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ON THE MINIMAL ENERGY SOLUTION IN A QUASILINEAR ELLIPTIC EQUATION

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Abstract. In this paper we seek a positive, radially symmetric and energy minimizing solution of an m-Laplacian equation, $-div(|\nabla u|^{m-2}|\nabla u) = h(u)$. In the variational sense, the solutions are the critical points of the associated functional called the energy, $J(v) = \frac{1}{m}\int_{\mathbf{R}^N}|\nabla v|^m - \int_{\mathbf{R}^N} H(v)dx$, where $H(v) = \int_0^v h(t)dt$. A positive, radially symmetric critical point of J can be obtained by solving the constrained minimization problem; minimize$\{\int_{\mathbf{R}^N}|\nabla u|^m dx \mid \int_{\mathbf{R}^N} H(u)dx = 1\}$. Moreover, the solution minimizes $J(v)$.

1. Introduction

In this paper we are concerned with the existence of positive, radially symmetric, nonincreasing solution of a quasilinear elliptic equation

$$-div(|\nabla u|^{m-2}|\nabla u) = h(u) \qquad \text{in } \mathbf{R}^N, \tag{I}$$

where $1 < m < N$ and the continuous function h satisfy the following conditions:

(h_1) $h(u) = 0$ for $u \le 0$
(h_2) $\lim_{u\to 0} h(u)/u^{m-1} = -\mu < 0$
(h_3) $\lim_{u\to\infty} h(u)/u^{m^*-1} = 0, \qquad m^* = \frac{mN}{N-m}$
(h_4) There exists $\xi > 0$ such that $\int_0^\xi h(u) > 0$.

This problem is m-Laplacian form related with the following elliptic partial differential equation

$$\begin{cases} -\Delta u = g(u) & \text{in } \mathbf{R}^N \\ u > 0. \end{cases} \tag{1}$$

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The problem (1) was investigated by Beristycky and Lion [1] under the following growth conditions imposed on g;

(g_1) $-\infty < \underline{\lim}_{t\to 0^+} g(t)/t \le \overline{\lim}_{t\to 0^+} g(t)/t = -m < 0$
(g_2) $-\infty \le \overline{\lim}_{t\to +\infty} g(t)/t^l = -m < 0$, where $l = \frac{N+2}{N-2}$,
(g_3) there exists $\zeta > 0$ such that $G(\zeta) = \int_0^\zeta g(t)dt > 0$.

They showed that (1) has a positive, radial C^2 solution decreasing with exponential decay rate. Equations of type (1) have been studied by a number of mathematicians, e.g., Dancer [3], Strauss [7] and Coleman, Glazer, Martin [2].

A function $u \in W^{1,m}(\mathbf{R}^N)$ is called a generalized solution of (I) if it satisfies, for all $v \in C_0^\infty(\mathbf{R}^N)$:

$$\int_{\mathbf{R}^N} |\nabla u|^{m-2} \nabla u \nabla v - \int_{\mathbf{R}^N} h(u) v dx = 0.$$

Also, note that each generalized solution u is a critical point of the functional J on $W^{1,m}(\mathbf{R}^N)$ defined by

$$J(v) = \frac{1}{m} \int_{\mathbf{R}^N} |\nabla v|^m - \int_{\mathbf{R}^N} H(v) dx,$$

where $H(v) = \int_0^v h(t)dt$. In relation to (I), our main result is as follows:

THEOREM 1.1. *Under the hypotheses* (h_1)-(h_4), (I) *possesses a solution* u *such that*

(i) $u > 0$ *on* $\mathbf{R}^N$,
(ii) u *is spherically symmetric and* u *decreases with respect to* $r = |x|$,
(iii) u *minimizes the energy* J, *that is,* $J(v) \ge J(u)$ *for any nontrivial* v *solving* (I).

2. Proof of Theorem 1.1

Our main idea for Theorem 1.1 is the constrained minimization method. That is, the problem;

$$\text{minimize} \left\{ \int_{\mathbf{R}^N} |\nabla u|^m dx \middle| \int_{\mathbf{R}^N} H(u) dx = 1 \right\}.$$

In exercising the constrained minimization method, the Pohozǎev's identity is crucial. So we first prove the Pohozǎev's identity for the equation (I).

THEOREM 2.1. (*Pohozăev's identity*). *Let $g : \mathbf{R} \to \mathbf{R}$ be continuous such that $g(0) = 0$ and let $G(u) = \int_0^u g(t)dt$ and $u \in L^\infty_{loc}(\mathbf{R}^N) \cup W^{1,m}_{loc}(\mathbf{R}^N)$ satisfy*

$$-div(|\nabla u|^{m-2}|\nabla u) = g(u) \qquad in \quad \mathbf{R}^N.$$

Assume further that $G(u) \in L^1(\mathbf{R}^N)$. Then u satisfy

$$\frac{N-m}{m}\int_{\mathbf{R}^N} |\nabla u|^m dx - N\int_{\mathbf{R}^N} G(u)dx = 0 \tag{2}$$

PROOF. If u is a solution of (I), on B_R

$$\int_{B_R} [div(|\nabla u|^{m-2}\nabla u)(x,\nabla u) + g(u)(x,\nabla u)]dx = 0. \tag{3}$$

Since $g(u)(x,\nabla u) = div(G(u)x) - NG(u)$,

$$\int_{B_R} g(u)(x,\nabla u)dx = -N\int_{B_R} G(u)dx + \int_{\partial B_R} G(u)\,(x,\nu)dS. \tag{4}$$

Now,

$$\begin{aligned}
&\int_{B_R} div(|\nabla u|^{m-2}(x,\nabla u)\nabla u)dx \\
&= \int_{B_R} D_i(|\nabla u|^{m-2}(x,\nabla u)D_i u)dx \\
&= \int_{B_R} [D_i(|\nabla u|^{m-2})D_i u + |\nabla u|^{m-2}D_{ii}u](x,\nabla u)dx \\
&\quad + \int_{B_R} |\nabla u|^{m-2}D_i(x,\nabla u)D_i u dx \\
&= \int_{B_R} div(|\nabla u|^{m-2}\nabla u)(x,\nabla u)dx + \int_{B_R} |\nabla u|^{m-2}D_i(x,\nabla u)D_i u dx.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\int_{B_R} div(|\nabla u|^{m-2}\nabla u)(x,\nabla u)dx \\
&= \int_{B_R} div(|\nabla u|^{m-2}\nabla u(x,\nabla u))dx - \int_{B_R} |\nabla u|^{m-2}D_i(x,\nabla u)D_i u dx,
\end{aligned} \tag{5}$$

On the other hand,

$$\int_{B_R} |\nabla u|^{m-2} D_i(x, \nabla u) D_i u dx$$

$$(6) \qquad = \int_{B_R} |\nabla u|^m dx + \int_{B_R} |\nabla u|^{m-2} D_i u D_{ij} u x_j dx$$

and observe that

$$\int_{B_R} |\nabla u|^{m-2} D_i u D_{ij} u x_j dx = \frac{1}{2} \int_{B_R} |\nabla u|^{m-2} D_j(|D_i u|^2) x_j dx$$

$$= \int_{B_R} |\nabla u|^{m-1} D_j(|\nabla u|) x_j dx$$

$$(7) \qquad = \frac{1}{m} \int_{\partial B_R} |\nabla u|^m x_j \nu_j dS - \frac{N}{m} \int_{B_R} (|\nabla u|^m) dx.$$

Apply the Divergence theorem after substituting (6) and (7) for (5). Then we have

$$(8) \quad \int_{B_R} div(|\nabla u|^{m-2} \nabla u)(x, \nabla u) dx$$

$$= \frac{N-m}{m} \int_{B_R} |\nabla u|^m dx + R \int_{\partial B_R} |\nabla u|^{m-2} \left| \frac{\partial u}{\partial \nu} \right|^2 dS - \frac{R}{m} \int_{\partial B_R} |\nabla u|^m dS.$$

By inserting (4) and (8) into (3), we obtain

$$(9) \quad \frac{N-m}{m} \int_{B_R} |\nabla u|^m dx - N \int_{B_R} G(u) dx$$

$$= R \left(\frac{1}{m} \int_{\partial B_R} |\nabla u|^m dS - \int_{\partial B_R} |\nabla u|^{m-2} |\partial u / \partial \nu|^2 dS - \int_{\partial B_R} G(u) dS \right).$$

So we have

$$(10) \quad \left| \frac{N-m}{m} \int_{B_R} |\nabla u|^m dx - N \int_{B_R} G(u) dx \right|$$

$$\leq R \left(C(m) \int_{\partial B_R} |\nabla u|^m + |G(u)| dS \right).$$

Since $\int_{\mathbf{R}^N} |\nabla u|^m + |G(u)| dx = \int_0^\infty \left(\int_{\partial B_R} |\nabla u|^m + |G(u)| dS \right) dR < +\infty$, there exists a sequence $R_n \to \infty$ such that, as $n \to \infty$,

$$R_n \int_{\partial B_{R_n}} |\nabla u|^m + |G(u)| dS \to 0.$$

Moreover, as $n \to \infty$,

$$\int_{B_{R_n}} |\nabla u|^m dx \to \int_{\mathbf{R}^N} |\nabla u|^m dx, \quad \int_{B_{R_n}} G(u)dx \to \int_{\mathbf{R}^N} G(u)dx.$$

With this sequence R_n, limiting n to ∞ in (10), we have

$$\frac{N-m}{m}\int_{\mathbf{R}^N} |\nabla u|^m \, dx \; - \; N\int_{\mathbf{R}^N} G(u)\, dx \; = \; 0.$$

□

PROOF OF THEOREM. Define $T(u) = \int_{\mathbf{R}^N} |\nabla u|^m dx$ and $V(u) = \int_{\mathbf{R}^N} H(u)dx$ where $H(u) = \int_0^u h(t)dt$ and $M = \{u \in W^{1,m}(\mathbf{R}^N) | V(u) = 1\}$. We consider the following constrained minimization problem;

$$\text{minimize}\{T(u) \mid u \in M\}.$$

First, it is simple to show that $V(u)$ is C^1-functional. Due to (h_4), it is easily checked that M is not empty (see [1]). Choose a minimizing sequence (u_n), that is,

$$\lim_{n\to\infty} T(u_n) = I \overset{def}{=} \inf_{u \in M} T(u)\,.$$

Let u_n^* be the Schwarz spherical decreasing rearrangement of $|u_n|$. Then $u_n^* \in W^{1,m}(\mathbf{R}^N)$, $V(u_n^*) = 1$ and $T(u_n^*) \le T(u_n)$ (confer Theorm1.25 in [4]). Hence (u_n^*) is again a minimizing sequence. Rewriting them by u_n, we may assume in the following that u_n is non-negative, spherically symmetric and non increasing in $|x|$. Now we assert that $\|u_n\|_{W^{1,m}(\mathbf{R}^N)}$ is bounded. Define $h_1(t) = \big(h(t) + \mu t^{m-1}\big)^+$ and $h_2(t) = \big(h(t) + \mu t^{m-1}\big)^-$, where $a^+ = max\{a, 0\}$ and $a^- = -min\{a, 0\}$. Then $h_i \ge 0$, where $i = 1, 2$ and $h = h_1 - h_2$. It follows from(h_1), (h_2) and (h_3) that

$$\lim_{t\to 0} \frac{h_1(t)}{t^{m-1}} = 0, \qquad \lim_{t\to\infty} \frac{h_1(t)}{t^{m^*-1}} = 0 \tag{11}$$

$$h_2(t) \ge \mu t^{m-1}. \tag{12}$$

Let $H_i(u) = \int_0^u h_i(t)\, dt$, $\quad i = 1, 2$. From (11) and (12), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$H_1(u) \le C_\varepsilon |u|^{m^*} + \varepsilon H_2(u), \quad u \in \mathbf{R}. \tag{13}$$

Since $\|\nabla u_n\|_{L^m}$ is bounded, we find that $\|u_n\|_{L^{m^*}}$ is bounded by the Sobolev imbedding theorem $(\mathcal{D}^{1,m}(\mathbf{R}^N) \hookrightarrow L^{m^*}(\mathbf{R}^N)$; see e.g., [8]). Since $V(u_n) = 1$, we have

$$\int_{\mathbf{R}^N} H_1(u_n)dx = \int_{\mathbf{R}^N} H_2(u_n)dx + 1. \tag{14}$$

We obtain from (13) and (14) with $\varepsilon = \frac{1}{2}$,

$$C + \frac{1}{2}\int_{\mathbf{R}^N} H_2(u_n)dx \geq \int_{\mathbf{R}^N} H_2(u_n)dx + 1.$$

Hence $\int_{\mathbf{R}^N} H_2(u_n)dx \leq C$, and by (12)

$$\frac{\mu}{m}\int_{\mathbf{R}^N} |u_n|^m dx \leq \int_{\mathbf{R}^N} H_2(u_n)dx < C.$$

Thus $\|u_n\|_{W^{1,m}(\mathbf{R}^N)}$ is bounded. Hence we may say, via subsequence argument, that there exists $u \in W^{1,m}(\mathbf{R}^N)$ such that $u_n \rightharpoonup u$ weekly in $W^{1,m}(\mathbf{R}^N)$. Observe that $u_n \to u$ a.e., and u is nonnegative, non increasing and spherically symmetric.

From now on, we are to show that u is positive. Note also that $\|u_n\|_{L^p(\mathbf{R}^N)} \leq C$ for any p, with $m \leq p \leq m^*$. Applying Strauss compactness lemma (see [7]), with $Q(t) = t^m + t^{m^*}$, we have

$$\int_{\mathbf{R}^N} H_1(u_n)\,dx \to \int_{\mathbf{R}^N} H_1(u)dx \quad \text{as } n \to \infty.$$

Applying Fatou's lemma to (14), we obtain

$$\int_{\mathbf{R}^N} H_1(u)dx \geq \int_{\mathbf{R}^N} H_2(u)dx + 1.$$

Hence $V(u) \geq 1$. Applying again Fatou's lemma, we have that $T(u) \leq I$. Now we assert that $V(u) = 1$, from which it follows that $T(u) = I$. Suppose to the contrary $V(u) > 1$ and put $u_\delta(x) = u(x/\delta)$. Then there exists δ, $0 < \delta < 1$, such that $V(u_\delta) = \delta^N V(u) = 1$. However with this u_δ, we have

$$0 < I \leq T(u_\delta) = \delta^{N-m}T(u) \leq \delta^{N-m}I < I,$$

which is absurd. Now since V and T are C^1-functionals on $W^{1,m}(\mathbf{R}^N)$, there exists a Lagrange multiplier θ such that

$$\frac{1}{m}DT(u) = \theta DV(u).$$

We assert that θ is positive. First note that $\theta \neq 0$ and suppose to the contrary that $\theta < 0$ and choose a function $\psi \in C_0^\infty(\mathbf{R}^N)$ such that $\langle DV(u), \psi\rangle = \int_{\mathbf{R}^N} h(u)\psi dx > 0$. Since

$$V(u + \varepsilon\psi) - V(u) = \varepsilon\langle DV(u), \psi\rangle + o(\varepsilon)$$

and

$$T(u + \varepsilon\psi) - T(u) = m\varepsilon\theta\langle DV(u), \psi\rangle + o(\varepsilon),$$

we find that there exists $\varepsilon > 0$ sufficiently small such that

$$V(w = u + \varepsilon\psi) > V(u) = 1$$

and

$$T(w = u + \varepsilon\psi) < T(u) = I.$$

Then again by changing the scale, there exists $\theta, 0 < \theta < 1$ such that $V(w_\theta) = 1$, whence we obtain

$$T(w_\theta) = \theta^{N-m}T(w) < I,$$

which is absurd. Hence $\theta > 0$ and u satisfies

$$-div(|\nabla u|^{m-2}\nabla u) = \theta h(u) \quad \text{in } \mathbf{R}^N,$$

and it is easy to see that $u_{\theta^{1/m}}(x)$, which is given by $u(\frac{x}{\theta^{1/m}})$ is a solution of (I) in the sense that

$$\int_{\mathbf{R}^N} |\nabla u_{\theta^{1/m}}|^{m-2}\nabla u_{\theta^{1/m}}\nabla v dx = \int_{\mathbf{R}^N} h(u_{\theta^{1/m}})v dx.$$

Finally we remark that $u_{\theta^{1/m}} \in C^{1,\alpha}$ by the regularity theory ([6]) and $u_{\theta^{1/m}} > 0$ by the Hopf boundary point lemma ([5]).

Now it remains to show (iii). Recall that if $\underline{u}$ is a critical point of the problem $\min\{T(u) \mid u \in M\}$, then there exists θ such that

$$-div(|\nabla \underline{u}|^{m-2}\nabla \underline{u}) = \theta h(\underline{u}) \quad \text{in } \mathbf{R}^N.$$

And if we set $u = \underline{u}_{\theta^{1/m}}$, u satisfies

$$-div(|\nabla u|^{m-2}\nabla u) = h(u) \quad \text{in } \mathbf{R}^N.$$

Now by Theorem 2.1, we have

$$T(u) = \frac{mN}{N-m}V(u). \tag{15}$$

Changing the scale of u yields

$$T(u) = \theta^{(N-m)/m}T(\underline{u}) \tag{16}$$

$$V(u) = \theta^{1/m}V(\underline{u}). \tag{17}$$

Noting that $V(\underline{u}) = 1$, we obtain $\theta = \frac{N-m}{mN}T(\underline{u})$ from (15), (16) and (17). Again by Pohozǎev's identity

$$J(u) = \frac{1}{m}T(u) - V(u)$$
$$= \left(\frac{1}{m} - \frac{N-m}{mN}\right)T(u)$$
$$= \frac{1}{N}\left(\frac{N-m}{mN}\right)^{(N-m)/m} T(\underline{u})^{N/m}. \tag{18}$$

Now let v be another solution of (I). By Pohozǎev's identity, we obtain

$$T(v) = \frac{mN}{N-m}V(v).$$

Let $\delta > 0$ be such that $V(v_\delta) = 1$. It is easy to see that $\delta = \left(\frac{N-m}{Nm}T(v)\right)^{-1/N}$. On the other hand,

$$T(v_\delta) = \delta^{N-m}T(v)$$
$$= \left(\frac{N-m}{Nm}\right)^{-(N-m)/N} T(v)^{m/N}.$$

Hence

$$J(v) = \frac{T(v)}{N} = \frac{1}{N}\left(\frac{N-m}{mN}\right)^{(N-m)/m} T(v_\delta)^{N/m}. \tag{19}$$

Since $\underline{u}$ is a minimum point of $T(u)$ on M, $T(u_\delta) \geq T(\underline{u})$. Hence from (18) and (19), we obtain the conclusion, $J(v) \geq J(\text{u})$. □

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