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ON q-ANALOG OF KUMMER'S THEOREM AND ITS CONTIGUOUS RESULTS

YONG SUP KIM, ARJUN K. RATHIE AND CHANG HYUN LEE

ABSTRACT. The aim of this paper is to derive the well-known q-analog of kummer's theorem by using q-integral representation. In addition to this, two results closely related to the q-kummer's theorem have also been obtained by the same method.

1. Introduction and results required

The q-hypergeometric series, a generalization of the hypergeometric series, considered by Heine 1846 is given by the formula

$$ {}_2\phi_1(a, b; c; q, z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n \tag{1.1} $$

$$ (c \neq q^{-m}; m = 0, 1, 2, \dots), $$

where the q-shifted factorial is defined by

$$ (a;q)_n = \begin{cases} 1, & n = 0, \\ \prod_{m=0}^{n-1} (1 - aq^m), & n = 1, 2, 3, \dots, \end{cases} \tag{1.2} $$
$$ (a;q)_\infty = \prod_{m=0}^{\infty} (1 - aq^m). $$

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The series (1.1) converges absolutely for $|z| < 1$ when $0 < |q| < 1$.

In 1973, using an elementary summation technique, Andrew [1] gave the q-analogue of the well-known q-kummer's theorem:

$$ {}_2\phi_1 \left[\begin{matrix} a, b \\ \frac{aq}{b} \end{matrix} ; q, -\frac{q}{b} \right] = \frac{(aq; q^2)_\infty (-q; q)_\infty (\frac{aq^2}{b^2}; q^2)_\infty}{(\frac{aq}{b}; q)_\infty (-\frac{q}{b}; q)_\infty} $$

or, equivalently,

$$ (1.3) \quad {}_2\phi_1 \left[\begin{matrix} q^a, q^b \\ q^{1+a-b} \end{matrix} ; q, -q^{1-b} \right] = \frac{(q^{a+1}; q^2)_\infty (-q; q)_\infty (q^{2+a-2b}; q^2)_\infty}{(q^{1+a-b}; q)_\infty (-q^{1-b}; q)_\infty}. $$

Thomae [6, Eq. (1.11.9)] rewrote the Heine's formula (1.1) in the q-integral form

(1.4)

$$ {}_2\phi_1 \left[\begin{matrix} q^a, q^b \\ q^c \end{matrix} ; q, q^z \right] = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{(tq^{z+a}; q)_\infty (tq; q)_\infty}{(tq^z; q)_\infty (tq^{c-b}; q)_\infty} d_q(t), $$

where $|\arg(1-z)| < \pi$ and $\Re e(c) > \Re e(b) > 0$, and the q-gamma function

$$ (1.5) \qquad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x} \qquad (0 < q < 1) $$

was introduced by Thomae [6] and later by Jackson [4].

Thomae [6, 7] and Jackson [5] introduced the q-integral

$$ (1.6) \qquad \int_0^1 f(t) d_q(t) = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n. $$

We shall also need the following elementary summation, (see Slater [8, p. 92, Eq.(3.2.2.11)])

$$ (1.7) \qquad \sum_{n=0}^{\infty} \frac{(A)_n T^n}{(q)_n} \equiv {}_1\phi_0 [A; q, t] = \frac{(AT)_\infty}{(T)_\infty}. $$

The aim of this paper is to derive (1.3) by using q-integral representation (1.4). In addition to this, two results closely related to (1.3) have also been obtained by the same technique.

2. Derivation of (1.3)

Interchanging a and b in (1.4), we have

$$ {}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^c \end{matrix}; q, q^z\right] = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(c-a)} \int_0^1 t^{a-1} \frac{(tq^{z+b};q)_\infty (tq;q)_\infty}{(tq^z;q)_\infty (tq^{c-a};q)_\infty} d_q(t), $$

which, upon taking $q^z = -q^{1-b}$, $q^c = q^{1+a-b}$, yields

$$ \begin{aligned} S :&= {}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^{1+a-b} \end{matrix}; q, -q^{1-b}\right] \\ &= \frac{\Gamma_q(1+a-b)}{\Gamma_q(a)\Gamma_q(1-b)} \int_0^1 t^{a-1} \frac{(-tq;q)_\infty (tq;q)_\infty}{(-tq^{1-b};q)_\infty (tq^{1-b};q)_\infty} d_q(t). \end{aligned} $$

If we use (1.5), then, after some simplification, we have

$$ S = \frac{(q^a;q)_\infty (q^{1-b};q)_\infty}{(q^{1+a-b};q)_\infty (q;q)_\infty} \frac{1}{1-q} \int_0^1 t^{a-1} \frac{(-tq;q)_\infty (tq;q)_\infty}{(-tq^{1-b};q)_\infty (tq^{1-b};q)_\infty} d_q(t). $$

Using (1.6), we get

$$ S = \frac{(q^a;q)_\infty (-q;q)_\infty}{(q^{1+a-b};q)_\infty (-q^{1-b})_\infty} \sum_{n=0}^{\infty} \frac{(-q^{1-b};q)_n (q^{1-b};q)_n}{(-q;q)_n (q;q)_n} q^{na}. $$

With the aid of $(-q;q)_n (q;q)_n = (q^2;q^2)_n$,

$$ S = \frac{(q^a;q)_\infty (-q;q)_\infty}{(q^{1+a-b};q)_\infty (-q^{1-b})_\infty} \sum_{n=0}^{\infty} \frac{q^{2-2b};q^2)_n}{(q^2;q^2)_n} q^{na}. $$

Using the q-binomial theorem (1.7), we get

$$ S = \frac{(q^a;q)_\infty (-q;q)_\infty (q^{2-2b+a};q^2)_\infty}{(q^{1+a-b};q)_\infty (-q^{1-b})_\infty (q^a;q^2)_\infty}. $$

Finally, using an elementary identity:

$$(q^a;q)_\infty = (q^a;q^2)_\infty (q^{a+1};q^2)_\infty,$$

we have

$${}_2\phi_1\left[\begin{matrix} q^a, q^b \\ q^{1+a-b} \end{matrix}; q; -q^{1-b}\right] = \frac{(q^{a+1};q^2)_\infty(-q;q)_\infty(q^{2-2b+a};q^2)_\infty}{(q^{1+a-b};q)_\infty(-q^{1-b};q)_\infty},$$

which can, upon writing a, b, c instead of q^a, q^b, q^c [Watson notation] respectively, be rewritten in an equivalent form:

$${}_2\phi_1\left[\begin{matrix} a, b \\ \frac{aq}{b} \end{matrix}; q, -\frac{q}{b}\right] = \frac{(aq:q^2)_\infty(-q;q)_\infty(\frac{aq^2}{b^2};q^2)_\infty}{(\frac{aq}{b};q)_\infty(-\frac{q}{b};q)_\infty}.$$

This completes the proof of (1.3). □

3. Contiguous results

By using the same technique as in Section 2, we can easily derive the following two very interesting contiguous results.

(3.1)

$${}_2\phi_1\left[\begin{matrix} a, b \\ \frac{aq^2}{b} \end{matrix}; q, -\frac{q}{b}\right] = \frac{(\frac{q^2}{b};q)_\infty(-q;q)_\infty}{(\frac{aq^2}{b};q)_\infty(\frac{q}{b};q)_\infty(-\frac{q}{b};q)_\infty} \times \left\{(aq;q^2)_\infty(\frac{aq^2}{b^2};q^2)_\infty - \frac{q}{b}(a;q^2)_\infty(\frac{aq^3}{b^2};q^2)_\infty\right\},$$

and

(3.2)

$${}_2\phi_1\left[\begin{matrix} a, b \\ \frac{a}{b} \end{matrix}; q, -\frac{q}{b}\right] = \frac{(-q;q)_\infty}{(\frac{a}{b};q)_\infty(-\frac{1}{b};q)_\infty} \times \left\{(aq;q^2)_\infty(\frac{a}{b^2};q^2)_\infty + \frac{1}{b}(a;q^2)_\infty(\frac{aq}{b^2};q^2)_\infty\right\}.$$

4. Proofs of (3.1) and (3.2)

Similarly as in Section 2, here, letting $c = 2 + a - b$ and $q^z = -q^{1-b}$ in (1.4), we have

$$R := {}_2\phi_1 \left[\begin{matrix} q^a, q^b \\ q^{2+a-b} \end{matrix} ; q, -q^{1-b} \right]$$
$$= \frac{\Gamma_q(2+a-b)}{\Gamma_q(a)\Gamma_q(2-b)} \int_0^1 \frac{t^{a-1}(-tq;q)_\infty (tq;q)_\infty}{(-tq^{1-b};q)_\infty (tq^{2-b};q)_\infty} d_q(t).$$

Making use of identities

$$(q^{n+1};q)_\infty = \frac{(q;q)_\infty}{(q;q)_n}$$

and

$$(q^{n+2-b};q)_\infty = \frac{(q^{1-b};q)_\infty}{(q^{1-b};q)_{n+1}}$$

with the help of (1.5) and (1.6), we obtain

$$R = \frac{(q^a;q)_\infty (q^{2-b};q)_\infty}{(q^{2+a-b};q)_\infty} \sum_{n=0}^{\infty} \frac{(-q;q)_\infty (-q^{1-b};q)_n (q^{1-b};q)_{n+1} (q^a)^n}{(-q;q)_n (q;q)_n (-q^{1-b};q)_\infty (q^{1-b};q)_\infty}.$$

By writing

$$(q^{1-b};q)_{n+1} = (q^{1-b};q)_n (1 - q^{1-b+n}),$$

we obtain

$$R = \frac{(q^a;q)_\infty (q^{2-b};q)_\infty (-q;q)_\infty}{(q^{2+a-b};q)_\infty (-q^{1-b};q)_\infty (q^{1-b};q)_\infty}$$
$$\times \left[\sum_{n=0}^{\infty} \frac{(q^{2(1-b)};q^2)_n}{(q^2;q^2)_n} (q^a)^n - q^{1-b} \sum_{n=0}^{\infty} \frac{(q^{2(1-b)};q^2)_n}{(q^2;q^2)_n} \left(q^{a+1}\right)^n \right].$$

Using

$$\sum_{n=0}^{\infty} \frac{(A;q)_n}{(q;q)_n} T^n = \frac{(AT;q)_\infty}{(T;q)_\infty}$$

$$= \frac{(q^a;q)_\infty (q^{2-b};q)_\infty (-q;q)_\infty}{(q^{2+a-b};q)_\infty (-q^{1-b};q)_\infty (q^{1-b};q)_\infty} \times \left[\frac{\left(q^{a+2(1-b)};q^2\right)_\infty}{(q^a;q^2)_\infty} - q^{1-b} \frac{\left(q^{a+1+2(1-b)};q^2\right)_\infty}{(q^{a+1};q^2)_\infty} \right].$$

Replacing q^a, q^b, q^c by a, b, c respectively, we get

$${}_2\phi_1 \left[\begin{matrix} a, b, \\ \frac{aq^2}{b} \end{matrix} ; q, -\frac{q}{b} \right] = \frac{(a;q)_\infty (\frac{a^2}{b};q)_\infty (-q;q)_\infty}{(\frac{aq^2}{b};q)_\infty (\frac{q}{b};q)_\infty (-\frac{q}{b};q)_\infty} \left[\frac{(\frac{aq^2}{b^2};q^2)_\infty}{(a;q^2)_\infty} - \frac{q}{b} \frac{(\frac{aq^3}{b^2};q^2)_\infty}{(aq;q^2)_\infty} \right].$$

Using $(a;q)_\infty = (a;q^2)_\infty (aq;q^2)_\infty$, we arrive at the right hand side of (3.1).

In the corresponding manner, the result (3.2) can also be obtained.

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Yong-Sup Kim
Department of Mathematics
Wonkwang University
Iksan 570-749, Korea
E-mail: yspkim@wonkwang.ac.kr

Arjun K. Rathie
Department of Mathematics
Dungar College (M.D.S. University)
Rajasthan State, India

Chang-Hyun Lee
Department of Mathematics
Seonam University
Namwon 590-711, Korea
E-mail: chlee@tiger.seonam.ac.kr