euclidean space and $G = \mathrm{Aff}(n)$, the group of affine transformations on $\mathbb{E}^n$. An affinely flat manifold also can be viewed as a projectively flat manifold whose holonomy preserves the set of points at infinity, $\mathbb{R}P_\infty^{n-1}$, by identifying $\mathbb{E}^n$ with the affine space given by $x_{n+1} = 1$ in $\mathbb{R}^{n+1}$ so that $\mathbb{R}P^n$ becomes a compactification of $\mathbb{E}^n$. Similarly all the Riemannian and pseudo-Riemannian space forms can be considered as a subclass of projectively flat manifolds, and also a subclass of affinely flat manifolds if they are flat.

The Euler characteristic of flat Riemannian or pseudo-Riemannian manifolds vanishes by Gauss-Bonnet-Chern theorem and it is natural to ask the same for affinely flat manifolds more generally and this is the content of Chern's conjecture. If a compact affinely flat manifold M is complete, then the conjecture is true by the work of Kostant and Sullivan [12], but note that the compactness does not necessarily imply the completeness in contrast to the Riemannian case. There has been various partial answers in different directions but the conjecture is not

completely resolved yet. As one of the corollaries of the Main Theorem, we show that the conjecture is true if the holonomy group of affinely flat manifold has an invariant probability measure generalizing the earlier result for amenable case as well as for radiant case in a unified way.

In Section 2, we will define a pull-back measure $f^*\lambda$ for a given local homeomorphism f from a manifold M to another manifold N having a measure λ. For (X, G)-manifold M and the corresponding developing map $D : \tilde{M} \to X$, $D^*\lambda$ is well-defined on $\tilde{M}$ whenever X has a measure λ. If λ is invariant under its holonomy action, we can also define a measure μ on M naturally induced from $D^*\lambda$ by covering projection $p : \tilde{M} \to M$. This is proved in Section 3. In Section 4, we will prove the Main Theorem. In Section 5, we will discuss the consequences and applications of the Main Theorem including the relation between the Euler characteristic and the developing maps for projectively flat manifolds and forementioned results for affinely flat manifolds.

2. Pull-back measure

Let (N, Ω, λ) denote a (finitely additive, resp.) measure space such that the σ-algebra (algebra, resp.) Ω contains all the open sets of N, that is, λ is a Borel measure if it is countably additive. In this paper we will also call such a measure finitely additive Borel measure when λ is only finitely additive. Let M and N be manifolds and $f : M \to N$ be a local homeomorphism. Then we can define a pull-back (finitely additive, resp.) Borel measure $f^*\lambda$ on M whenever such a (finitely additive, resp.) Borel measure λ is given on N.

An open covering $B = \{B_i \,|\, \overline{B_i} \text{ compact}\}_{i \in \mathbb{N}}$ of M will be called a covering *adapted to local homeomorphism* f if $f|_{B_i}$ is a homeomorphism for each i.

DEFINITION 2.1. Let $B = \{B_i\}$ be an adapted covering. Define
$$\Omega_B = \{\, A \subset M \,:\, f(A \cap B_i) \text{ is } \lambda\text{-measurable for all } i \,\}.$$

LEMMA 2.1. *Let* $B = \{B_i\}$ *be an adapted covering. Then* Ω_B *is a* σ-*algebra.* (Ω_B *is an algebra if* λ *is finitely additive.*)

Proof. Obviously $\varnothing$ and M are contained in Ω_B. For each subset A of M, we have the following equality:
$$f(A^c \cap B_i) = f|_{B_i}(B_i \setminus (A \cap B_i)) = f(B_i) \setminus f(A \cap B_i) \text{ for all } i.$$

So the measurability of $f(A \cap B_i)$ implies the measurability of $f(A^c \cap B_i)$ for all i. Therefore $A^c \in \Omega_B$ whenever $A \in \Omega_B$.

Let $\{S_j\}_{j\in I} \subset \Omega_B$ where I is finite if λ is finitely additive and I is equal to $\mathbb{N}$ if λ is countably additive. Then $f\big((\cup_{j\in I}S_j) \cap B_i\big) = f\big(\cup_{j\in I}(S_j \cap B_i)\big) = \cup_{j\in I}f(S_j \cap B_i)$ for all i. Therefore $f\big((\cup_{j\in I}S_j)\cap B_i\big)$ is measurable for all i and thus $\cup_{j\in I}S_j$ is contained in Ω_B. $\square$

DEFINITION 2.2. Let $B = \{B_i\}$ be an adapted covering and $B'_k = B_k \setminus \cup_{i=1}^{k-1}B_i$. Define a set function $\lambda_B : \Omega_B \to \{x \in \mathbb{R} | x \geq 0\} \cup \{+\infty\}$ as follows.

$$\lambda_B(A) = \sum_{k=1}^{\infty} \lambda(f(A \cap B'_k)).$$

λ_B is well-defined by the following reason. Firstly we see the following equalities from the fact that f is a homeomorphism on B_j for each j:

$$\begin{aligned}
f(B'_k \cap B_j) &= f|_{B_j}\big((B_k \setminus \cup_{i=1}^{k-1}B_i) \cap B_j\big)\\
&= f|_{B_j}\big((B_k \cap B_j) \setminus (\cup_{i=1}^{k-1}B_i) \cap B_j\big)\\
&= f(B_k \cap B_j) \setminus f((\cup_{i=1}^{k-1}B_i) \cap B_j) \text{ for each } j.
\end{aligned}$$

Since $(\cup_{i=1}^{k-1}B_i) \cap B_j$ is open and $f|_{B_j}$ is a homeomorphism for each j, $f((\cup_{i=1}^{k-1}B_i)\cap B_j)$ is also open and thus λ-measurable. This implies that $f(B'_k \cap B_j)$ is λ-measurable for all j and so $B'_k \in \Omega_B$. So $A \cap B'_k \in \Omega_B$ for all $A \in \Omega_B$ and hence $f(A \cap B'_k) = f(A \cap B'_k \cap B_k)$ is λ-measurable. Therefore λ_B is well-defined.

LEMMA 2.2. (M, Ω_B, λ_B) is a (finitely additive) Borel measure space.

Proof. Let A be an open subset of M. Notice that Ω contains all the open sets of N and hence $f(A \cap B_i) \in \Omega$. Therefore all the open sets of M are contained in Ω_B. Now, it suffices to show the countable (finite) additivity of λ. Let $\{S_i\}_{i\in I} \subset \Omega_B$ be a disjoint collection, where I is finite if λ is finitely additive and I is equal to $\mathbb{N}$ if λ is countably additive. Then we have

$$\begin{aligned}
\lambda_B(\cup_{i\in I}S_i) &= \sum_{k=1}^{\infty} \lambda\big(f((\cup_{i\in I}S_i) \cap B'_k)\big)\\
&= \sum_{k=1}^{\infty} \lambda\big(\cup_{i\in I}f|_{B_k}(S_i \cap B'_k)\big)
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \sum_{i \in I} \lambda\big(f(S_i \cap B_k')\big)$$

$$= \sum_{i \in I} \sum_{k=1}^{\infty} \lambda\big(f(S_i \cap B_k')\big)$$

$$= \sum_{i \in I} \lambda_B(S_i).$$

Notice that the third equality holds since $f|_{B_k}$ is a homeomorphism. $\square$

LEMMA 2.3. *Let $\{B_i\}$ and $\{C_i\}$ be adapted coverings and $\{\Omega_B, \lambda_B\}$ and $\{\Omega_C, \lambda_C\}$ be the corresponding measure systems on M. Then $\Omega_B = \Omega_C$ and $\lambda_B = \lambda_C$.*

Proof. (i) $\Omega_B = \Omega_C$: For each C_k and for all i, $f(C_k \cap B_i)$ is open and thus $f(C_k \cap B_i)$ is λ-measurable. Hence $C_k \in \Omega_B$ for all k. Suppose $A \in \Omega_B$. To prove $A \in \Omega_C$, it suffices to show $f(A \cap C_i)$ is λ-measurable for all i by definition. Observe that

$$f(A \cap C_i) = f\big((A \cap C_i) \cap (\cup_{k=1}^{\infty} B_k)\big)$$

$$= f\big(\cup_{k=1}^{\infty} (A \cap C_i \cap B_k)\big)$$

$$= \cup_{k=1}^{\infty} f(A \cap C_i \cap B_k).$$

The last union is in fact a finite union since $\overline{C_i}$ is compact. Since $C_i \in \Omega_B$ for all i, we get $A \cap C_i \in \Omega_B$ for all i and so $f(A \cap C_i \cap B_k)$ is λ-measurable for all i, k. Therefore $f(A \cap C_i)$ is measurable whether λ is finitely additive or not. This shows $\Omega_B \subset \Omega_C$ and similarly $\Omega_C \subset \Omega_B$
(ii) $\lambda_B = \lambda_C$: For each $A \in \Omega_B = \Omega_C$, we have

$$\lambda_B(A) = \sum_{k=1}^{\infty} \lambda\big(f(A \cap B_k')\big)$$

$$= \sum_{k=1}^{\infty} \lambda\big(f((A \cap B_k') \cap (\cup_{j=1}^{\infty} C_j'))\big)$$

$$= \sum_{k=1}^{\infty} \lambda\big(\cup_{j=1}^{\infty} f(A \cap B_k' \cap C_j')\big)$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda\big(f(A \cap B_k' \cap C_j')\big)$$

$$= \sum_{j=1}^{\infty}(\sum_{k=1}^{\infty} \lambda(f(A \cap C'_j \cap B'_k)))$$

$$= \sum_{j=1}^{\infty} \lambda(f(A \cap C'_j))$$

$$= \lambda_C(A),$$

where the fourth, fifth and the last equalities hold even if λ is finitely additive since the summation $\sum_{j=1}^{\infty}$ and $\sum_{k=1}^{\infty}$ are in fact finite sums. □

From Lemma 2.2 and 2.3, we have the following theorem.

THEOREM 2.1. *Let $f : M \to N$ be a local homeomorphism and λ a (finitely additive, resp.) Borel measure on N. Then there exist a (finitely additive, resp.) measure μ on M such that*

(i) *every open subset of M is measurable,*

(ii) *for μ-measurable subset A of M (with $\overline{A}$ compact, resp.) $\mu(A) = \lambda(f(A))$ if $f|_A$ is a homeomorphism.*

Furthermore, μ is unique in the following sense: If μ' is another measure satisfying the above property, then $\mu(A) = \mu'(A)$ for each open set A (with $\overline{A}$ compact, resp.) of M.

Proof. The existence of such a measure is proved by Lemma 2.2, 2.3 and the definition of μ respectively. So the only thing left to prove is the uniqueness. Let A be a open subset of M and $\{B_i\}$ be an adapted covering. Then A and B_i are μ'-measurable and $\mu(A) = \sum_k \mu(A \cap B'_k) = \sum_k \lambda(f(A \cap B'_k)) = \sum_k \mu'(A \cap B'_k) = \mu'(\cup_k(A \cap B'_k)) = \mu'(A)$. Notice that $\sum_k$ is in fact a finite sum if $\overline{A}$ is compact and thus the first and fourth equalities also hold in the finitely additive case. Also the second and third equalities hold by the property (ii). □

Up to now we have defined a pull-back Borel measure on M when a local homeomorphism $f : M \to N$ and a Borel measure λ on N are given. Now, we will denote the pull-back measure by $f^*\lambda$. Notice, for any measurable subset A of M, $f^*\lambda(A) = \sum_i \lambda(f(A \cap U'_i))$ where $\{U_i\}$ is any adapted covering of M and $U'_k = U_k \setminus \cup_{i=1}^{k-1} U_i$. Denote the σ-algebra corresponding to $f^*\lambda$ by $f^*\Omega$.

REMARK 2.1. If f is a homeomorphism, then $f^*\Omega = f^{-1}\Omega$ and $f^*\lambda(A) = \lambda(f(A))$ for all $A \in f^*\Omega$ when λ is countably additive. But this does not hold if λ is finitely additive. In fact, $\text{id}^*\lambda \neq \lambda$ if λ is finitely additive and M is not compact. For example, if $M = \mathbb{R}$ and

λ is any finitely additive translation invariant probability measure of $\mathbb{R}$ (the amenability of $\mathbb{R}$ ensures the existence of such a measure), then any bounded measurable subset of $\mathbb{R}$ has a measure 0 and thus $\mathrm{id}^*\lambda \equiv 0$ by the definition of $\mathrm{id}^*\lambda$. This strange phenomenon arises since our measure $\mathrm{id}^*\lambda$ is pulled back only locally and then is given as the sum of these local measures not reflecting the global nature of the original measure λ.

THEOREM 2.2. *Suppose topological groups G and H act continuously on M and N respectively. Let $(\phi, f) : (G, M) \to (H, N)$ be an equivariant pair where $\phi : G \to H$ is a homomorphism and $f : M \to N$ is a local homeomorphism, and λ be an $H-$invariant Borel measure. Then $f^*\lambda$ is $G-$invariant.*

Proof. Let A be a measurable subset of M and $\{B_i\}$ be an adapted covering of M. Then for each fixed $g \in G$, $\{gB_i\}$ is also an adapted covering of M. Therefore we have

$$
\begin{aligned}
f^*\lambda(gA) &= \sum \lambda\big(f(gA \cap (gB_i)')\big) \\
&= \sum \lambda\big(f(gA \cap gB_i')\big) \\
&= \sum \lambda\big(f(g(A \cap B_i'))\big) \\
&= \sum \lambda\big(\phi(g)f(A \cap B_i')\big) \\
&= \sum \lambda\big(f(A \cap B_i')\big) \\
&= f^*\lambda(A).
\end{aligned}
$$

Notice the fifth equality holds since λ is $H-$invariant. $\qquad\square$

3. Holonomy invariant measure

Let $p : \widetilde{M} \to M$ be a regular covering map and λ be a Borel measure on $\widetilde{M}$. Assume λ is invariant under the action of the deck transformation group. Then we will define a Borel measure μ on M such that $p^*\mu = \lambda$.

An open covering $B = \{B_i\}_{i \in \mathbb{N}}$ of M will be called a covering *adapted to covering map* p, if B_i is evenly covered and $\overline{B_i}$ is compact for each i.

DEFINITION 3.1. Let $B = \{B_i\}$ be an adapted covering. Define

$$
\Omega_B = \{A \subset M \mid \text{for each } i, \left(p|_{\widetilde{B}_i}\right)^{-1}(A \cap B_i) \text{ is } \lambda\text{-measurable on } \widetilde{M}
$$

$$
\text{for some lifting } \widetilde{B_i} \}.
$$

Note that since p is regular and λ is invariant under the action of the deck transformation group, $A \in \Omega_B$ implies $\left(p|_{\widetilde{B_i}}\right)^{-1}(A \cap B_i)$ is λ-measurable for any lifting $\widetilde{B_i}$, that is Ω_B is well-defined independently of lifting. The following Lemmas 3.1, 3.2 and 3.3 are easily proved by the same argument as in Section 2 and thus we will omit the proofs.

LEMMA 3.1. *Let* $B = \{B_i\}$ *be an adapted covering of* M. *Then* Ω_B *is a σ-algebra. (* Ω_B *is an algebra if λ is finitely additive.)*

DEFINITION 3.2. Let $B = \{B_i\}$ be an adapted covering of M. Define a function

$$\mu_B : \Omega_B \to \{x \in \mathbb{R} : x \geq 0\}\cup\{+\infty\} \text{ by } \mu_B(A)=\sum_{k=1}^{\infty} \lambda\left(\left(p|_{\widetilde{B_k}}\right)^{-1}(A\cap B_k')\right),$$

where $B_k' = B_k \setminus \cup_{i=1}^{k-1}B_i$.

It is easy to prove $B_k' \in \Omega_B$ by similar argument as in Section 2. Hence, μ_B is well defined independently of the choice of the lifting $\widetilde{B_i}$ since λ is invariant under the action of the deck transformation group.

LEMMA 3.2. *Let* $B = \{B_i\}$ *be an adapted covering of* M. *Then* (M, Ω_B, μ_B) *is a (finitely additive) Borel measure space.*

LEMMA 3.3. *Let both* $\{B_i\}$ *and* $\{C_i\}$ *be adapted coverings of* M *and* $\{\Omega_B, \mu_B\}$ *and* $\{\Omega_C, \mu_C\}$ *be the corresponding measure systems on* M. *Then* $\Omega_B = \Omega_C$ *and* $\mu_B = \mu_C$.

From the above Lemmas, we have the following theorem.

THEOREM 3.1. *Let* $p : \widetilde{M} \to M$ *be a regular covering and* λ *be a (finitely additive, resp.) Borel measure on* $\widetilde{M}$. *Assume that λ is invariant under the action of the deck transformation group. Then there exists a (finitely additive, resp.) Borel measure μ on M such that $p^*\mu = \lambda$. Furthermore μ is unique in the following sense: if μ' is another (finitely additive, resp.) Borel measure such that $p^*\mu' = \lambda$, then $\mu(A) = \mu'(A)$ for each open subset A (with $\overline{A}$ compact, resp.) of M.*

Proof. The existence is obvious by Lemmas 3.2 and 3.3. Let A be an open subset of M and $\{B_i\}$ be an adapted covering. Choose a lifting $\widetilde{B_i}$ for each i. Then $\mu(A) = \sum_k \mu(A \cap B_k') = \sum_k p^*\mu'\left(\left(p|_{\widetilde{B_k}}\right)^{-1}(A \cap B_k')\right) = \sum_k \mu'(A\cap B_k') = \mu'(A)$. Notice that the summation $\sum_k$ is in fact a finite sum if $\overline{A}$ is compact and thus the first and fifth equalities also hold in the finitely additive case. Also the second and third equalities hold since $\overline{A \cap B_k'}$ is compact. $\qquad\square$

THEOREM 3.2. *Let M be a (X, G)–manifold, D : $\widetilde{M} \to X$ be a developing map with holonomy group H, p : $\widetilde{M} \to M$ be a covering map and λ be an H–invariant (finitely additive, resp.) Borel measure on X. Then there exists a (finitely additive, resp.) Borel measure μ on M such that $p^*\mu = D^*\lambda$.*

Proof. By Theorem 2.2, $D^*\lambda$ is invariant under the action of deck transformation since λ is H-invariant. So there exists a Borel measure μ on M such that $p^*\mu = D^*\lambda$ by Theorem 3.1. $\qquad\qquad\square$

The measure μ described in Theorem 3.2 will be called the *induced measure*. Recall that a (G, X)–manifold is a smooth manifold which has a cover of coordinate charts by open subsets in X whose coordinate transition functions are locally restrictions of the elements of G. To distinguish from topological chart, we will call a coordinate chart constituting a (G, X)–manifold as a *geometric chart*.

REMARK 3.1. Let A be a subset of M so that A is contained in some evenly covered geometric chart. Assume that $\overline{A}$ is compact if λ is finitely additive. Then $\mu(A) = \lambda(D\tilde{A})$ where $D\tilde{A}$ is any developing image of a lifting $\tilde{A}$ of A.

4. Proof of the main theorem

We use a generalized *Gauss-Bonnet formula* in terms of angles of simplices in a triangulation to prove the theorem as in [9]. The notion of angle is not well defined in general, but we can do define an angle using an holonomy invariant measure.

Let $(\mathbb{R}P^n, \lambda)$ be a finitely additive probability Borel measure. We will use the same symbol λ to denote the pull-back measure induced on $\mathbb{S}^n$ which is invariant under the antipodal map so that $\lambda(\mathbb{S}^n) = 2$.

Let s^n be a spherical simplex lying in the standard unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ so that each of its $(n\text{-}1)$-dimensional faces is a part of great hyperplanes P_i, $i = 1, 2, \cdots, n+1$, in general position. A great hyperplane is the intersection of $\mathbb{S}^n$ with an n-dimensional subspace of $\mathbb{R}^{n+1}$. Let f_i be the characteristic function of *positive* side of P_i which, by definition, is the half of $\mathbb{S}^n$ bisected by P_i that contains s^n.

If $s^r < s^n$ is an r-dimensional face of s^n given by $s^r = P_{i_1} \cap \cdots \cap P_{i_{n-r}} \cap s^n$, the *angle* at s^r in s^n, denoted by $\alpha(s^r, s^n)$, is defined as

$$\alpha(s^r, s^n) = \frac{1}{2} \int_{S^n} f_{i_1} \cdots f_{i_{n-r}} d\lambda.$$

Clearly,

$$\lambda(s^n) = \int_{S^n} f_1 \cdots f_{n+1} \, d\lambda$$

$$\lambda(\bar{s}^n) = \int_{S^n} (1 - f_1) \cdots (1 - f_{n+1}) \, d\lambda,$$

where $\bar{s}^n$ is the antipodal image of s^n. Then

$$\lambda(s^n) = \lambda(\bar{s}^n) = \int_{S^n} (1 - f_1) \cdots (1 - f_{n+1}) \, d\lambda$$

$$= \sum_{r=0}^{n} \sum_{s^r < s^n} (-1)^{n-r} 2\alpha(s^r, s^n) + (-1)^{n+1} \int_{S^n} f_1 \cdots f_{n+1} \, d\lambda$$

and hence we get the *Spherical Gauss-Bonnet formula* [7]

$$2 \sum_{r=0}^{n} \sum_{s^r < s^n} (-1)^{n-r} \alpha(s^r, s^n) = (1 + (-1)^n) \lambda(s^n),$$

or equivalently,

$$2 \sum_{r=0}^{n} \sum_{s^r < s^n} (-1)^r \alpha(s^r, s^n) = (1 + (-1)^n) \lambda(s^n).$$

Let $k(s^n) = \sum_{r=0}^{n} \sum_{s^r < s^n} (-1)^r \alpha(s^r, s^n)$. Then the above formula implies $k(s^n)$ equals zero when n is odd and equals $\lambda(s^n)$ when n is even.

Let M be a closed projectively flat manifold with a *geometric triangulation* K consisting of simplices σ^n whose developing images are spherical simplices. Let $D : \tilde{M} \to \mathbb{R}P^n$ and H be the corresponding developing map and holonomy group respectively. Let λ be an H-invariant finitely additive probability measure on $\mathbb{R}P^n$. Then for each face $\sigma^r < \sigma^n$ the *angle* at σ^r in σ^n, denoted by $\tilde{\alpha}(\sigma^r, \sigma^n)$, is defined as

$$\tilde{\alpha}(\sigma^r, \sigma^n) = \alpha(s_0^r, s_0^n),$$

where (s_0^r, s_0^n) is a developing image of a lifting of a pair (σ^r, σ^n) in $\tilde{M}$.

Note that $\tilde{\alpha}(\sigma^r, \sigma^n)$ is well defined by the following reason. $\alpha(s_0^r, s_0^n)$ is actually the measure of some subset of $\mathbb{R}P^n$ and for any other choice of a lifting and its developing image s_1 there exists $h \in H$ so that $s_1^n = h(s_0^n)$ and $s_1^r = h(s_0^r)$. Therefore we get $\alpha(s_0^r, s_0^n) = \alpha(h(s_0^r), h(s_0^n)) = \alpha(s_1^r, s_1^n)$ since λ is H-invariant. But the angle depends on the developing map.

From now on, we'll simply denote the angle $\tilde{\alpha}$ by α by abusing notation.

Let

$$S(\sigma^i) = \sum_{\sigma^i < \sigma^n} \alpha(\sigma^i, \sigma^n),$$

$$k(\sigma^n) = \sum_{r=0}^{n} (-1)^r \sum_{\sigma^r < \sigma^n} \alpha(\sigma^r, \sigma^n),$$

and for a vertex ν in K, let

$$d(\nu) = \sum_{r=0}^{n} \frac{(-1)^r}{r+1} \sum_{\nu \in \sigma^r} (1 - S(\sigma^r)).$$

Then it is basically a rearrangement of the angle terms to verify the following *polyhedral Gauss-Bonnet formula* for the Euler characteristic $\chi(M)$:

$$\sum_{\nu \in K} d(\nu) + \sum_{\sigma^n \in K} k(\sigma^n) = \chi(M).$$

See [8] for a proof and see [10] for a motivation and geometric meanings of the terms in the formula.

Let $\mathrm{PGL}(n+1, \mathbb{R})$, as usual, be the projective general linear group, i.e., $\mathrm{GL}(n+1, \mathbb{R})/\mathbb{R}^* I$, where $\mathbb{R}^*$ is the set of nonzero elements of $\mathbb{R}$. If $V, W \subset \mathbb{R}^{n+1}$ is a nonzero linear subspace, we denote by $[V] \subset \mathbb{R}P^n$ its image in $\mathbb{R}P^n$ and write $W \lneq V$ if W is a proper subspace of V.

Let $S = \{V \lneq \mathbb{R}^{n+1} \mid \lambda([V]) > 0 \text{ and } \lambda([W]) = 0 \text{ for any } W \lneq V\}$. Then S is a disjoint union of S_i with $i \in \{0, 1, \cdots, n-1\}$, where

$$S_i = \{V \in S \mid \dim V = i+1\}.$$

Let $S_{i,j} = \{V \in S_i \mid \lambda([V]) > \frac{1}{j}\}$. Then $|S_{i,j}| < j$, since $\lambda(\mathbb{R}P^n) = 1$ and $\lambda([V \cap W]) = 0$ if V and W belong to $S_{i,j}$ and $V \neq W$. Therefore $S_i = \cup_{j=1}^{\infty} S_{i,j}$ is countable and so is $S = \cup_{i=0}^{n-1} S_i$. Therefore we have the following properties:

(i) $|S|$ is countable.

(ii) If $X \lneq \mathbb{R}^{n+1}$ and X is transversal to each element of S, then $\lambda([X]) = 0$.

Therefore (i) implies that we can choose a geometric triangulation K on M by a small perturbation such that every hyperplane in $\mathbb{R}P^n$ containing a developing image of some $(n-1)$-dimensional geometric simplex in K is transversal to S and so it has a measure zero by (ii).

We now prove the Main Theorem: $S(\sigma^i) = 1$ for any geometric simplex σ^i in K by the above consideration and thus $d(\nu) = 0$ for all vertex $\nu \in K$. Let s_0^n be any developing image of σ^n. Then $k(\sigma^n) = k(s_0^n)$

by definition of $\alpha(\sigma^r, \sigma^n)$ and thus we get $k(\sigma^n) = k(s_0^n) = \lambda(s_0^n)$. We may assume σ^n is evenly covered and lies in some geometric chart, $\lambda(s_0^n) = \mu(\sigma^n)$ by Remark in Section 3. Therefore $k(\sigma^n) = \mu(\sigma^n)$ for all n-simplex $\sigma^n \in K$. Now by the *polyhedral Gauss-Bonnet Theorem*,

$$\chi(M) = \sum_{\sigma^n \in K} \mu(\sigma^n).$$

But we have chosen a triangulation so that the faces of σ^n have measure zero and hence

$$\sum_{\sigma^n \in K} \mu(\sigma^n) = \mu(M).$$

This completes the proof. $\square$

5. Consequences and applications

The right hand side $\mu(M)$ of the formula in the Main Theorem is supposed to depend on the holonomy invariant measure chosen and on the developing map, namely the projectively flat structure of M. But the theorem says that in fact it does not, and is always equal to the Euler characteristic of M, a topological invariant. Furthermore, there is no reason, a priori, that the total measure of M, $\mu(M)$ should be an integer. The topology, geometry and the measure related to M are interlocked by the formula and we can expect interesting applications from these observations. We will see some of the immediate consequences and applications in this section.

Let M be a closed projectively flat manifold with amenable holonomy group H and m an invariant mean on $B(H)$, the space of all bounded functions on H, see [5] for definitions of amenable group and invariant mean. We may assume that m is right invariant since H is a group, that is, $m(f_s) = m(f)$ for all $s \in H$, where f_s is a bounded function on H given by $f_s(t) = f(ts)$. Then we can define an H-invariant finitely additive probability measure on $\mathbb{R}P^n$ as follows. Choose any probability measure λ_0 on $\mathbb{R}P^n$. Then for any λ_0-measurable subset E of $\mathbb{R}P^n$ we can define a bounded function $f_E : H \to [0,1]$ by

(5.1) $f_E(h) = \lambda_0(h(E))$

for all $h \in H$. Now define a new measure λ on $\mathbb{R}P^n$ by

$$\lambda(E) = m(f_E)$$

for all λ_0-measurable subset $E \subset \mathbb{R}P^n$. Then, by the property of invariant mean, λ is a finitely additive H–invariant probability measure on $\mathbb{R}P^n$. More precisely,

$$\lambda(hE) = m(f_{hE}) = m((f_E)_h) = m(f_E) = \lambda(E)$$

since

$$f_{hE}(h') = \lambda_0(h'(hE)) = \lambda_0(h'h)E) = (f_E)(h'h) = (f_E)_h(h'),$$

and the property $m(1) = 1$ implies that λ is a finitely additive probability measure on $\mathbb{R}P^n$.

THEOREM 5.1. *Let M be an even dimensional closed projectively flat manifold with holonomy group H. Suppose there exist an H-invariant finitely additive probability Borel measure λ on $\mathbb{R}P^n$. Then we have the following.*

(i) *$\chi(M)$ is nonnegative.*

(ii) *If the developing map is injective, then $\chi(M) = 0$.*

(iii) *If $\chi(M) = 0$, then the developing map is not surjective.*

(iv) *If the invariant measure λ is countably additive, then $\chi(M) = 0$ if and only if $\lambda(\Omega) = 0$ and $\chi(M) > 0$ if and only if $\lambda(\Omega) = 1$, where Ω is the developing image.*

Proof. To prove this theorem, it suffices to prove (ii), (iii) and (iv) because (i) is the immediate consequence of the Main Theorem.

(ii) Let F be an open fundamental domain of M in $\tilde{M}$. Let $D : \tilde{M} \rightarrow \mathbb{R}P^n$ be the corresponding developing map and $\phi : \pi_1(M) \rightarrow H$ be the holonomy representation. Suppose $\phi(\xi) = \mathrm{id}$ for some $\xi \in \pi_1(M)$. Then $D(\xi x) = \phi(\xi)D(x) = D(x)$ for all $x \in \tilde{M}$. Since D is injective, $\xi x = x$ for all $x \in \tilde{M}$, i.e., $\xi = \mathrm{id}$. Therefore ϕ becomes an isomorphism. Note that H is non-trivial: If it were trivial, M is simply connected and D becomes a homeomorphism since M is compact and D is an injective local homeomorphism. But this is impossible since $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$ for $n \geq 2$. Let $h \in H$ be a non-identity element. Then by the injectivity of D, $D(F) \cap h(D(F)) = \varnothing$. Since $\lambda(D(F)) = \lambda(h(D(F)))$ and $\lambda(D(F)) + \lambda(h(D(F))) \leq 1$, the Euler characteristic, being a nonnegative integer, has to be zero.

(iii) Let μ be the induced measure. If $\chi(M) = 0$, then the Main Theorem implies that $\mu(M) = 0$ and thus for each $x \in D(\tilde{M})$ there exists an open neighborhood U_x of x such that $\lambda(U_x) = 0$ by the definition of the induced measure. Therefore any compact subset E in $D(\tilde{M})$ has measure zero. Suppose D is surjective. Then $\mathbb{R}P^n = D(\tilde{M})$. But $\mathbb{R}P^n$

is compact and thus $\lambda(\mathbb{R}P^n) = 0$. This contradicts that λ is a probability measure.

(iv) Let μ be the induced measure on M. $\chi(M) = 0$ implies $\mu(M) = 0$ by the Main Theorem and thus again for each $x \in \Omega$ there exists an open neighborhood U_x of x such that $\lambda(U_x) = 0$ by definition of the induced measure. Therefore any compact subset E in Ω has measure zero and thus $\lambda(\Omega) = 0$ by the countable additivity of λ. The converse is clear. Therefore $\chi(M) > 0$ implies $\lambda(\Omega) \neq 0$. Suppose $\lambda(\Omega^c) = \alpha\,(\alpha > 0)$. By considering another invariant measure $\tilde{\lambda} = (1/\alpha)\lambda|_{\Omega^c}$ supported on the complement of Ω, we get $\chi(M) = 0$. This is a contradiction. So $\lambda(\Omega^c) = 0$. □

Theorem 5.1 (ii) says for instance that the holonomy group of even dimensional hyperbolic manifold can not have a finitely additive invariant probability measure. (But it does have complex invariant probability measure.) A much broader class of convex projectively flat manifolds should have the same property. And in this case the holonomy group of such manifolds can not be amenable since amenability enables one to construct an invariant probability measure starting from any probability measure by averaging process. But in general the holonomy group of projectively flat manifold is far from being amenable even when it has an invariant probability measure. The case of amenable holonomy group is an interesting special case and we can obtain a sharper result as the following Theorem 5.2 shows.

THEOREM 5.2. *Let M be an even dimensional closed projectively flat manifold with amenable holonomy group. Then the followings are equivalent.*

(i) *The developing map is not onto.*

(ii) $\chi(M) = 0$.

(iii) *There exists finitely additive invariant probability Borel measure λ on $\mathbb{R}P^n$ such that $\lambda(K) = 0$ for any compact subset K of the developing image.*

(iv) *For any finitely additive invariant probability Borel measure λ on $\mathbb{R}P^n$, $\lambda(K) = 0$ whenever K is a compact subset of the developing image.*

(v) *There exists a countably additive invariant probability Borel measure λ on $\mathbb{R}P^n$ such that $\lambda(\Omega) = 0$, where Ω is the developing image of $\tilde{M}$.*

(vi) *For any countably additive invariant probability Borel measure μ, $\mu(\Omega) = 0$.*

Proof. (i)$\Rightarrow$(ii); Let λ_0 be the Dirac measure concentrated at a point x_0 outside the developing image. Let m be an invariant mean on $B(H)$. Then we can define a measure λ on $\mathbb{R}P^n$ by $\lambda(E) = m(f_E)$ for each subset $E \subset \mathbb{R}P^n$, where f_E is defined as the equation (5.1). Then λ is an invariant finitely additive probability measure and for each subset E contained in the developing image, $\lambda(E) = 0$. Therefore $\chi(M) = 0$ by the Main Theorem.

(ii)$\Rightarrow$(i) has already been shown in Theorem 4.1 (iii).

Since the holonomy group is amenable, there exists a finitely additive invariant probability Borel measure by averaging and furthermore there also exists a countably additive invariant probability Borel measure by compactness of $\mathbb{R}P^n$.

(ii)$\Rightarrow$(iv), (vi); Suppose that λ_1 is a finitely additive invariant probability Borel measure on $\mathbb{R}P^n$ and λ_2 is a countably additive invariant probability Borel measure. Let μ_1 and μ_2 be the corresponding induced measure on M respectively. Then $\mu_1(M) = \mu_2(M) = 0$ since $\chi(M) = 0$. By the definition of the induced measure, for each x in the developing image, there exist an open neighborhood U_x such that $\lambda_1(U_x) = \lambda_2(U_x) = 0$. Therefore $\lambda_i(K) = 0 \, (i = 1, 2)$ for any compact subset K of the developing image and furthermore $\lambda_2(\Omega) = 0$ since λ_2 is countably additive.

(iv)$\Rightarrow$(iii) and (vi)$\Rightarrow$(v) are true since the holonomy group is amen -able.

(iii) and (v) each imply (ii) by the Main Theorem. $\square$

Another interesting special case in which the existence of H-invariant probability measure is guaranteed is where the holonomy group H has a fixed point or more generally has a finite orbit. In this case, we have the following theorem.

THEOREM 5.3. *Let M be an even dimensional closed projectively flat manifold with holonomy group H. Suppose H has a finite invariant set I. Then we have the followings.*

(i) *$I \subset D(\tilde{M})$ if and only if $\chi(M) > 0$.*

(ii) *$I \cap D(\tilde{M}) = \varnothing$ if and only if $\chi(M) = 0$.*

In particular, if H has a fixed point outside the developing image, then the Euler characteristic of M must vanish.

Proof. Define an invariant probability Borel measure λ on $\mathbb{R}P^n$ by $\lambda(E) = \sum_{a \in E \cap I} \frac{1}{n}$ for any subset E of $\mathbb{R}P^n$ where n is the cardinal number of I. Let μ be the induced measure on M. Let $I_1 = I \cap D(\tilde{M})$

and $I_2 = I \setminus I_1$. Suppose that neither I_1 nor I_2 is empty. Let $\lambda_1 = \frac{1}{\chi(I_1)}\lambda|_{I_1}$ and $\lambda_2 = \frac{1}{\chi(I_2)}\lambda|_{I_2}$ and μ_1 and μ_2 are the corresponding induced measures on M, respectively. Then $\mu_1(M) > 0$ and $\mu_2(M) = 0$. This is a contradiction. Therefore either $I_1 = \varnothing$ or $I_2 = \varnothing$. If $I_1 = \varnothing$, i.e., $I \cap D(\tilde{M}) = \varnothing$, then $\chi(M) = \mu_2(M) = 0$. Otherwise, i.e., $I \subset D(\tilde{M})$, then $\chi(M) = \mu(M) = \mu_1(M) > 0$. Since $\chi(M) \geq 0$, the converses are proved immediately. $\square$

For the case of an affinely flat manifold M, a fixed point can not lie in the developing image by the result of Fried, Goldman and Hirsch [3] and hence $\chi(M)$ vanishes if holonomy group H has a fixed point, that is, if M is a radiant affine manifold. The vanishing of $\chi(M)$ was observed by Kobayashi [11] using the Euler vector field, which gives a non-vanishing H-invariant vector field on $D(\tilde{M})$ and thus a non-vanishing vector field on M.

If M is a closed affinely flat manifold, we can go further to show the following Theorem 5.4 giving an affirmative answer for the Chern conjecture when the holonomy group of M has an invariant finitely additive probability measure on $\mathbb{R}P^n$. In fact, we do not know whether an affine manifold always have such a measure on $\mathbb{R}P^n$. Anyway, the theorem generalizes the earlier result of Hirsch and Thurston for the amenable holonomy case [6] and of Kobayashi for the radiant case in a unified way. If one can show directly that the holonomy group of a complete affine manifold has an invariant probability measure on $\mathbb{R}P^n$, then the theorem would also cover the result of Kostant and Sullivan. In fact, if Auslander conjecture is true, that is, if the fundamental group of a complete closed affinely flat manifold is virtually solvable, then this is an amenable case and has an invariant probability measure.

The holonomy group H of affinely flat manifold of M acts on $\mathbb{E}^n$ as affine transformations. Recall that $\mathbb{E}^n$ is given by $x_{n+1} = 1$ in $\mathbb{R}^{n+1}$. The linear parts of these affine transformations are well-defined and form a group called the linear holonomy group. If we projectivize the linear holonomy group, we obtain an action of the projectivized linear holonomy group on the projective space, denoted by $\mathbb{R}P^{n-1}$, of the vector space $\mathbb{R}^n$ associated to the affine space $\mathbb{E}^n$. From the Main Theorem, we see immediately that if the projectivized linear holonomy group has an invariant probability Borel measure on $\mathbb{R}P^{n-1}$ then $\chi(M) = 0$, since such a measure can be regarded as a holonomy invariant probability Borel measure on $\mathbb{R}P^n$ supported on $\mathbb{R}P_\infty^{n-1}$ which is disjoint from the affine space $\mathbb{E}^n$.

THEOREM 5.4. *Let M be an even dimensional closed affinely flat manifold with holonomy group H. If H has an invariant finitely additive probability measure on $\mathbb{R}P^n$, the compactification of $\mathbb{E}^n$, then $\chi(M) = 0$.*

Proof. Note that there exists a countably additive H-invariant probability measure on $\mathbb{R}P^n$ by Propositions A.1 and A.2 in Appendix. Consider M as a projectively flat manifold so that $H \subset \mathrm{Aff}(n,\mathbb{R}) \subset \mathrm{PGL}(n+1,\mathbb{R})$. By Furstenberg Theorem [16, Cor 3.2.2], either (i) $\overline{H}$ is compact or (ii) there is a proper subspace V_0 such that $\lambda[V_0] > 0$ and V_0 is invariant by a subgroup of H with finite index. An affinely flat manifold also can be viewed as a $(\mathbb{S}^n, \mathrm{P}^+\mathrm{GL}(n+1,\mathbb{R}))$ manifold, where $\mathrm{P}^+\mathrm{GL}(n+1,\mathbb{R}) \cong \mathrm{GL}(n+1,\mathbb{R})/\mathbb{R}^+$. Let $\mathrm{SL}^\pm(n+1,\mathbb{R}) = \{\, A \in \mathrm{GL}(n+1,\mathbb{R}) \mid \det A = \pm 1 \,\}$. Then $\mathrm{P}^+\mathrm{GL}(n+1,\mathbb{R}) \cong \mathrm{SL}^\pm(n+1,\mathbb{R})$. Notice that $\mathrm{PGL}(n+1,\mathbb{R}) \cong \mathrm{SL}(n+1,\mathbb{R})$ if n is even. Let $q : \mathrm{SL}^\pm(n+1,\mathbb{R}) \to \mathrm{PGL}(n+1,\mathbb{R})$ be the covering homomorphism and $p : \mathbb{S}^n \to \mathbb{R}P^n$ be the usual covering map. Let $D : \tilde{M} \to \mathbb{R}P^n$ be the developing map and $\tilde{D} : \tilde{M} \to \mathbb{S}^n$ be its lifting so that $\tilde{D} \circ p = D$. Let $\tilde{H} \subset \mathrm{P}^+\mathrm{GL}(n+1,\mathbb{R})$ be the holonomy group corresponding to $\tilde{D}$ so that it is the lifting of H.

If $\overline{H}$ is compact then $q^{-1}(\overline{H})$ is compact in $\mathrm{GL}(n+1,\mathbb{R})$ since $\mathrm{SL}^\pm(n+1,\mathbb{R})$ is closed. Therefore there exists $q^{-1}(\overline{H})$-invariant inner product on $\mathbb{R}^{n+1}$ and thus we may assume via conjugation that there exists a $\tilde{H}$-invariant Riemmanian metric ϕ on $\mathbb{S}^n$. Since $\tilde{D}^*\phi$ is deck transformation invariant Riemmanian metric on $\tilde{M}$, there is a Riemmanian metric ψ on M such that the covering map becomes a local isometry. Since M is compact, ψ is complete and $\tilde{D}^*\phi$ is also complete and thus $\tilde{D}$ becomes a covering map. Therefore $\tilde{M}$ is homeomorphic to $\mathbb{S}^n$ and thus M is a spherical space form. But an affinely flat manifold cannot be a spherical space form. So there is a proper subspace V_0 such that $\lambda[V_0] > 0$ and V_0 is invariant by a subgroup of H with finite index. Let V_0 be of minimal demension among all linear subspace with $\lambda[V_0] > 0$ and V_0 is invariant by a subgroup of H with finite index. We may assume $[V_0]$ is invariant by H. If $V_0 \cap \mathbb{E}^n = \varnothing$ (recall that $\mathbb{E}^n$ is the affine space given by $x_{n+1} = 1$ in $\mathbb{R}^{n+1}$), then $\chi(M) = 0$ by considering an H-invariant probability measure $\lambda' = (1/\lambda[V_0])\lambda|_{[V_0]}$. Now assume that $V_0 \cap \mathbb{E}^n \neq \varnothing$. If $\dim V_0 = 1$, then $V_0 \cap \mathbb{E}^n$ is a fixed point and thus M is radiant. Assume $\dim V_0 \geq 2$. Consider $[V_0]$ and $H' = H|_{[V_0]}$. Again by Furstenberg Theorem, either $\overline{H'}$ is compact or there exist a proper subspace W of V_0 such that $\lambda[W] > 0$ and $[W]$ is invariant by a subgroup of H' with finite index. But by minimality of V_0, $\overline{H'}$ is necessarily compact and thus $q^{-1}(\overline{H'})$ is compact in $\mathrm{GL}(m,\mathbb{R})$ where $m = \dim V_0$. So there exists a

$q^{-1}(\overline{H'})$-invariant inner product on V_0 and this gives an inner product on $\mathbb{R}^{n+1}$ such that $q^{-1}(H)$ acts by orthogonal transformation leaving V_0 invariant. Let $W = V_0 \cap \mathbb{E}^n$ and $W_0 = V_0 \cap (\mathbb{R}^n \times \{0\})$. Observe that W_0 is the subspace of V_0 contained in $\mathbb{R}^n \times \{0\}$ which is obtained by translating W along $W_0^\perp$ in V_0. Then $[W]$, $[W_0]$ and $[W_0^\perp]$ are all H-invariant. Furthermore $[W_0^\perp] \cap [W] = \{p\}$ and p is also invariant and thus M is radiant. Since a fixed point can not lie in the developing image in the case of an affinely flat manifold, $\chi(M) = 0$ by considering the Dirac measure concentrated at p. $\qquad\square$

Appendix A.

In this appendix, we show that for each finitely additive probability Borel measure on a compact Hausdorff space there exists a countably additive probability Borel measure corresponding to the measure and furthermore the corresponding countably additive measure is G-invariant if G acts on X and the finitely additive measure is G-invariant.

PROPOSITION A.1. *Let μ_f be a finitely additive probability Borel measure on a compact Hausdorff space X . Then there exists a countably additive probability Borel measure μ_c on X, which corresponds to the measure μ_f.*

Proof. Let $B(X, \Sigma)$ be the Banach space consisting of all uniform limits of finite linear combination of characteristic functions of sets in Borel algebra Σ. Then the dual space of $B(X, \Sigma)$ is isometrically isomorphic to the Banach space $ba(X, \Sigma)$ consisting of all bounded finitely additive measures on Σ (See Theorem IV.5.1 in [1]). In this correspondence, a probability measure μ_f in $ba(X, \Sigma)$ corresponds to a positive linear functional Λ_{μ_f} on $B(X, \Sigma)$ and $\Lambda_{\mu_f}(\chi_X) = 1$ for the characteristic function $\chi_X \in B(X, \Sigma)$. Since Σ is the Borel algebra on X and $B(X, \Sigma)$ is complete with respect to the supremum norm, the Banach space $C(X)$ consisting of all continuous functions on *compact* space X is a Banach subspace of $B(X, \Sigma)$. So the restriction $\Lambda_{\mu_f}|_{C(X)}$ of Λ_{μ_f} is a positive linear functional on $C(X)$ with $\Lambda_{\mu_f}|_{C(X)}(\chi_X) = 1$ since $\chi_X \in C(X)$. Consequently, we have a countably additive probability Borel measure μ_c on X corresponding to $\Lambda_{\mu_f}|_{C(X)}$ by the Riesz Representation Theorem. This completes the proof. $\qquad\square$

REMARK A.1. This correspondence does not imply that $\mu_f(E) = \mu_c(E)$ for all subset E of X which is contained in the Borel algebra.

For example, consider a finitely additive translation invariant probability Borel measure μ_f. In fact, μ_f can be regarded as a finitely additive probability Borel measure on the closed interval $[-\infty, +\infty]$, the compactification of $\mathbb{R}^1$, such that $\mu_f(\{-\infty\}) = \mu_f(\{+\infty\}) = 0$. But for the corresponding countably additive probability measure μ_c, $\mu_c(\{-\infty, +\infty\}) = 1$. In fact $\mu_f(I) = 0$ for any bounded interval $I \subset \mathbb{R}$ and this implies that $\mu_c(\mathbb{R}) = 0$ using the Monotone Convergence Theorem.

PROPOSITION A.2. *Let the group G act on a compact metric space X and μ_f be a G-invariant finitely additive Borel measure on X. Then the countably additive probability measure μ_c which corresponds to μ_f is also G-invariant.*

Proof. G-invariance of μ_f implies that $\mu_f(E) = \mu_f(gE)$ for all measurable E and $g \in G$ and it follows that $\int f d\mu_f = \int g \cdot f d\mu_f$ for any $f \in C(X)$ where $(g \cdot f)(x) = f(g^{-1}x)$. Since $\mu_f = \mu_c$ on $C(X)$, $\int f d\mu_c = \int g \cdot f d\mu_c$ for any $f \in C(X)$, which in turn implies that

$$\mu_c(E) = \int \chi_E d\mu_c = \int g^{-1} \cdot \chi_E d\mu_c = \int \chi_{gE} d\mu_c = \mu_c(gE)$$

for all measurable E and $g \in G$ by the Monotone Convergence Theorem.
□

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