Commun. Korean Math. Soc. **18** (2003), No. 1, pp. 181–192

THE NUMBER OF INDEPENDENT DOMINATING SETS OF LABELED TREES

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ABSTRACT. We count the numbers of independent dominating sets of rooted labeled trees, ordinary labeled trees, and recursive trees, respectively.

1. Introduction

A set S of vertices of a graph G is *independent* if no two vertices of S are adjacent by an edge in G. A set S of vertices of a graph G is *dominating* if every vertex of G not in S is adjacent to a vertex in S. An *independent dominating set* of a graph G is a set of vertices of G that is both independent and dominating in G. A set S of vertices of a graph G is an independent dominating set if and only if it is a maximal independent set of G [1]. Notice that a set S of vertices of a graph G is a maximal independent set in G if and only if the induced subgraph by S in the complement $\overline{G}$ of G is a clique in $\overline{G}$. Therefore, the study of maximal independent (and so independent dominating) sets is closely related to that of cliques. For definitions and notation not given here, see [2] and [3].

In 1965, J. W. Moon and L. Moser [6] found the largest number of independent dominating sets in a graph on n vertices, and determined the extremal graphs. In 1986, H. S. Wilf [8] determined the largest number of independent dominating sets of vertices that any tree of n vertices can have and derived a linear time algorithm for the computation of the number of independent dominating sets of any given tree. It is natural to raise following question: What is the average number of independent

Received April 1, 2002.

2000 Mathematics Subject Classification: 05C69, 05C30, 05C05.

Key words and phrases: independent dominating set, maximal independent set, rooted labeled tree, ordinary labeled tree, recursive tree.

This research was partially supported by the research fund. of the University of Seoul, 2001.

dominating sets in trees with n vertices? Our object is to determine the numbers y_n of independent dominating sets of rooted labeled, ordinary labeled, recursive trees with n vertices, respectively. Therefore, dividing y_n by the number of corresponding trees with n vertices, we can find the average number of independent dominating sets in trees with n vertices mentioned above.

2. Preliminaries

Let T be a rooted tree with root r and let S be a maximal independent set of T. Then S either does or does not contain the root r. We shall say that S is a *type A maximal independent set* or a *type B maximal independent set* according as S does or does not contain the root r. We shall say that a set S of vertices of a rooted tree T with root r is a *type C maximal independent set* in T if $S \subseteq V(T) - N[r]$ and S is a maximal independent set of the subgraph $T - r$. Here, $N[r]$ denotes the set of all neighbors of r together with r. Notice that this type C maximal independent set S is independent but not maximal independent in T. If we remove the root r of a rooted tree T, along with the edges incident with r, we obtain a (possibly empty) collection of disjoint rooted trees (or *branches*) whose roots were originally joined to r.

LEMMA 1. *Let T be a rooted tree with root r and let T have branches $B_1, \cdots, B_k$. Suppose that S_i is a set of vertices in B_i for $i = 1, \cdots, k$.*

(1) *If S_i is either a type B or a type C maximal independent set in B_i for $i = 1, \cdots, k$, then $\bigcup_{i=1}^{k} S_i \cup \{r\}$ is a type A maximal independent set of T.*

(2) *If at least one of S_i's is a type A maximal independent set and the others (possibly none) are type B maximal independent sets in the branches, then $\bigcup_{i=1}^{k} S_i$ is a type B maximal independent set of T.*

(3) *If all of S_i's are type B maximal independent sets in the branches, then $\bigcup_{i=1}^{k} S_i$ is a type C maximal independent set in T.*

PROOF. (1) It is obvious that $\bigcup_{i=1}^{k} S_i \cup \{r\}$ is an independent set of T containing the root r of T. To show that it is a maximal independent set of T, we let b be a vertex in T but not in $\bigcup_{i=1}^{k} S_i \cup \{r\}$. Then b must be in some branch B_i but not in S_i.

If S_i is a type B maximal independent set in B_i, then $S_i \cup \{b\}$ is not independent in B_i since S_i is a maximal independent set in B_i. Thus $\bigcup_{i=1}^{k} S_i \cup \{r\} \cup \{b\}$ is not independent in T.

If S_i is a type C maximal independent set in B_i, then either (i) b is in $B_i - r_i$ or (ii) $b = r_i$, where r_i is the root of B_i. In case of (i), $S_i \cup \{b\}$ is not independent in B_i since S_i is a maximal independent set in $B_i - r_i$. Thus $\bigcup_{i=1}^{k} S_i \cup \{r\} \cup \{b\}$ is not independent in T. In case of (ii), $\bigcup_{i=1}^{k} S_i \cup \{r\} \cup \{b\}$ is not independent in T since r and b are adjacent.

(2) It is obvious that $\bigcup_{i=1}^{k} S_i$ is an independent set of T that does not contain the root r of T. To show that it is a maximal independent set of T, we let b be a vertex in T but not in $\bigcup_{i=1}^{k} S_i$. Then b must be either r or a vertex in some branch B_i but not in S_i. Let S_j be a type A maximal independent set in branch B_j whose root is r_j for some j. Notice that S_j contains r_j.

If $b = r$, then b is adjacent to r_j and thus $\bigcup_{i=1}^{k} S_i \cup \{b\}$ is not independent in T.

If b is in some branch B_i but not in S_i, then $S_i \cup \{b\}$ is not independent in B_i since S_i is a maximal independent set in B_i. Thus $\bigcup_{i=1}^{k} S_i \cup \{b\}$ is not independent in T.

(3) It is obvious that $\bigcup_{i=1}^{k} S_i \subseteq V(T) - N[r]$ and that it is independent in $T - r$. To show that it is a maximal independent set of $T - r$, we let b be a vertex in $T - r$ but not in $\bigcup_{i=1}^{k} S_i$. Then b must be in some branch B_j but not in S_j. Since S_j is a maximal independent set in B_j, $S_j \cup \{b\}$ is not independent in B_j. Thus $\bigcup_{i=1}^{k} S_i \cup \{b\}$ is not independent in $T - r$. □

LEMMA 2. *Let T be a rooted tree with root r and let T have branches $B_1, \cdots, B_k$. Suppose that S is a maximal independent set of T and let $S_i = S \cap V(B_i)$ for $i = 1, \cdots, k$.*

(1) If S is a type A maximal independent set of T, then S_i is either a type B or a type C maximal independent set in the branch B_i of height at least one and $S_i = \emptyset$ in the branch B_i of height zero.

(2) If S is a type B maximal independent set of T, then at least one of S_i's is a type A maximal independent set and the others (possibly none) are type B maximal independent sets in the branches.

(3) If S is a type C maximal independent set in T, then all of S_i's are type B maximal independent sets in the branches.

PROOF. Notice that S_i is an empty set or an independent set in B_i for each i. Let r_i denote the root of branch B_i.

(1) If B_i has height of zero, then clearly $S_i = \emptyset$. Let the height of B_i be at least one. Since S does contain r, each S_i does not contain r_i. If S_i contains a vertex of B_i in $N(r_i) = N[r_i] - \{r_i\}$, then it is easy to

verify that S_i is a type B maximal independent set in B_i. If S_i contains no vertices of B_i in $N(r_i)$, then it is easy to verify that B_i must have height of at least two and that S_i is a type C maximal independent set in B_i.

(2) Recall that a set of vertices of T is a maximal independent set in T if and only if it is an independent dominating set in T. Since S does not contain r, some of S_i's should contain the roots of the branches to dominate r. If S_i contains r_i, then it is easy to verify that S_i is a type A maximal independent set in B_i. If S_i does not contain r_i, then it is easy to verify that S_i is a type B maximal independent set in B_i.

(3) Since S does not contains $r, r_1, \cdots, r_k$, each B_i must have height of at least one and each S_i must be a type B maximal independent set in B_i. □

We need the following for computation in next sections.

LEMMA 3. [3] *If*

$$\sum_{n=0}^{\infty} A_n x^n = \exp\{\sum_{n=1}^{\infty} a_n x^n\},$$

then

$$A_n = a_n + \frac{1}{n}\sum_{k=1}^{n-1} k a_k A_{n-k}$$

for $n \geq 2$. □

LEMMA 4. [7] *Let* $E_f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$ *denote an exponential generating function and let* S *be a finite set. Given functions* $f, g : \mathbb{N} \to \mathbb{N}$, *where* $\mathbb{N}$ *denotes the set of nonnegative integers, define new functions* h_1, h_2, h_3, *and* h_4 *as follows:*

$$\begin{aligned} h_1(|S|) &= f(|S|) + g(|S|), \\ h_2(|S|) &= |S| f(|S| - 1), \\ h_3(|S|) &= f(|S| + 1), \\ h_4(|S|) &= |S| f(|S|). \end{aligned}$$

Then

$$
\begin{aligned}
E_{h_1}(x) &= E_f(x) + E_g(x), \\
E_{h_2}(x) &= xE_f(x), \\
E_{h_3}(x) &= \frac{dE_f(x)}{dx}, \\
E_{h_4}(x) &= x\frac{dE_f(x)}{dx}.
\end{aligned}
$$

□

3. Rooted labeled trees

Let y_n denote the number of maximal independent (or independent dominating) sets in rooted labeled trees with n vertices and let a_n, b_n, and c_n denote the numbers of these that are of type A, type B, and type C, respectively. Let

$$y = y(x) = \sum_{n=1}^{\infty} y_n \frac{x^n}{n!}$$

and let the exponential generating functions $a = a(x)$, $b = b(x)$, and $c = c(x)$ be defined similarly for the type A, type B, and type C maximal independent sets, respectively.

THEOREM 5. *The exponential generating functions $y(x)$, $a(x)$, $b(x)$, and $c(x)$ satisfy the relations*

$$y = a + b, \tag{3.1}$$

$$a = xe^{x+b+c}, \tag{3.2}$$

$$b = x(e^y - e^b), \tag{3.3}$$

$$c = x(e^b - 1). \tag{3.4}$$

PROOF. It is easy to check that $a_1 = 1$, $b_1 = 0$, and $c_1 = 0$.

Since every maximal independent set of a rooted labeled tree T either does or does not contain the root of T, it follows clearly that

$$y = a + b.$$

Suppose that S is a type A maximal independent set of a rooted labeled tree T. If T has k branches $B_1, \cdots, B_k$, then each of $S_1 = S \cap V(B_1), \cdots, S_k = S \cap V(B_k)$ is a type B or type C maximal independent set in the branches $B_1, \cdots, B_k$, respectively. The exponential generating function for collections of k type B or type C maximal independent sets is

$$\frac{\{(x+b)+c\}^k}{k!} = \frac{(x+b+c)^k}{k!};$$

the term x in $(x+b)$ is present to count the case that a branch has a single vertex. It follows, therefore, that

$$a = x + x\sum_{k=1}^{\infty}\frac{(x+b+c)^k}{k!} = x\sum_{k=0}^{\infty}\frac{(x+b+c)^k}{k!} = xe^{x+b+c};$$

the first term x is present since $a_1 = 1$ and the factor x in the second term is present to recover the root of T.

If S is a type B maximal independent set of T, then at least one of S_i's must be a type A maximal independent set and the others must be type B maximal independent sets in the branches. The exponential generating function for collections of k members of these sets is

$$\frac{(a+b)^k - b^k}{k!} = \frac{y^k - b^k}{k!}.$$

It follows, therefore, that

$$b = 0 + x\sum_{k=1}^{\infty}\frac{y^k - b^k}{k!} = x(\sum_{k=0}^{\infty}\frac{y^k}{k!} - \sum_{k=0}^{\infty}\frac{b^k}{k!}) = x(e^y - e^b);$$

the term 0 is present since $b_1 = 0$ and the factor x is again present to recover the root of T.

Similarly, if S is a type C maximal independent set in T, then all of S_i's must be type B maximal independent sets in the branches. The exponential generating function for non-empty collections of k members of these sets is $b^k/k!$. It follows, therefore, that

$$c = 0 + x\sum_{k=1}^{\infty}\frac{b^k}{k!} = x(e^b - 1);$$

the term 0 is present since $c_1 = 0$ and the factor x is again present to recover the root of T. □

Here and in what follows, we can find no way to derive explicit formulas for the number y_n of independent dominating sets of various types of rooted trees. However, it is possible to derive recursive formulas for y_n but they are quite messy.

The following is straightforward from relations (3.1) through (3.4) and Lemma 3.

COROLLARY 6. *The number y_n of independent dominating sets in rooted trees with n labeled vertices satisfies the following recurrence relations: For $n \geq 1$,*

$$y_n = a_n + b_n,$$

where

$$a_{n+1} = (n+1)(b_n + c_n + \delta_{1n}) + \frac{n+1}{n} \sum_{k=1}^{n-1} \binom{n}{k-1} (b_k + c_k + \delta_{1k}) a_{n-k+1},$$

$$b_{n+1} = (n+1)a_n + \frac{n+1}{n} \sum_{k=1}^{n-1} \binom{n}{k-1} (a_k b_{n-k+1} + a_k c_{n-k+1} + b_k b_{n-k+1}),$$

$$c_{n+1} = (n+1)b_n + \frac{n+1}{n} \sum_{k=1}^{n-1} \binom{n}{k-1} b_k c_{n-k+1},$$

with $a_1 = 1$, $b_1 = 0$, and $c_1 = 0$. □

The entries in the following list were obtained using Corollary 6 and verified by examining the diagrams of the trees with up to six vertices given in [5; p. 2].

n	a_n	b_n
1	1	0
2	2	2
3	9	9
4	88	88
5	1045	1105
6	16536	17556
7	316099	341089
8	7140848	7765584
9	186011865	203725953

4. Ordinary labeled trees

Let z_n denote the number of independent dominating sets in rooted labeled trees with n vertices. If y_n denotes the number of independent

10	5486810320	6042331180
11	180830782231	200026701721
12	6584668606968	7310802147672
13	262545783146101	292410335375041
14	11376422212207976	12704437837102116
15	532311068751115515	595825893395602185

n	c_n	y_n
1	0	1
2	0	4
3	6	18
4	36	176
5	500	2150
6	7710	34092
7	147882	657188
8	3351992	14906432
9	87278760	389737818
10	2577703770	11529141500
11	84985001750	380857483952
12	3096079087332	13895470754640
13	123490438973772	554956118521142
14	5352557860340054	24080860049310092
15	250512328307110170	1128136962146717700

dominating sets of ordinary (or unrooted) labeled trees with n vertices, then it follows that $y_n = z_n/n$ since a rooted labeled tree T and the unrooted labeled tree that is obtained by ignoring the name 'root' from T have the same number of independent dominating sets.

The entries in the following list were obtained using $y_n = z_n/n$ and verified by examining the diagrams of the trees with up to six vertices given in [5; p. 2].

n	a_n	b_n
1	1	0
2	1	1
3	3	3
4	22	22
5	209	221
6	2756	2926
7	45157	48727
8	892606	970698
9	20667985	22636217
10	548681032	604233118
11	16439162021	18184245611
12	548722383914	609233512306
13	20195829472777	22493102721157
14	812601586586284	907459845507294
15	35487404583407701	39721726226373479

n	c_n	y_n
1	0	1
2	0	2
3	2	6
4	9	44
5	100	430
6	1285	5682
7	21126	93884
8	418999	1863304
9	9697640	43304202
10	257770377	1152914150
11	7725909250	34623407632
12	258006590611	1157955896220
13	9499264536444	42688932193934
14	382325561452861	1720061432093578
15	16700821887140678	75209130809781180

Here, let us see what the average number of independent dominating sets in labeled trees with n vertices.

n	1	2	3	4	5	6	7	8	9	10	11	12
average	1	2	2	2.7	3.4	4.3	5.5	7.1	9.0	11.5	14.6	18.7

5. Recursive trees

A tree with n labeled vertices is a *recursive tree* if $n = 1$ or if $n > 1$ and the tree is obtained by joining the nth vertex to some vertex of a recursive tree with $n - 1$ labeled vertices. We regard these trees as being rooted at the vertex labeled. The branches of a recursive tree may themselves be regarded as recursive trees if we relabel the vertices in each branch in increasing order. The number of recursive trees with n vertices is $(n-1)!$ (see [4]).

Let y_n denote the number of maximal independent (or independent dominating) sets in recursive trees with n vertices and let a_n, b_n, and c_n denote the numbers of these that are of type A, type B, and type C, respectively. Let

$$y = y(x) = \sum_{n=1}^{\infty} y_n \frac{x^n}{n!}$$

and let the exponential generating functions $a = a(x)$, $b = b(x)$, and $c = c(x)$ be defined similarly for the type A, type B, and type C maximal independent sets, respectively.

THEOREM 7. *The exponential generating functions $y(x)$, $a(x)$, $b(x)$, and $c(x)$ satisfy the relations*

$$y = a + b, \tag{5.1}$$

$$a' = \frac{da}{dx} = e^{x+b+c}, \tag{5.2}$$

$$b' = \frac{db}{dx} = e^{y} - e^{b}, \tag{5.3}$$

$$c' = \frac{dc}{dx} = e^{b} - 1. \tag{5.4}$$

PROOF. It is easy to check that $a_1 = 1$, $b_1 = 0$, and $c_1 = 0$.

The relations (5.1) through (5.4) can be proved using Lemma 4 and the same argument as in the proof of Theorem 5. □

The following is straightforward from relations (5.1) through (5.4) and Lemma 3.

COROLLARY 8. *The number y_n of independent dominating sets in recursive trees with n labeled vertices satisfies the following recurrence relations: For $n \geq 1$,*

$$y_n = a_n + b_n,$$

where

$$a_{n+1} = b_n + c_n + \delta_{1n} + \sum_{k=1}^{n-1} \binom{n-1}{k-1} (b_k + c_k + \delta_{1k}) a_{n-k+1},$$

$$b_{n+1} = a_n + \sum_{k=1}^{n-1} \binom{n-1}{k-1} (a_k b_{n-k+1} + a_k c_{n-k+1} + b_k b_{n-k+1}),$$

$$c_{n+1} = b_n + \sum_{k=1}^{n-1} \binom{n-1}{k-1} b_k c_{n-k+1},$$

with $a_1 = 1$, $b_1 = 0$, *and* $c_1 = 0$. □

The entries in the following list were obtained using Corollary 8 and verified by examining the diagrams of the recursive trees with up to five vertices.

n	a_n	b_n	c_n	y_n
1	1	0	0	1
2	1	1	0	2
3	2	2	1	4
4	7	9	2	16
5	33	45	12	78
6	198	294	65	492
7	1437	2247	484	3684
8	12245	20093	4032	32338
9	119768	204272	39315	324040
10	1322325	2333579	430036	3655904
11	16263969	29565681	5257544	45829650
12	220505888	411597480	70825471	632103368
13	3266999755	6244693089	1043535446	9511692844
14	52512518673	102560002461	16685417968	155072521134
15	910097291970	1812782990586	287781565169	2722880282556

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