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DEFORMATION SPACES OF 3-DIMENSIONAL FLAT MANIFOLDS

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ABSTRACT. The deformation spaces of the six orientable 3-dimensional flat Riemannian manifolds are studies. It is proved that the Teichmüller spaces are homeomorphic to the Euclidean spaces. To state more precisely, let Φ denote the holonomy group of the manifold. Then the Teichmüller space is homeomorphic to (1) $\mathbb{R}^6$ if Φ is trivial, (2) $\mathbb{R}^4$ if Φ is cyclic with order two, (3) $\mathbb{R}^2$ if Φ is cyclic of order 3, 4 or 6, and (4) $\mathbb{R}^3$ if $\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

1. Preliminaries

Let $\mathrm{Isom}(\mathbb{R}^n)$ denote the group of isometries of the Euclidean space $\mathbb{R}^n$. So,

$$\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{O}(n),$$

where $\mathrm{O}(n)$ is the n-dimensional orthogonal group. $\mathrm{Isom}(\mathbb{R}^n)$ is a subgroup of the affine group

$$\mathcal{A}(n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R}).$$

A subgroup Π of $\mathrm{Isom}(\mathbb{R}^n)$ is said to be a *crystallographic* group if Π is cocompact and discrete. A torsion free crystallographic group is called a *Bieberbach group.* If Π is a Bieberbach group of dimension n, then the quotient space $\mathbb{R}^n/\Pi$ is a Riemannian manifold of sectional curvature $\kappa = 0$. Conversely, a flat closed Riemannian manifold of dimension n is necessarily a quotient space of $\mathbb{R}^n$ by a Bieberbach group of dimension n, see [4].

In this paper, we focus on the 3-dimensional manifolds which are closed and Riemannian flat. We use the notation $\mathcal{I}$ for the isometry

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group $\mathrm{Isom}(\mathbb{R}^3)$. So,

$$\mathcal{I} = \mathrm{Isom}(\mathbb{R}^3) = \mathbb{R}^3 \rtimes \mathrm{O}(3).$$

The following Bieberbach's second theorem is crucial for us. See [3] or [4].

THEOREM 1.1 (Bieberbach). *Any isomorphism between two crystallographic groups is a conjugation by an element of the affine group.*

There are only 10 affine diffeomorphism classes of connected closed 3-dimensional flat manifolds, six of which are orientable and the others are not. A Bieberbach group Π contains a unique maximal normal abelian subgroup $\mathbb{Z}^3$, fitting the following commutative diagram of groups with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle\theta} & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \rtimes \mathrm{O}(3) & \longrightarrow & \mathrm{O}(3) & \longrightarrow & 1, \end{array}$$

where Φ is called the *holonomy group* of Π. It is a finite group and $\Phi \to \mathrm{O}(3)$ is injective.

We shall investigate the various deformation spaces associated with closed 3-dimensional flat Riemannian manifolds. Firstly, the space of discrete representations, *the Weil space*, is defined as follows:

$$\begin{aligned} \mathcal{R}(\Pi;\mathcal{I}) = {} & \text{the space of all injective homomorphisms } \theta \text{ of } \Pi \text{ into } \mathcal{I} \\ & \text{such that } \theta(\Pi) \text{ is discrete in } \mathcal{I} \text{ and } \mathcal{I}/\theta(\Pi) \text{ is compact.} \end{aligned}$$

If Π is a Bieberbach group, every element of $\mathcal{R}(\Pi;\mathcal{I})$ gives rise to a flat Riemannian manifold. Let $\theta, \theta' \in \mathcal{R}(\Pi;\mathcal{I})$. If an affine map $f : \mathbb{R}^3 \to \mathbb{R}^3$ conjugates $\theta(\Pi)$ into $\theta'(\Pi)$, then it induces an affine diffeomorphism from $\mathbb{R}^3/\theta(\Pi)$ to $\mathbb{R}^3/\theta'(\Pi)$. For $g \in \mathcal{I}$, $\mu(g)$ denotes the conjugation by g. We denote the group of inner automorphisms of $\mathcal{I}$ by $\mathbf{Inn}(\mathcal{I})$. This group acts on the space $\mathcal{R}(\Pi;\mathcal{I})$ from the left by

$$\begin{gathered} \mathbf{Inn}(\mathcal{I}) \times \mathcal{R}(\Pi;\mathcal{I}) \to \mathcal{R}(\Pi;\mathcal{I}) \\ (\mu(g), \theta) \longmapsto \mu(g) \circ \theta. \end{gathered}$$

The orbit space of this action is called the *Teichmüller space*. That is,

$$\mathcal{T}(\Pi;\mathcal{I}) = \mathbf{Inn}(\mathcal{I}) \backslash \mathcal{R}(\Pi;\mathcal{I}).$$

If $\theta, \theta' \in \mathcal{R}(\Pi;\mathcal{I})$ represent the same point in $\mathcal{T}(\Pi;\mathcal{I})$, then $\theta' = \mu(g) \circ \theta$ for some $g \in \mathcal{I}$. This implies

$$g \circ \theta(\alpha) = \theta'(\alpha) \circ g$$

for all $\alpha \in \Pi$. Then, g induces a map

$$\theta(\Pi)\backslash\mathbb{R}^3 \longrightarrow \theta'(\Pi)\backslash\mathbb{R}^3$$

which is an isometry.

2. Spaces of discrete representation

The next theorem says that there are only six 3-dimensional orientable flat manifolds.

THEOREM 2.1 ([4]). *There are just 6 affine diffeomorphism classes of compact connected orientable flat 3-dimensional Riemannian manifolds. They are represented by the manifolds $\mathbb{R}^3/\Pi$ where Π is one of the 6 groups given below. Here $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ and $\boldsymbol{t}_3$ are translations by $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ respectively and $\Phi = \Pi/\mathbb{Z}^3$ is the holonomy.*

(1) $\Phi = \{1\}$. *Π is generated by the translations $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3\}$ with $\{\mathbf{a}_i\}$ linearly independent.*

(2) $\Phi = \mathbb{Z}_2$. *Π is generated by $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3, \alpha\}$ where $\alpha^2 = \boldsymbol{t}_1, \alpha\boldsymbol{t}_2\alpha^{-1} = \boldsymbol{t}_2^{-1}$ and $\alpha\boldsymbol{t}_3\alpha^{-1} = \boldsymbol{t}_3^{-1}$, $\mathbf{a}_1$ is orthogonal to $\mathbf{a}_2$ and $\mathbf{a}_3$ while $\alpha = (\boldsymbol{t}_{\mathbf{a}_1/2}, A)$ with $A(\mathbf{a}_1) = \mathbf{a}_1, A(\mathbf{a}_2) = -\mathbf{a}_2, A(\mathbf{a}_3) = -\mathbf{a}_3$.*

(3) $\Phi = \mathbb{Z}_3$. *Π is generated by $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3, \alpha\}$ where $\alpha^3 = \boldsymbol{t}_1, \alpha\boldsymbol{t}_2\alpha^{-1} = \boldsymbol{t}_3$ and $\alpha\boldsymbol{t}_3\alpha^{-1} = \boldsymbol{t}_2^{-1}\boldsymbol{t}_3^{-1}$; $\mathbf{a}_1$ is orthogonal to $\mathbf{a}_2$ and $\mathbf{a}_3$, $\|\mathbf{a}_2\| = \|\mathbf{a}_3\|$ and $\{\mathbf{a}_2, \mathbf{a}_3\}$ is a hexagonal plane lattice, and $\alpha = (\boldsymbol{t}_{\mathbf{a}_1/3}, A)$ with $A(\mathbf{a}_1) = \mathbf{a}_1, A(\mathbf{a}_2) = \mathbf{a}_3$ and $A(\mathbf{a}_3) = -\mathbf{a}_2 - \mathbf{a}_3$.*

(4) $\Phi = \mathbb{Z}_4$. *Π is generated by $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3, \alpha\}$ where $\alpha^4 = \boldsymbol{t}_1, \alpha\boldsymbol{t}_2\alpha^{-1} = \boldsymbol{t}_3$ and $\alpha\boldsymbol{t}_3\alpha^{-1} = \boldsymbol{t}_2^{-1}$; $\{\mathbf{a}_i\}$ are mutually orthogonal with $\|\mathbf{a}_2\| = \|\mathbf{a}_3\|$ while $\alpha = (\boldsymbol{t}_{\mathbf{a}_1/4}, A)$ with $A(\mathbf{a}_1) = \mathbf{a}_1, A(\mathbf{a}_2) = \mathbf{a}_3$ and $A(\mathbf{a}_3) = -\mathbf{a}_2$.*

(5) $\Phi = \mathbb{Z}_6$. *Π is generated by $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3, \alpha\}$ where $\alpha^6 = \boldsymbol{t}_1, \alpha\boldsymbol{t}_2\alpha^{-1} = \boldsymbol{t}_3$ and $\alpha\boldsymbol{t}_3\alpha^{-1} = \boldsymbol{t}_2^{-1}\boldsymbol{t}_3$; $\mathbf{a}_1$ is orthogonal to $\mathbf{a}_2$ and $\mathbf{a}_3$, $\|\mathbf{a}_2\| = \|\mathbf{a}_3\|$ and $\{\mathbf{a}_2, \mathbf{a}_3\}$ is a hexagonal plane lattice, and $\alpha = (\boldsymbol{t}_{\mathbf{a}_1/6}, A)$ with $A(\mathbf{a}_1) = \mathbf{a}_1, A(\mathbf{a}_2) = \mathbf{a}_3$ and $A(\mathbf{a}_3) = \mathbf{a}_3 - \mathbf{a}_2$.*

(6) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ *and Π is generated by $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3, \alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_3\alpha_2\alpha_1 = \boldsymbol{t}_1\boldsymbol{t}_3$ and for $i = 1, 2, 3$*

$$\alpha_i^2 = \boldsymbol{t}_i \quad \text{and} \quad \alpha_i\boldsymbol{t}_j\alpha_i^{-1} = \boldsymbol{t}_j^{-1}(i \neq j).$$

The $\{\mathbf{a}_i\}$ are mutually orthogonal and

$\alpha_1 = (A_1, \boldsymbol{t}_{\mathbf{a}_1/2})$ *with* $A_1(\mathbf{a}_1) = \mathbf{a}_1, A_1(\mathbf{a}_2) = -\mathbf{a}_2, A_1(\mathbf{a}_3) = -\mathbf{a}_3$;

$\alpha_2 = (A_2, \boldsymbol{t}_{(\mathbf{a}_2+\mathbf{a}_3)/2})$ *with* $A_2(\mathbf{a}_1) = -\mathbf{a}_1, A_2(\mathbf{a}_2) = \mathbf{a}_2, A_2(\mathbf{a}_3) = -\mathbf{a}_3$;

$\alpha_3 = (A_3, \boldsymbol{t}_{(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)/2})$ *with* $A_3(\mathbf{a}_1) = -\mathbf{a}_1, A_3(\mathbf{a}_2) = -\mathbf{a}_2, A_3(\mathbf{a}_3) = \mathbf{a}_3$.

We need some notation. Let X be a 3×3 matrix whose column vectors are the $\mathbf{x}_i$ $(i = 1, 2, 3)$. Then $X^T X$ is symmetric of which the (i, j) entry is the inner product $\langle \mathbf{x}_i, \mathbf{x}_j \rangle$ of two vectors $\mathbf{x}_i$ and $\mathbf{x}_j$. For subgroups $\mathrm{H}_1, \mathrm{H}_2$ of G, we denote

$$\mathrm{H}_1 \cdot \mathrm{H}_2 = \{h_1 \cdot h_2 \mid h_1 \in \mathrm{H}_1, h_2 \in \mathrm{H}_2\}.$$

Note that $\mathrm{H}_1 \cdot \mathrm{H}_2$ is not a subgroup but a subspace of G. Of course H_1 and H_2 may have a nontrivial subgroup in common.

Consider a 3×3 non-singular matrix A. Let $\mathcal{X}(A)$ be the space consisting of 3×3 invertible matrices by which the conjugates of A are orthogonal. That is

$$\begin{aligned}\mathcal{X}(A) &= \{X \in \mathrm{GL}(3, \mathbb{R}) \mid XAX^{-1} \in \mathrm{O}(3)\} \\ &= \{X \in \mathrm{GL}(3, \mathbb{R}) \mid (X^T X)A = A(X^T X)\}.\end{aligned}$$

It is easy to see that $\mathcal{X}(P^{-1}AP) = \mathcal{X}(A) \cdot P$.

THEOREM 2.2. *Let $\Pi \subset \mathbb{R}^3 \rtimes \mathrm{SO}(3)$ be a 3-dimensional Bieberbach group with holonomy group $\Phi \subset \mathrm{SO}(3)$. Let*

1. *$\mathcal{X}(\Phi) = \{X \in \mathrm{GL}(3, \mathbb{R}) \mid XAX^{-1} \in \mathrm{O}(3)$ for all $A \in \Phi\}$, and*
2. *$(\mathbb{R}^3)^\Phi$ be the fixed point set of the Φ action on $\mathbb{R}^3$.*

Then $\mathcal{R}(\Pi; \mathcal{I}) = \mathbb{R}^3 \rtimes \mathcal{X}(\Phi)/(\mathbb{R}^3)^\Phi$.

PROOF. Let $\theta_0 : \Pi \hookrightarrow \mathcal{I}$ be the embedding given in Theorem 2.1, and let $\theta \in \mathcal{R}(\Pi; \mathcal{I})$. By Theorem 1.1, they are conjugate by an affine motion. That is, there exists an element $\xi = (\mathbf{x}, X) \in \mathrm{Aff}(3) = \mathbb{R}^3 \rtimes \mathrm{GL}(3, \mathbb{R})$ such that $\theta(\Pi) = \xi \cdot \theta_0(\Pi) \cdot \xi^{-1}$. So we need to find all elements $\xi \in \mathrm{Aff}(3)$ which conjugates $\theta_0(\Pi)$ into $\mathcal{I}$.

Note that the fact $\xi \cdot \theta_0(\Pi) \cdot \xi^{-1} \subset \mathcal{I}$ depends only on the matrix part of ξ. That is $\xi \cdot \theta_0(\Pi) \cdot \xi^{-1} \subset \mathcal{I}$ if and only if $X \Phi X^{-1} \subset \mathrm{O}(3)$, or equivalently, $X \in \mathcal{X}(\Phi)$. Observe that $\mathcal{X}(\Phi)$ is not a subgroup in general, but is a nice algebraic sub-variety of $\mathrm{GL}(3, \mathbb{R})$. Thus the space of all $\xi \in \mathrm{Aff}(3)$ which conjugates $\theta_0(\Pi)$ into $\mathcal{I}$ is

$$\mathbb{R}^3 \rtimes \mathcal{X}(\Phi).$$

Suppose now two elements $(\mathbf{d}_1, D_1), (\mathbf{d}_2, D_2) \in \mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$ yield the same representation. We must have

$$(\mathbf{d}_1, D_1)(\mathbf{x}, X)(\mathbf{d}_1, D_1)^{-1} = (\mathbf{d}_2, D_2)(\mathbf{x}, X)(\mathbf{d}_2, D_2)^{-1}$$

for all $(\mathbf{x}, X) \in \Pi$. Since Π contains a lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ (in fact, $\Pi/\mathbb{Z}^3 = \Phi$), the above equality for $(\mathbf{x}, X) = (\mathbf{z}, I)$ for all $\mathbf{z} \in \mathbb{Z}^3$ ensures that

$D_1 = D_2 (= D)$. Now let us let

$$(\mathbf{d}_2, D) = (\mathbf{d}_1, D)(\mathbf{c}, I).$$

Then

$$\begin{aligned}(\mathbf{d}_1, D)(\mathbf{x}, X)(\mathbf{d}_1, D)^{-1} &= (\mathbf{d}_2, D)(\mathbf{x}, X)(\mathbf{d}_2, D)^{-1} \\ &= ((\mathbf{d}_1, D)(\mathbf{c}, I))(\mathbf{x}, X)((\mathbf{d}_1, D_1)(\mathbf{c}, I))^{-1} \\ &= (\mathbf{d}_1, D)\Big((\mathbf{c}, I)(\mathbf{x}, X)(\mathbf{c}, I)^{-1}\Big)(\mathbf{d}_1, D)^{-1}.\end{aligned}$$

Therefore,

$$(\mathbf{c}, I)(\mathbf{x}, X)(\mathbf{c}, I)^{-1} = (\mathbf{x}, X)$$

for all $(\mathbf{x}, X) \in \Pi$. This readily implies that $X\mathbf{c} = \mathbf{c}$. Thus, $\mathbf{c} \in (\mathbb{R}^3)^\Phi$, the subspace of $\mathbb{R}^3$ which is fixed by every element of Φ. Note that this is the centralizer $\mathcal{C}_{\mathrm{Aff}(\mathbb{R}^3)}(\theta_0(\Pi))$ of $\theta_0(\Pi)$ in the group Aff(3).

We can summarize this as follows: The group $\mathcal{C} = (\mathbb{R}^3)^\Phi$ acts on $\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$ on the right by

$$(\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)) \times \mathcal{C} \to \mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$$

$$((\mathbf{x}, X), (\mathbf{c}, I)) \longmapsto (\mathbf{x}, X) \cdot (\mathbf{c}, I) = (\mathbf{x} + X\mathbf{c}, X).$$

Note that $(\mathbf{x}, X)$ and $(\mathbf{x}, X) \cdot (\mathbf{c}, I)$ yield the same representation. More precisely, for any $u \in \theta_0(\Pi)$,

$$\begin{aligned}\mu((\mathbf{x}, X) \cdot (\mathbf{c}, I))(u) &= \mu(\mathbf{x}, X)(\mu(\mathbf{c}, I)(u)) \\ &= \mu(\mathbf{x}, X)(u)\end{aligned}$$

because $(\mathbf{c}, I)$ centralizes $\theta_0(\Pi)$. The space of representations is thus the orbit space of the action of $(\mathbb{R}^3)^\Phi$ on $\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$; that is, $\mathcal{R}(\Pi; \mathcal{I}) = \mathbb{R}^3 \rtimes \mathcal{X}(\Phi)/(\mathbb{R}^3)^\Phi$. □

PROPOSITION 2.3. *Let $\Pi \subset \mathbb{R}^3 \rtimes \mathrm{SO}(3)$ be a 3-dimensional Bieberbach group with holonomy group $\Phi \subset \mathrm{SO}(3)$. Then, with respect to the representation given in Theorem 2.1,*

1. *If Φ is trivial, then $\mathcal{X}(\Phi) = \mathrm{GL}(3, \mathbb{R})$ and $(\mathbb{R}^3)^\Phi = \mathbb{R}^3$.*
2. *If $\Phi = \mathbb{Z}_2$, then $\mathcal{X}(\Phi) = \mathrm{O}(3) \cdot (\mathbb{R}^* \times \mathrm{GL}(2, \mathbb{R}))$ and $(\mathbb{R}^3)^\Phi = \mathbb{R}\boldsymbol{e}_1 \cong \mathbb{R}^1$.*
3. *If $\Phi = \mathbb{Z}_3$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$, then $\mathcal{X}(\Phi) = \mathrm{O}(3) \cdot (\mathbb{R}^* \times (\mathbb{R}^+ \times \mathrm{O}(2)))$ and $(\mathbb{R}^3)^\Phi = \mathbb{R}\boldsymbol{e}_1 \cong \mathbb{R}^1$.*
4. *If $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathcal{X}(\Phi) = \mathrm{O}(3) \cdot (\mathbb{R}^*)^3)$ and $(\mathbb{R}^3)^\Phi = 0$.*

PROOF. (1) Case of $\Phi = \{1\}$: Obvious.

(2) Case of $\Phi = \mathbb{Z}_2$: The holonomy group $\mathbb{Z}_2$ is generated by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. We have to find the matrix X such that XAX^{-1} is orthogonal, which is equivalent to $(X^TX)A = A(X^TX)$. This implies that two inner products $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$ and $\langle \mathbf{x}_1, \mathbf{x}_3 \rangle$ must be zero. Hence,

$$\begin{aligned} \mathcal{X}(\Phi) &= \{X = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] \in \mathrm{GL}(3, \mathbb{R}) \ \mid \mathbf{x}_1 \perp \mathbf{x}_2 \text{ and } \mathbf{x}_1 \perp \mathbf{x}_3\} \\ &= \mathrm{O}(3) \cdot \left\{ \begin{bmatrix} a & O \\ O & B \end{bmatrix} \mid a \in \mathbb{R}^* \text{ and } B \in \mathrm{GL}(2, \mathbb{R}) \right\} \\ &= \mathrm{O}(3) \cdot (\mathbb{R}^* \times \mathrm{GL}(2, \mathbb{R})), \end{aligned}$$

where $\mathbb{R}^*$ means the set of all non-zero real numbers. Note that a 3-dimensional space $\mathrm{O}(3)$ and a 5-dimensional space $\mathbb{R}^* \times \mathrm{GL}(2, \mathbb{R})$ intersects the common space $\mathbb{Z}_2 \times \mathrm{O}(2)$ which is 1-dimensional, and $\mathcal{X}(\Phi)$ has 4-components. And so $\mathrm{O}(3) \cdot (\mathbb{R}^* \times \mathrm{GL}(2, \mathbb{R}))$ is 7-dimensional. $(\mathbb{R}^3)^\Phi = \mathbb{R}$ consists of the first axis.

(3) Cases of $\Phi = \mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$: Denote

$$\rho(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

the rotation matrix of rotation of $\mathbb{R}^2$ about the origin through θ. Now we can take embeddings $\theta_0 : \Pi \to \mathcal{I}$ as follows : Let $\Phi = \mathbb{Z}_n$ (for $n = 3, 4$ or 6). Then

$$\begin{aligned} \theta_0(\mathbf{t}_1) &= (\mathbf{e}_1, I) \\ \theta_0(\mathbf{t}_2) &= (A\mathbf{e}_2, I) \\ \theta_0(\mathbf{t}_3) &= (A^2\mathbf{e}_2, I) \\ \theta_0(\alpha) &= (\frac{1}{n}\ \mathbf{e}_1, A), \end{aligned}$$

where $A = \left[\begin{array}{c|c} 1 & \\ \hline & \rho(\frac{2\pi}{n}) \end{array}\right]$. The defining condition $X \in \mathcal{X}(\Phi)$ is $X\rho(\theta)X^{-1} \in \mathrm{O}(3)$. Clearly $(X\rho(\theta)X^{-1})^T(X\rho(\theta)X^{-1}) = I$ if and only if $\rho(-\theta)X^T X\rho(\theta) = X^TX$ i.e., $(X^TX)\rho(\theta) = \rho(\theta)(X^TX)$. The most general X^TX is of the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix},$$

where a and b are nonzero real numbers. So the nonzero column vectors of X satisfies that $\mathbf{x}_i \perp \mathbf{x}_j (i \neq j)$ and $\|\mathbf{x}_2\| = \|\mathbf{x}_3\|$. This implies that

$$\mathcal{X}(\Phi) = \mathrm{O}(3) \cdot (\mathbb{R}^* \times (\mathbb{R}^+ \times \mathrm{O}(2))).$$

The 3-dimensional spaces $\mathrm{O}(3)$ and $\mathbb{R}^* \times (\mathbb{R}^+ \times \mathrm{O}(2))$ have intersection $\mathbb{Z}_2 \times \mathrm{O}(2)$ which is 1-dimensional, and the product has 4-components. Hence $\mathcal{X}(\Phi)$ is 5-dimensional. As in the case of $\Phi = \mathbb{Z}_2$, the centralizer $(\mathbb{R}^3)^\Phi = \{(\mathbf{x}, I) | \mathbf{x} = [*, 0, 0]^T\} \approx \mathbb{R}\mathbf{e}_1$.

(4) Case of $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: The holonomy group $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. We look for the space $\mathcal{X}(\Phi) = \{X \in \mathrm{GL}(3, \mathbb{R}) \mid XA_iX^{-1} \text{ is orthogonal for } i = 1, 2\}$. XA_iX^{-1} is orthogonal for $1 \leq i \leq 2$ if and only if X^TX is diagonal. Hence

$$\begin{aligned} \mathcal{X}(\Phi) &= \{X \in \mathrm{GL}(3, \mathbb{R}) \mid \mathbf{x}_i \perp \mathbf{x}_j \text{ if } i \neq j\} \\ &= \mathrm{O}(3) \cdot (\mathbb{R}^*)^3, \end{aligned}$$

where $(\mathbb{R}^*)^3$ is the diagonal matrices of non-zero determinant. The 3-dimensional spaces $\mathrm{O}(3)$ and $(\mathbb{R}^*)^3$ have intersection $(\mathbb{Z}_2)^3$, consisting of all diagonal matrices with entries ± 1. This space is 0-dimensional. And so $\mathcal{X}(\Phi)$ is 6-dimensional. Clearly, the centralizer $(\mathbb{R}^3)^\Phi$ is trivial. □

COROLLARY 2.4. *Let M be a 3-dimensional orientable flat manifold with $\pi_1(M) = \Pi$. Then*

1. *If Φ is trivial, then $\mathcal{R}(\Pi; \mathcal{I}) = \mathrm{GL}(3, \mathbb{R})$, and so it is a 9-dimensional space.*
2. *If $\Phi = \mathbb{Z}_2$, then $\mathcal{R}(\Pi; \mathcal{I}) = \mathbb{R}^3 \rtimes (\mathrm{O}(3) \cdot (\mathbb{R}^* \times \mathrm{GL}(2, \mathbb{R})))/\mathbb{R} \rtimes \{\mathrm{I}\}$, and so it is a 9-dimensional space.*
3. *If $\Phi = \mathbb{Z}_3, \mathbb{Z}_4$ or $\mathbb{Z}_6$, then $\mathcal{R}(\Pi; \mathcal{I}) = \mathbb{R}^3 \rtimes (\mathrm{O}(3) \cdot (\mathbb{R}^* \times (\mathbb{R}^+ \times \mathrm{O}(2)))/\mathbb{R} \rtimes \{\mathrm{I}\}$, and so it is a 7-dimensional space.*
4. *If $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathcal{R}(\Pi; \mathcal{I}) = \mathbb{R}^3 \rtimes (\mathrm{O}(3) \cdot (\mathbb{R}^*)^3)$ and so it is a 9-dimensional space.*

REMARK 2.5. In the cases of (2) and (3)of Corollary 2.4, note that the (right) action of $\mathbb{R} \rtimes \{I\} \cong \mathbb{R}$ on $\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$ is twisted. In other words, one cannot write the orbit space as $(\mathbb{R}^3 \rtimes \mathcal{X}(\Phi))/\mathbb{R} \rtimes \{I\} \approx \mathbb{R}^2 \rtimes \mathcal{X}(\Phi)$. This is because

$$(\mathbf{x}, X) \cdot (\mathbf{c}, I) = (\mathbf{x} + X\mathbf{c}, X)$$

which is different from $(\mathbf{x}+\mathbf{c}, X)$. However the action is free and proper so that the orbit space is a manifold.

3. Teichmüller spaces

THEOREM 3.1. *Let M be a 3-dimensional orientable flat manifold with $\Pi(M) = \Pi$. Then the Teichmüller spaces are as follow:*

1. *If Φ is trivial, then $\mathcal{T}(\Pi;\mathcal{I}) = \mathrm{O}(3)\backslash\mathrm{GL}(3,\mathbb{R}) \approx \mathbb{R}^6$.*
2. *If $\Phi = \mathbb{Z}_2$, then $\mathcal{T}(\Pi;\mathcal{I}) = \mathbb{R}^+ \times (\mathrm{O}(2)\backslash\mathrm{GL}(2,\mathbb{R})) \approx \mathbb{R}^+ \times \mathbb{R}^3 \approx \mathbb{R}^4$.*
3. *If $\Phi = \mathbb{Z}_3, \mathbb{Z}_4$ or $\mathbb{Z}_6$, then $\mathcal{T}(\Pi;\mathcal{I}) = (\mathbb{R}^+)^2 \approx \mathbb{R}^2$.*
4. *If $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathcal{T}(\Pi;\mathcal{I}) = (\mathbb{R}^*)^3/(\mathbb{Z}_2)^3 = (\mathbb{R}^+)^3 \approx \mathbb{R}^3$.*

PROOF. The group of isometries $\mathcal{I} = \mathbb{R}^3 \rtimes \mathrm{O}(3)$ acts on $\mathcal{R}(\Pi;\mathcal{I})$ on the left by conjugation, and the orbit space is the Teichmüller space of Π. On the space $\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$-level, this action is just a multiplication from the left. On the other hand, we had an action of $(\mathbb{R}^3)^\Phi$ by right multiplication to get the orbit space $\mathcal{R}(\Pi;\mathcal{I})$. Clearly, these two action of $\mathcal{I}$ and $(\mathbb{R}^3)^\Phi$ on $\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$ commute with each other. Furthermore, from $\mathbb{R}^3 \rtimes \{I\} \subset \mathbb{R}^3 \rtimes \mathrm{O}(3)$, every orbit must contain whole $\mathbb{R}^3$. Therefore,

$$\begin{aligned}\mathcal{T}(\Pi;\mathcal{I}) &= \mathcal{I}\backslash(\mathbb{R}^3 \rtimes \mathcal{X}(\Phi))/(\mathbb{R}^3)^\Phi \\ &= \mathrm{O}(3)\backslash(\mathbb{R}^3 \rtimes \{I\}\backslash\mathbb{R}^3 \rtimes \mathcal{X}(\Phi))/(\mathbb{R}^3)^\Phi \\ &= \mathrm{O}(3)\backslash\mathcal{X}(\Phi).\end{aligned}$$

Also recall that $\mathrm{O}(n)$ is a maximal compact subgroup of $\mathrm{GL}(n,\mathbb{R})$, and $\mathrm{O}(n)\backslash\mathrm{GL}(n,\mathbb{R}) \approx \mathbb{R}^{\frac{n(n+1)}{2}}$. □

REMARK 3.2. Here we studied the deformation spaces of groups $\Pi \subset \mathcal{I}$. Since all of our groups lie in $\mathcal{I}_0 = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$ (the connected component of the identity element, which is the orientation-preserving isometries), one may want to understand the deformation spaces with respect to $\mathcal{I}_0$ (instead of $\mathcal{I}$).

REMARK 3.3. For a Bieberbach group Π, we have worked with a specific representation $\theta_0 : \Pi \longrightarrow \mathcal{I}$ to calculate the Teichmüller space. What will happen to these deformation spaces if we use a different representation? Let $\theta_1 : \Pi \longrightarrow \mathcal{I}$ be another discrete cocompact representation. Then, there exists $\sigma = (\mathbf{d}, D) \in \mathrm{Aff}(3)$ such that

$$\theta_1(\mathbf{a}, A) = (\mathbf{d}, D)\theta_0(\mathbf{c}, A)(\mathbf{d}, D)^{-1}$$

for every $(\mathbf{a}, A) \in \Pi$. For simplicity, let us denote $\theta_0(\Pi)$ and $\theta_1(\Pi)$ by Π and Π' respectively. Also their holonomies are denoted by Φ and Φ'. Then, clearly,

$$\mathcal{X}(\Phi') = \mathcal{X}(D\Phi D^{-1}) = \mathcal{X}(\Phi)D^{-1}$$

so that

$$\mathbb{R}^3 \rtimes \mathcal{X}(\Phi') \;=\; \mathbb{R}^3 \rtimes \mathcal{X}(\Phi) \cdot D^{-1} \;=\; (\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)) \cdot (\mathbf{d}, D)^{-1}.$$

The left actions of $\mathcal{I}$ do not change. The normalizers are

$$N(\Pi') = (\mathbf{d}, D) \cdot N(\Pi) \cdot (\mathbf{d}, D)^{-1}.$$

In particular, the centralizers are

$$(\mathbb{R}^3)^{\Phi'} \times \{I\} = (\mathbf{d}, D) \cdot (\mathbb{R}^3)^{\Phi'} \times \{I\} \cdot (\mathbf{d}, D)^{-1} = D((\mathbb{R}^3)^{\Phi}) \rtimes \{I\}.$$

The diagram

$$\begin{array}{ccc}
\Big(\mathbb{R}^3 \rtimes \mathcal{X}(\Phi),\ N(\Phi)\Big) & \longrightarrow & \mathbb{R}^3 \rtimes \mathcal{X}(\Phi) \\
\downarrow & & \downarrow \\
\Big(\mathbb{R}^3 \rtimes \mathcal{X}(\Phi'),\ N(\Phi')\Big) & \longrightarrow & \mathbb{R}^3 \rtimes \mathcal{X}(\Phi')
\end{array}$$

which is given by

$$\begin{array}{ccc}
((\mathbf{b}, B), (\mathbf{n}, N)) & \longrightarrow & (\mathbf{b}, B) \cdot (\mathbf{n}, N) \\
\downarrow & & \downarrow \\
((\mathbf{b}, B)\sigma^{-1}, \sigma(\mathbf{n}, N)\sigma^{-1}) & \longrightarrow & ((\mathbf{b}, B) \cdot (\mathbf{n}, N))\sigma^{-1}
\end{array}$$

is commutative. This shows that the deformations spaces are precisely the right translates of the original deformation spaces by $(\mathbf{d}, D)^{-1}$.

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