Commun. Korean Math. Soc. **18** (2003), No. 1, pp. 133–149

LOCAL STABILITY OF ENDEMIC STATES FOR AN EPIDEMIC MODEL WITH EXTERNAL FORCE OF INFECTION

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ABSTRACT. Local stability of steady states of an epidemic model is considered. An age structured S-I-R epidemic model with separable inter-cohort force of infection with external force is considered. Stability result for the nontrivial steady states is obtained.

1. Introduction

In this paper we consider the stability of steady states of an age-structured S-I-R epidemic model with external force of infection.

In S-I-R model, the population is structured into three disjoint classes according to disease status: susceptibles, infecteds, and removeds. Susceptibles are the individuals who are not infected with the disease but may become infected through contact with infected individuals or by the external force. Infecteds are the individuals who are currently infected, and removeds are the individuals who may not contract or transmit the disease. Removeds usually consist of the individuals who are immunized.

Age structured S-I-R models are suitable for most common childhood diseases (measles, chickenpox, rubella), as well as for many sexually transmitted diseases which impart immunity (syphilis, chlamydia), and also for those diseases, like HIV/AIDS or mad cow disease, which lead to definitive isolation or death.

A lot of papers can be found for the S-I-R models *without* external force of infection [1, 3, 4, 5, 8, 9]. Since this problem is intimately associated with the study of long-time behavior of solutions of the model,

Received April 10, 2002 .
2000 Mathematics Subject Classification: 92B05.
Key words and phrases: epidemic, endemic, S-I-R, stability, intercohort, external force.
This work was supported by Korea Research Foundation Grant (KRF-2001-003-D00015).

existing literature's main concerns are existence, uniqueness and stability change of the steady state solutions.

The long term behavior has some important epidemiological implications such as suggesting whether an outbreak of a particular infection may result in an endemic situation or the infection will die out.

Recent paper [2] deals with the S-I-R model with external force as an important part of the force of infection. Existence and uniqueness of steady state solutions are obtained there.

Even if infection of the human disease mainly occurs between humans through physical contacts, there are lots of other ways of infection. For instance, one can be infected with HIV/AIDS through blood transfusion. Mad cow disease is another good example: infected animals can be the main source of the infection in this case.

In fact, other than infected human individuals, there are still a lot of sources of infection. We call them external force of infection.

In this paper we shall show that the endemic state of that model is locally asymptotically stable. Our main result will be give in section 3, Theorem 3.1.

2. S-I-R model with external force of infection

We consider the following system of integro-differential equations.

$$\tag{2.1} \begin{cases} \dfrac{\partial s}{\partial t} + \dfrac{\partial s}{\partial a} + \mu(a)s = -\lambda(a,t)s, \\ \dfrac{\partial i}{\partial t} + \dfrac{\partial i}{\partial a} + \mu(a)i = \lambda(a,t)s - \gamma(a)i, \\ \dfrac{\partial r}{\partial t} + \dfrac{\partial r}{\partial a} + \mu(a)r = \gamma(a)i, \\ s(0,t) = \displaystyle\int_0^{a_\dagger} \beta(a)\big(s(a,t) + r(a,t) + (1-q)i(a,t)\big)\,da, \\ i(0,t) = q\displaystyle\int_0^{a_\dagger} \beta(a)i(a,t)\,da, \\ r(0,t) = 0, \\ i(a,0) = i_0(a), \quad s(a,0) = s_0(a), \quad r(a,0) = r_0(a). \end{cases}$$

Here a is the age of individuals, and t is the time. Also, $s(a,t)$, $i(a,t)$ and $r(a,t)$, respectively, denotes the age-specific density of susceptible, infected, and removed individuals.

$\beta(a)$ is the birth rate and $\mu(a)$ is the death rate of the population. $q \in [0,1]$ is the vertical transmission parameter, i.e. the probability that the disease be transmitted from parent to newborn, $\gamma(\cdot)$ is the removal rate of infected individuals, and $\lambda(a,t)$ is the force of infection. Note that since we assume the condition $r(0,t) = 0$, our model assumes that there is no vertical transmission of immunity.

Our main concern is the stability of an *endemic* state of the model. Endemic state is a steady state solution of the model for which the density of infected individuals does not vanish identically (see [2]).

Summing the equations in (2.1) we obtain the following problem for the population density $p(a,t) = s(a,t) + i(a,t) + r(a,t)$,

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} + \mu(a)p = 0, \\ p(0,t) = \displaystyle\int_0^{a_\dagger} \beta(a)p(a,t)da, \\ p(a,0) = p_0(a). \end{cases} \tag{2.2}$$

This is the standard McKendrick-Von Foester equation.

We make the following usual hypotheses for this problem,

$$\beta(\cdot) \in L^\infty([0,a_\dagger)), \qquad \beta(a) \geq 0 \text{ in } [0,a_\dagger), \tag{2.3}$$

$$\mu(\cdot) \in L^1_{\text{loc}}([0,a_\dagger)), \qquad \mu(a) \geq 0 \text{ in } [0,a_\dagger), \tag{2.4}$$

$$\int_0^{a_\dagger} \mu(a)d\sigma = \infty. \tag{2.5}$$

Here $a_\dagger$ is the maximum age an individual of the population may reach and it may be either finite or infinite. If $a_\dagger = \infty$, we also assume that

$$\text{there exists } A > 0 \text{ such that } \beta(a) = 0 \text{ for } a \geq A. \tag{2.6}$$

Furthermore, in order to deal with a steady state population, we assume that the net reproductive rate of the population is equal to 1 and that the total population is at an equilibrium. This means that

$$\int_0^{a_\dagger} \beta(a)e^{-\int_0^a \mu(\sigma)d\sigma}da = 1, \qquad p(a,t) = p_\infty(a) = b_0\pi(a), \tag{2.7}$$

where

$$\pi(a) = e^{-\int_0^a \mu(\sigma)d\sigma}. \tag{2.8}$$

Note that the function $\pi(a)$ is the probability that an individual at age 0 can survive until age a. Since no individual may live past age $a_\dagger$, (2.5) is needed.

Clearly, we have to take initial data such that
(2.9)
$$s_0(a) \geq 0, \quad i_0(a) \geq 0, \quad r_0(a) \geq 0, \quad s_0(a) + i_0(a) + r_0(a) = p_\infty(a),$$

which gives

$$b_0 = \frac{\int_0^{a_\dagger} s_0(a)da + \int_0^{a_\dagger} i_0(a)da + \int_0^{a_\dagger} r_0(a)da}{\int_0^{a_\dagger} \pi(a)da}. \tag{2.10}$$

We also assume that

$$\gamma(\cdot) \in L^\infty([0, a_\dagger)), \qquad \gamma(a) \geq 0 \text{ in } [0, a_\dagger) \tag{2.11}$$

and consider the following form for the force of infection:

$$\lambda(a,t) = \kappa(a) \int_0^{a_\dagger} h(\sigma) i(\sigma, t)\, d\sigma + g(a), \tag{2.12}$$

where h is the age-specific infectiousness, κ the age-specific contagion rate, g the external force of infection. They satisfy the following conditions:
(2.13)
$$h(\cdot),\ \kappa(\cdot),\ g(\cdot) \in L^\infty([0, a_\dagger)) \ \text{ and } \ h(a),\ \kappa(a),\ g(a) \geq 0, \ \text{ on } \ [0, a_\dagger).$$

We also assume that none of $\beta(\cdot)$, $\mu(\cdot)$, $\gamma(\cdot)$, $h(\cdot)$, $\kappa(\cdot)$, $g(\cdot)$ is identically zero.

We note that by assumption (2.7), the fourth equation in (2.1) becomes

$$\begin{aligned} s(0,t) &= \int_0^{a_\dagger} \beta(a)\, p_\infty(a)\, da - q \int_0^{a_\dagger} \beta(a)\, i(a,t)\, da \\ &= b_0 - q \int_0^{a_\dagger} \beta(a)\, i(a,t)\, da, \end{aligned} \tag{2.14}$$

so that the equations involving the variable $r(a,t)$ in (2.1) can be removed since $s(a,t)$ and $i(a,t)$ are sufficient to determine the evolution of the whole system. Thus, in the rest of the paper, we will be concerned with the following reduced system derived from (2.1) and (2.7):

$$
(2.15)\quad \begin{cases}
\dfrac{\partial s}{\partial t}(a,t) + \dfrac{\partial s}{\partial a}(a,t) + \mu(a)s(a,t) = -\lambda(a,t)s(a,t), \\
\dfrac{\partial i}{\partial t}(a,t) + \dfrac{\partial i}{\partial a}(a,t) + \mu(a)i(a,t) = \lambda(a,t)s(a,t) - \gamma(a)i(a,t), \\
s(0,t) = b_0 - i(0,t), \\
i(0,t) = q\displaystyle\int_0^{a_\dagger} \beta(a)i(a,t)da, \\
i(a,0) = i_0(a), \quad s(a,0) = s_0(a).
\end{cases}
$$

Now we will give a brief review of existence and uniqueness results of the model, which are stated in [2].

First, we consider the following problem, which is obtained by removing the variable t in (2.15):

$$
(2.16)\quad \begin{cases}
\text{i)} & \dfrac{\partial s}{\partial a} + \mu(a)s(a) = -\big(J\kappa(a) + g(a)\big)\, s(a), \\
\text{ii)} & \dfrac{\partial i}{\partial a} + \mu(a)i(a) = \big(J\kappa(a) + g(a)\big)\, s(a) - \gamma(a)\, i(a), \\
\text{iii)} & J = \displaystyle\int_0^{a_\dagger} h(a)\, i(a)\, da, \\
\text{iv)} & s(0) = b_0 - i(0), \\
\text{v)} & i(0) = q\displaystyle\int_0^{a_\dagger} \beta(a)\, i(a)\, da.
\end{cases}
$$

It is easy to see that the problem admits the disease-free equilibrium $s^*(a) = p_\infty(a)$, $i^*(a) \equiv 0$, if and only if $g(a) \equiv 0$. Since we are assuming that g is not identically zero, we have to concentrate on the search of endemic states, that is nonnegative solutions for which $i^*(a)$ does not vanish identically.

In order to investigate the existence of such solutions, we modify problem (2.16) by taking the following new variables, the age profiles respectively of infecteds and susceptibles:

$$
(2.17)\qquad u(a) = \frac{i(a)}{p_\infty(a)}; \qquad v(a) = \frac{s(a)}{p_\infty(a)}.
$$

With these definitions, problem (2.16) becomes

$$(2.18)\qquad \begin{cases} \text{i)} & \dfrac{dv}{da} = -\big(J\kappa(a) + g(a)\big)\, v(a), \\ \text{ii)} & \dfrac{du}{da} = \big(J\kappa(a) + g(a)\big)\, v(a) - \gamma(a)\, u(a), \\ \text{iii)} & J = b_0 \displaystyle\int_0^{a_\dagger} h(\sigma)\, \pi(\sigma)\, u(\sigma)\, d\sigma, \\ \text{iv)} & v(0) = 1 - X, \\ \text{v)} & X = q \displaystyle\int_0^{a_\dagger} \beta(\sigma)\, \pi(\sigma)\, u(\sigma)\, d\sigma. \end{cases}$$

Note that integration of (2.18.i) gives

$$(2.19)\qquad v(a) = (1 - X)e^{-\int_0^a (J\kappa(\sigma)+g(\sigma))d\sigma}.$$

Then, substituting (2.19) into (2.18.ii) and integrating the equation we have

$$\begin{aligned} u(a) =& Xe^{-\int_0^a \gamma(\sigma)d\sigma} + (1 - X) \int_0^a \big(J\kappa(\sigma) \\ &+ g(\sigma)\big)e^{-\int_\sigma^a \gamma(s)ds - \int_0^\sigma (J\kappa(s)+g(s))ds} d\sigma. \end{aligned} \qquad (2.20)$$

Substituting (2.19) and (2.20) into (2.18.iii) and (2.18.v) we get the following relations,

$$(2.21)\qquad \begin{cases} J = XG + (1 - X)\big(JM(J) + D(J)\big), \\ X = XR + (1 - X)\big(JL(J) + C(J)\big), \end{cases}$$

where we have introduced the following notation:

$$(2.22)\qquad G = b_0 \int_0^{a_\dagger} h(a)\pi(a)\Gamma(a)\, da,$$

$$(2.23)\qquad R = q \int_0^{a_\dagger} \beta(a)\pi(a)\, \Gamma(a)\, da,$$

$$(2.24)\qquad M(J) = b_0 \int_0^{a_\dagger} h(a)\, \pi(a)\, F(a, J)\, da,$$

$$(2.25)\qquad L(J) = q \int_0^{a_\dagger} \beta(a)\, \pi(a)\, F(a, J)\, da,$$

$$(2.26)\qquad D(J) = b_0 \int_0^{a_\dagger} h(a)\, \pi(a)\, H(a, J)\, da,$$

$$(2.27)\qquad C(J) = q \int_0^{a_\dagger} \beta(a)\, \pi(a)\, H(a, J)\, da,$$

and

$$(2.28)\qquad \Gamma(a) = e^{-\int_0^a \gamma(\sigma)d\sigma},$$

$$(2.29)\qquad F(a,J) = \int_0^a \kappa(\sigma)e^{-\int_\sigma^a \gamma(s)ds - \int_0^\sigma (J\kappa(s)+g(s))ds}d\sigma,$$

$$(2.30)\qquad H(a,J) = \int_0^a g(\sigma)e^{-\int_\sigma^a \gamma(s)ds - \int_0^\sigma (J\kappa(s)+g(s))ds}d\sigma.$$

We seek solutions of (2.21) such that $J \geq 0$ and $0 \leq X \leq 1$. (Note that $X = u(0)$ and $0 \leq u(0) \leq 1$.) In fact, any such a pair (X^*, J^*) provides a nonnegative solution of (2.18) via (2.19) and (2.20).

Note that if the following conditions are satisfied, then (2.21) reduces to a single equation with two unknowns J and X:

$$(2.31)\qquad R = 1 \quad \text{and} \quad L(J) = C(J) = 0 \quad \text{for all } J.$$

Hence in this case (2.21) has continuum of solutions (see [2]). In order to rule out such a pathological case, the following assumption is required in the rest of the paper.

(2.32) All the relations in (2.31) do not hold simultaneously.

Also note that if we have an endemic equilibrium (J^*, X^*) with $J^* = 0$, we can immediately show that $X^* = 0$ and $C(\cdot) \equiv D(\cdot) \equiv 0$ by (2.18 iii), (2.20), (2.26) and (2.27). Since this situation is rather special, we also rule out such a case in this paper. Actually we shall assume the following in the rest of the paper:

(2.33) $D(\cdot)$ is not identically zero.

Note that the above assumption is equivalent to the following:

$$a_h^+ > a_g^-, \text{ where}$$
$$a_h^+ = \text{Inf}\{A : h(a) = 0 \text{ a.e. in } [A, a_\dagger)\},$$
$$a_g^- = \text{Sup}\{A : g(a) = 0 \text{ a.e. in } [0, A]\}.$$

Hence, in biological point of view, assumption (2.33) is not so restrictive.

Under the assumptions (2.32) and (2.33) we can further reduce the system (2.21) to a single equation. In fact, solving the second equation for X we obtain

$$X = \frac{JL(J) + C(J)}{1 - R + JL(J) + C(J)}, \tag{2.34}$$

which, when substituted into the other equation yields:

$$(1 - R)\big(J - JM(J) - D(J)\big) + \big(J - G\big)\big(JL(J) + C(J)\big) = 0. \tag{2.35}$$

Thus we consider the continuous function

$$\phi(J) = \big(1 - R\big)\big(J - JM(J) - D(J)\big) + \big(J - G\big)\big(JL(J) + C(J)\big) \tag{2.36}$$

and we analyze its behavior in the interval $(0, \infty)$. The following existence and uniqueness results are given in [2].

THEOREM 2.1. *An endemic state always exists.*

THEOREM 2.2. *If $q = 0$, then the endemic state is unique.*

Note that the assumption, $q = 0$, in theorem 2.2 implies that there is no vertical transmission of the disease.

3. Stability analysis

In this section we shall consider the local stability of the model.

First we take the following new variables, respectively called the age profiles of infecteds and susceptibles:

$$u(a,t) = \frac{i(a,t)}{p_\infty(a)},$$
$$v(a,t) = \frac{s(a,t)}{p_\infty(a)}.$$

Then (2.15) reduces to the following system:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial a} = -\kappa(a)J(t)v(a,t) - g(a)v(a,t), \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} = \kappa(a)J(t)v(a,t) + g(a)v(a,t) - \gamma(a)u(a,t), \\ v(0,t) = 1 - u(0,t), \\ u(0,t) = q\displaystyle\int_0^{a_\dagger} \beta(a)u(a,t)\pi(a)da, \\ u(a,0) = u_0(a), \\ v(a,0) = v_0(a) \end{cases} \tag{3.1}$$

with

$$J(t) = b_0 \int_0^{a_\dagger} h(\sigma)u(\sigma,t)\pi(\sigma)d\sigma.$$

Integrating v along the characteristics in (3.1), we get

$$(3.2) \qquad v(a,t) = \begin{cases} v_0(a-t)e^{-\int_0^t [\kappa(\sigma+a-t)J(\sigma)+g(\sigma+a-t)]d\sigma} & \text{for } a \geq t, \\ Y(t-a)e^{-\int_0^a [\kappa(\sigma)J(\sigma+t-a)+g(\sigma)]d\sigma} & \text{for } t \geq a, \end{cases}$$

where

$$Y(t) = v(0,t).$$

Using (3.2) and integrating u along the characteristics in (3.1), we have
(3.3)

$$u(a,t) = \begin{cases} e^{-\int_0^t \gamma(s+a-t)ds}\Big[u_0(a-t) + v_0(a-t) \\ \quad \times \int_0^t [\kappa(\sigma+a-t)J(\sigma) + g(\sigma+a-t)] \\ \quad \times e^{\int_0^\sigma [\gamma(s+a-t)-\kappa(s+a-t)J(s)-g(\sigma+a-t)]ds}d\sigma\Big] & \text{for } a \geq t, \\ e^{-\int_0^a \gamma(\sigma)d\sigma}\Big[X(t-a) + Y(t-a) \\ \quad \times \int_0^a [\kappa(\sigma)J(\sigma+t-a) + g(\sigma)] \\ \quad \times e^{\int_0^\sigma [\gamma(s)-\kappa(s)J(s+t-a)-g(s)]ds}d\sigma\Big] & \text{for } t \geq a, \end{cases}$$

where

$$X(t) = u(0,t).$$

Using this relation, together with

$$X(t) = q\int_0^{a_\dagger} \beta(a)u(a,t)\pi(a)da,$$

we obtain the following expressions for $t \geq a_\dagger$:

$$X(t) = q\int_0^{a_\dagger} \beta(a)\pi(a)\Gamma(a)\Big[X(t-a) + (1-X(t-a)) \tag{3.4}$$
$$\times \int_0^a \{\kappa(\sigma)J(\sigma+t-a)+g(\sigma)\}$$
$$\times\, e^{\int_0^\sigma[\gamma(s)-\kappa(s)J(s+t-a)-g(s)]ds}d\sigma\Big]\,da,$$

$$J(t) = b_0\int_0^{a_\dagger} h(a)\pi(a)\Gamma(a)\Big[X(t-a) + (1-X(t-a)) \tag{3.5}$$
$$\times \int_0^a \{\kappa(\sigma)J(\sigma+t-a)+g(\sigma)\}$$
$$\times\, e^{\int_0^\sigma[\gamma(s)-\kappa(s)J(s+t-a)-g(s)]ds}d\sigma\Big]\,da.$$

In order to linearize (3.4) and (3.5), we let

$$X(t) = X^* + x(t), \qquad J(t) = J^* + j(t), \tag{3.6}$$

where X^* and J^* satisfy (2.21), (2.34) and (2.35).

After a long calculation, we get the following expressions:

$$\begin{cases} x(t) = \int_0^t Q_1(a)x(t-a)da + \int_0^t Q_2(a)j(t-a)da, \\ j(t) = \int_0^t Q_3(a)x(t-a)da + \int_0^t Q_4(a)j(t-a)da, \end{cases} \tag{3.7}$$

where the convolution kernels Q_1, Q_2, Q_3 and Q_4 are given by

$$Q_1(a) = w(a)\left[1 - \int_0^a N(\sigma, J^*)E(\sigma, J^*)d\sigma\right],$$
$$Q_2(a) = \int_a^{a_\dagger} w(s)(1-X^*)\kappa(s-a)$$
$$\times\left[E(s-a, J^*) - \int_{s-a}^s N(\sigma, J^*)E(\sigma, J^*)d\sigma\right] ds,$$
$$Q_3(a) = \tilde{w}(a)\left[1 - \int_0^a N(\sigma, J^*)E(\sigma, J^*)d\sigma\right],$$
$$Q_4(a) = \int_a^{a_\dagger} \tilde{w}(s)(1-X^*)\kappa(s-a)$$
$$\times\left[E(s-a, J^*) - \int_{s-a}^s N(\sigma, J^*)E(\sigma, J^*)d\sigma\right] ds,$$

with

$$N(\sigma, J) = J\kappa(\sigma) + g(\sigma),$$
$$E(a, J) = e^{\int_0^a [\gamma(s) - N(s,J)]ds},$$
$$w(a) = q\,\beta(a)\pi(a)\Gamma(a),$$
$$\tilde{w}(a) = b_0\,h(a)\pi(a)\Gamma(a).$$

Taking Laplace transforms in (3.7), we obtain the following characteristic equation (see [7]):

$$Q(\lambda) = \left(1 - \hat{Q}_1(\lambda)\right)\left(1 - \hat{Q}_4(\lambda)\right) - \hat{Q}_2(\lambda)\hat{Q}_3(\lambda) = 0, \tag{3.8}$$

where

$$\hat{Q}_i(\lambda) = \int_0^\infty e^{-\lambda a} Q_i(a)da, \quad i = 1, 2, 3, 4,$$

represents the Laplace transform of $Q_i(a)$.

Our problem is to investigate the stability of steady state solutions of (3.1) through the analysis of (3.8). We shall use the following general result (see [7]).

If all the roots of (3.8) *have negative real part, then the steady state corresponding to* (X^*, J^*) *is locally asymptotically stable. If there are roots with positive real part, then it is unstable.*

In order to obtain the local stability results, we shall need the following assumption (see [4, 8]):

$$u^*(a_\dagger) < e^{-\int_0^{a_\dagger} \gamma(\sigma)d\sigma}. \tag{3.9}$$

Under the assumption (3.9), we can establish the following three lemmas.

LEMMA 3.1. (3.9) *is equivalent to each of the following.*

$$\int_0^{a_\dagger} \gamma(\sigma)e^{\int_\sigma^{a_\dagger}(N(s,J^*) - \gamma(s))ds}d\sigma < 1, \tag{3.10}$$

$$\int_0^{a_\dagger} N(\sigma, J^*)e^{\int_0^\sigma(\gamma(s) - N(s,J^*))ds}d\sigma < 1, \tag{3.11}$$

$$\int_0^{s} \gamma(\sigma)e^{\int_\sigma^s(N(\tau,J^*) - \gamma(\tau))d\tau}d\sigma < 1, {}^\forall s \in [0, a_\dagger]. \tag{3.12}$$

PROOF. Note that $u^*(a)$ can be computed as follows.

(3.13)

$$u^*(a) = X^* e^{-\int_0^a \gamma(\sigma)d\sigma} + (1-X^*) \int_0^a N(\sigma, J^*) e^{-\int_\sigma^a \gamma(s)ds - \int_0^\sigma N(s,J^*)ds} d\sigma.$$

Hence

$$u^*(a_\dagger) = e^{-\int_0^{a_\dagger} \gamma(\sigma)d\sigma} \times \left[X^* + (1-X^*) \int_0^{a_\dagger} N(\sigma, J^*) e^{\int_0^\sigma (\gamma(s) - N(s,J^*))ds} d\sigma\right].$$

Thus (3.9) is equivalent to

$$X^* + (1-X^*) \int_0^{a_\dagger} N(\sigma, J^*) e^{\int_0^\sigma (\gamma(s)-N(s,J^*))ds} d\sigma < 1. \tag{3.14}$$

But

$$\begin{aligned} &\int_0^{a_\dagger} N(\sigma, J^*) e^{\int_0^\sigma (\gamma(s)-N(s,J^*))ds} d\sigma \\ &= \int_0^{a_\dagger} e^{\int_0^\sigma \gamma(s)ds} \frac{d}{d\sigma}\left(-e^{\int_0^\sigma N(s,J^*)ds}\right) d\sigma \\ &= 1 - e^{\int_0^{a_\dagger} (\gamma(s)-N(s,J^*))ds} + \int_0^{a_\dagger} \gamma(\sigma) e^{\int_0^\sigma (\gamma(s)-N(s,J^*))ds} d\sigma. \end{aligned} \tag{3.15}$$

Therefore (3.14) is equivalent to

$$\int_0^{a_\dagger} \gamma(\sigma) e^{\int_0^\sigma (\gamma(s)-N(s,J^*))ds} d\sigma < e^{\int_0^{a_\dagger} (\gamma(s)-N(s,J^*))ds}.$$

(Note that $X^* \neq 1$ by (3.14).) This shows that (3.9) is equivalent to (3.10).

Also note that (3.10) is equivalent to (3.11) by (3.15).

Finally, if we let

$$f(s) = \int_0^s \gamma(\sigma) e^{\int_\sigma^s (N(\tau,J^*)-\gamma(\tau))d\tau} d\sigma,$$

then $f(0) = 0$ and $f(a_\dagger) < 1$ by (3.10). Moreover, by a simple calculation, we have

$$f'(s) = \gamma(s) + \{N(s, J^*) - \gamma(s)\} f(s).$$

Now, either $N(s, J^*) - \gamma(s) \geq 0$ and consequently $f'(s) \geq 0$, or $N(s, J^*) - \gamma(s) < 0$ and, in this case, if $0 \leq f(s) < 1$,

$$f'(s) \geq \gamma(s) + N(s, J^*) - \gamma(s) = N(s, J^*) \geq 0. \tag{3.16}$$

Hence, in both cases, $f(s)$ is a non-decreasing function whenever $0 \leq f(s) < 1$. This is enough to prove (3.12): in fact, $f(0) = 0$ and $f(a_\dagger) < 1$ so that if $f(s) \geq 1$ for some s, then there should be a point $s_0 > s$ such that $f(s_0) < 1$ but $f'(s_0) < 0$. This contradicts (3.16). □

LEMMA 3.2. *Assume* (3.9) *holds. Then all the functions* $Q_1(\cdot)$, $Q_2(\cdot)$, $Q_3(\cdot)$, $Q_4(\cdot)$ *are nonnegative.*

PROOF. We shall prove it only for $Q_1(\cdot)$ and $Q_2(\cdot)$. The proof for the remaining kernels is very similar. From (3.11),

$$Q_1(a) = w(a)\left[1 - \int_0^a N(\sigma, J^*)E(\sigma, J^*)d\sigma\right] \geq 0.$$

Concerning $Q_2(\cdot)$, it suffices to show that

$$E(s-a, J^*) - \int_{s-a}^{s} N(\sigma, J^*)E(\sigma, J^*)d\sigma > 0. \tag{3.17}$$

But

$$\begin{aligned}
&E(s-a, J^*) - \int_{s-a}^{s} N(\sigma, J^*)E(\sigma, J^*)d\sigma \\
=&e^{\int_0^{s-a}[\gamma(\tau)-N(\tau,J^*)]d\tau} - \int_{s-a}^{s} N(\sigma, J^*)e^{\int_0^{\sigma}[\gamma(\tau)-N(\tau,J^*)]d\tau}d\sigma \\
=&e^{\int_0^{s-a}[\gamma(\tau)-N(\tau,J^*)]d\tau} + \int_{s-a}^{s} e^{\int_0^{\sigma}\gamma(\tau)d\tau}\frac{d}{d\sigma}e^{-\int_0^{\sigma}N(\tau)d\tau}d\sigma \\
=&e^{\int_0^{s}[\gamma(\tau)-N(\tau,J^*)]d\tau}\left(1 - \int_{s-a}^{s}\gamma(\sigma)e^{\int_{\sigma}^{s}[N(\tau,J^*)-\gamma(\tau)]d\tau}d\sigma\right),
\end{aligned}$$

and the lemma follows from (3.12). □

LEMMA 3.3. *Assume that* (3.9) *holds. Then* $Q(0) > 0$.

PROOF. We have

(3.18)

$$\begin{aligned}\int_0^\infty Q_1(a)\,da &= \int_0^\infty w(a)\,da - \int_0^\infty w(a)\int_0^a E(\sigma,J^*)N(\sigma,J^*)\,d\sigma da\\ &= R - J^*\int_0^\infty w(a)\int_0^a E(\sigma,J^*)\kappa(\sigma)\,d\sigma da\\ &\quad - \int_0^a w(a)\int_0^a E(\sigma,J^*)g(\sigma)\,d\sigma da\\ &= R - J^*L(J^*) - C(J^*).\end{aligned}$$

Hence, from lemma 3.2, we get

$$0 \le \int_0^\infty Q_1(a)\,da \le R \le 1. \tag{3.19}$$

Similarly we get

$$0 \le \int_0^\infty Q_3(a)\,da = G - J^*M(J^*) - D(J^*) \le G. \tag{3.20}$$

Since we are assuming that $D(\cdot)$ is not zero, we get the following strict inequality from (3.20):

$$J^*M(J^*) < G. \tag{3.21}$$

Also we have

(3.22)

$$\begin{aligned}0 &\le \int_0^\infty Q_2(a)\,da\\ &= \int_0^\infty\int_a^{a_\dagger} w(s)(1-X^*)\kappa(s-a)\\ &\qquad \times \left[E(s-a,J^*) - \int_{s-a}^s N(\sigma,J^*)E(\sigma,J^*)d\sigma\right] ds\,da,\\ &= (1-X^*)\int_0^\infty\int_a^{a_\dagger} w(s)\kappa(s-a)E(s-a,J^*)\,ds\,da\\ &\quad - (1-X^*)\int_0^\infty\int_a^{a_\dagger} w(s)\kappa(s-a)\int_{s-a}^s N(\sigma,J^*)E(\sigma,J^*)\,d\sigma\,ds\,da\end{aligned}$$

$$
\begin{aligned}
&= (1 - X^*) \int_0^{a_\dagger} w(s) \int_0^s \kappa(s-a) E(s-a, J^*)\, da\, ds \\
&\quad - (1 - X^*) \int_0^{a_\dagger} w(s) \int_0^s \kappa(s-a) \int_{s-a}^s N(\sigma, J^*) E(\sigma, J^*)\, d\sigma\, da\, ds \\
&= (1 - X^*) \int_0^{a_\dagger} w(s) \int_0^s \kappa(\sigma) E(\sigma, J^*)\, d\sigma\, ds \\
&\quad - (1 - X^*) \int_0^{a_\dagger} w(s) \int_0^s \kappa(s-a) \int_{s-a}^s N(\sigma, J^*) E(\sigma, J^*)\, d\sigma\, da\, ds \\
&\le (1 - X^*) L(J^*).
\end{aligned}
$$

Similarly,

$$
0 \le \int_0^\infty Q_4(a)\, da \le (1 - X^*) M(J^*). \tag{3.23}
$$

Hence, from the relations (3.18) through (3.23), we get the following:
(3.24)

$$
\begin{aligned}
Q(0) &= \Big(1 - \int_0^\infty Q_1(a)\, da\Big)\Big(1 - \int_0^\infty Q_4(a)\, da\Big) \\
&\quad - \int_0^\infty Q_2(a)\, da \int_0^\infty Q_3(a)\, da \\
&\ge \Big(1 - R + J^* L(J^*) + C(J^*)\Big)\Big(1 - (1 - X^*) M(J^*)\Big) \\
&\quad - (1 - X^*) L(J^*)\Big(G - J^* M(J^*) - D(J^*)\Big) \\
&\ge \Big(1 - R + J^* L(J^*) + C(J^*)\Big)\Big(1 - M(J^*)\Big) \\
&\quad - L(J^*)\Big(G - J^* M(J^*) - D(J^*)\Big) \\
&= \frac{1}{J^*}\Big(J^* + J^{*2} L(J^*) + J^* C(J^*) \\
&\quad - J^* M(J^*) + J^* R M(J^*) - J^* G L(J^*) \\
&\qquad - R J^* + J^* L(J^*) D(J^*) - J^* M(J^*) C(J^*)\Big) \\
&= \frac{1}{J^*}\Big(\phi(J^*) + D(J^*) - R D(J^*) + G C(J^*) \\
&\qquad + J^* L(J^*) D(J^*) - J^* M(J^*) C(J^*)\Big) \\
&= \frac{1}{J^*}\Big((1 - R) D(J^*) + C(J^*)\big(G - J^* M(J^*)\big) + J^* L(J^*) D(J^*)\Big) \\
&> 0.
\end{aligned}
$$

Note that the last inequality follows from (2.32), (2.33) and (3.21). □

Now we are ready for our main theorem:

THEOREM 3.1. *Assume that* (3.9) *holds. Then any endemic equilibrium* (X^*, J^*) *is locally asymptotically stable.*

PROOF. For any λ with $\Re\lambda \geq 0$, (3.19) and lemma 3.3 imply that

$$\begin{aligned} &|Q(\lambda)| \\ &\geq |1-\hat{Q}_1(\lambda)||1-\hat{Q}_4(\lambda)| - |\hat{Q}_2(\lambda)||\hat{Q}_3(\lambda)| \\ &\geq (1-|\hat{Q}_1(\lambda)|)(1-|\hat{Q}_4(\lambda)|) - |\hat{Q}_2(\lambda)||\hat{Q}_3(\lambda)| \\ &\geq \left(1-\int_0^\infty Q_1(a)da\right)\left(1-\int_0^\infty Q_4(a)da\right) \\ &\quad - \int_0^\infty Q_2(a)da \int_0^\infty Q_3(a)da \\ &= Q(0) \\ &> 0. \end{aligned}$$

Thus there is no root λ of the characteristic equation having $\Re\lambda \geq 0$.□

4. Concluding remarks

We have considered an age-structured epidemic model of S-I-R type with external force of infection.

We obtained the stability result which says that the endemic states are locally asymptotically stable for almost all cases. This result is quite different from that of the usual S-I-R model without external force of infection. Note that the existence of external force makes the steady state stable.

We have used (3.9) to get the stability result in our model. In general, we do not know whether the sufficient condition we use in order to prove the local asymptotic stability of endemic equilibria is also necessary.

General results concerning stability of endemic equilibria are still open. As a future work, complete stability analysis might be a reasonable choice.

References

[1] V. Andreassen, *Instability in an SIR-model with age-dependent susceptibility*, Mathematical Population Dynamics: Analysis of Heterogeneity, vol. 1, Theory of Epidemics, Proc. "3rd International Conference on Mathematical Population Dynamics", Pau, France, 1992, Wuerz Publishing Ltd., Winnipeg, Canada, (1995), 3–14.

[2] Y. Cha, *Existence and uniqueness of endemic states for an epidemic model with external force of infection*, Commun. Korean Math. Soc. **17** (2002), no. 1, 175–187.

[3] Y. Cha, M. Iannelli, and F. Milner, *Existence and uniqueness of endemic states for the age-structured S-I-R epidemic model*, Mathematical Biosciences **150** (1998), 177–190.

[4] ———, *Stability change of an epidemic model*, Dynamic Systems and Applications **9** (2000), 361–376.

[5] M. Chipot, M. Iannelli, and A. Pugliese, *Age structured SIR epidemic model with intra-cohort transmission*, Mathematical Population Dynamics: Analysis of Heterogeneity, vol. 1, Theory of Epidemics, Proc. "3rd International Conference on Mathematical Population Dynamics", Pau, France, 1992, Wuerz Publishing Ltd., Winnipeg, Canada, (1995), 52–65.

[6] K. Dietz and D. Schenzle, *Proportionate mixing models for age-dependent infection transmission*, J. Math. Biol. **22** (1985), 117–120.

[7] M. Iannelli, *Mathematical Theory of Age-Structured Population Dynamics*, Applied Mathematics Monographs, Giardini Editori, Pisa. (1995).

[8] H. Inaba, *Threshold and stability results for an age-structured epidemic model*, J. Math. Biol. **28** (1990), 411–434.

[9] H. Thieme, *Stability change for the endemic equilibrium in age-structured models for the spread of S-I-R type infectious diseases*, Differential Equations Models in Biology, Epidemiology and Ecology, Lectures Notes in Biomathematics **92**, Springer Verlag, (1991), 139–158.

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