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SOME INEQUALITIES FOR THE FINITE HILBERT TRANSFORM OF A PRODUCT

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Abstract. Some inequalities for the Hilbert transform of the product of two functions are given.

1. Introduction

Let $\Omega = (-1, 1)$ where $1 \leq p < \infty$, the usual $\mathfrak{L}^p$−space with respect to the Lebesgue measure λ restricted to the open interval Ω will be denoted by $\mathfrak{L}^p(\Omega)$.

We define a linear operator T (see [24]) from the vector space $\mathfrak{L}^1(\Omega)$ into the vector space of all λ-measurable functions on Ω as follows. Let $f \in \mathfrak{L}^1(\Omega)$. The Cauchy principal value

$$\frac{1}{\pi} PV \int_{-1}^{1} \frac{f(\tau)}{\tau - t} d\tau = \lim_{\varepsilon \downarrow 0} \left[\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1} \right] \frac{f(\tau)}{\pi(\tau - t)} d\tau \tag{1.1}$$

exists for λ-almost every $t \in \Omega$.

We denote the left-hand side of (1.1) by $(Tf)(t)$ for each $t \in \Omega$ for which $(Tf)(t)$ exists. The so-defined function Tf, which we call the *finite Hilbert Transform* of f, is defined λ-almost everywhere on Ω and is λ-measurable; (see for example [1, Theorem 8.1.5]). The resulting linear operator T will be called the *finite Hilbert transform operator* or Cauchy kernel operator.

It is known that $\mathfrak{L}^1(\Omega)$ is not invariant under T, namely, $T(\mathfrak{L}^1(\Omega)) \not\subset \mathfrak{L}^1(\Omega)$ [17, Proof of Theorem 1 (b)].

The following basic results are well known and their proofs may be found in Propositions 8.1.9 and 8.2.1 of [1] respectively.

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THEOREM 1. (M. Riesz) *Let* $1 < p < \infty$. *Then* $T(\mathfrak{L}^p(\Omega)) \subset \mathfrak{L}^p(\Omega)$ *and the linear operator*

$$T_p : f \mapsto Tf,\ f \in \mathfrak{L}^p(\Omega)$$

on $\mathfrak{L}^p(\Omega)$ *is continuous.*

THEOREM 2. (Parseval) *Let* $1 < p < \infty$ *and* $q = \frac{p}{p-1}$. *Then*

$$\int_{-1}^{1} (fTg + gTf)\, d\lambda = 0 \tag{1.2}$$

for every $f \in \mathfrak{L}^p(\Omega)$ *and* $g \in \mathfrak{L}^q(\Omega)$.

We introduce the following definition.

DEFINITION 1. A function $f : \Omega \to \mathbb{C}$ is said to be *α–Hölder continuous* $(0 < \alpha \leq 1)$ in a subinterval Ω_0 of Ω if there exists a constant $c > 0$, dependent upon Ω_0, such that

$$|f(s) - f(t)| \leq c\,|s - t|^{\alpha}, \quad s, t \in \Omega_0. \tag{1.3}$$

A function on Ω is said to be *locally α-Hölder continuous* if it is α-Hölder continuous in every compact subinterval of Ω. We denote by $H_{loc}^{\alpha}(\Omega)$ the space of all locally α-Hölder continuous functions on Ω.

The class of Hölder continuous functions on Ω is independent because the finite Hilbert transform of such a function exists everywhere on Ω (see [15, Section 3.2] or [21, Lemma II.1.1]).

This is in contrast to the λ-almost everywhere existence of the finite Hilbert transform of functions in $\mathfrak{L}^1(\Omega)$.

There are continuous functions $f \in \mathfrak{L}^1(\Omega)$ such that $(Tf)(t)$ does not exist at some point $t \in \Omega$. An example is given by the function f defined by (see [24])

$$f(t) = \begin{cases} 0 & \text{if } -1 < t \leq 0, \\ \frac{1}{\ln t - \ln 2} & \text{if } 0 < t < 1. \end{cases}$$

It readily follows that $(Tf)(0)$ does not exist.

In paper [24] it is proved amongst others the following result.

THEOREM 3. (Okada-Elliot) *The space* $\mathfrak{L}^p(\Omega) \cap H_{loc}^{\alpha}(\Omega)$ *is invariant under the finite Hilbert transform operator* T *and the restriction of* T *to that space is continuous whenever* $1 < p < \infty$. *This, however, is not true when* $p = 1$.

We consider the finite Hilbert transform on the open interval (a,b)

$$(Tf)(a,b;t) := \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau, \quad t \in (a,b).$$

The following theorem holds (see [11]).

THEOREM 4. *Let $f : [a,b] \to \mathbb{R}$ be α-H-Hölder continuous on (a,b), i.e.,*

(1.4)

$$|f(t) - f(s)| \le H |t-s|^{\alpha} \quad \text{for all } t,s \in (a,b), \ \alpha \in (0,1], \ H > 0.$$

Then we have the estimate

$$\begin{aligned} &\left|(Tf)(a,b;t) - \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right)\right| \\ \le\ & \frac{H}{\alpha\pi}\left[(t-a)^{\alpha} + (b-t)^{\alpha}\right] \le \frac{H2^{1-\alpha}}{\alpha\pi}(b-a)^{\alpha}, \end{aligned} \tag{1.5}$$

for all $t \in (a,b)$.

The following result holds for monotonic functions (see [11]).

THEOREM 5. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing (nonincreasing) function on $[a,b]$. If the finite Hilbert transform $(Tf)(a,b,\cdot)$ exists in every $t \in (a,b)$, then*

$$(Tf)(a,b;t) \ge (\le) \frac{1}{\pi} f(t) \ln\left(\frac{b-t}{t-a}\right) \tag{1.6}$$

for all $t \in (a,b)$.

Now, if we assume that the mapping $f : (a,b) \to \mathbb{R}$ is convex on (a,b), then it is locally Lipschitzian on (a,b) and then the finite Hilbert transform of f exists in every point $t \in (a,b)$.

The following result holds (see [11]).

THEOREM 6. *Let $f : (a,b) \to \mathbb{R}$ be a convex mapping on (a,b). Then we have*

$$\begin{aligned} & \frac{1}{\pi}\left[l(t)(b-t) + \int_a^t l(s)\,ds + f(t)\ln\left(\frac{b-t}{t-a}\right)\right] \\ \le\ & (Tf)(a,b;t) \\ \le\ & \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right) + l(t)(t-a) + \int_t^b l(s)\,ds\right], \end{aligned} \tag{1.7}$$

where $l(s) \in \left[f'_-(s), f'_+(s)\right]$, $s \in (a,b)$.

The following more practical result also holds [11]:

COROLLARY 1. *Let $f : (a,b) \to \mathbb{R}$ be a differentiable convex function on (a,b). Then we have the inequality*

$$\begin{aligned}(1.8)\qquad & \frac{1}{\pi}\left[f(t) - f(a) + f'(t)(b-t) + f(t)\ln\left(\frac{b-t}{t-a}\right)\right] \\ \le\ & (Tf)(a,b;t) \\ \le\ & \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right) + f(b) - f(t) + f'(t)(t-a)\right]\end{aligned}$$

for all $t \in (a,b)$.

In this paper we point out some inequalities for the finite Hilbert transform of the product of two functions.

For a comprehensive number of results on the numerical approximation of the Cauchy principal value integrals, see [2]-[10], [13]-[14], [16], [18]-[20], [22]-[23], [25]-[27].

2. The results

The following lemma holds.

LEMMA 1. *If f and g are locally Hölder continuous on $[a,b]$, then fg is also locally Hölder continuous on $[a,b]$ and:*

$$\begin{aligned}(2.1)\quad & T(fg)(a,b;t) \\ =\ & f(t)T(g)(a,b;t) + g(t)T(f)(a,b;t) - \frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right) \\ & + \frac{1}{\pi}PV\int_a^b \frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau - t}d\tau\end{aligned}$$

for any $t \in (a,b)$.

PROOF. Assume that for a subinterval $[c,d] \subseteq [a,b]$, we have

$$(2.2)\qquad |f(s) - f(u)| \le L_1 |s-u|^{r_1} \quad \text{for any} \quad s,u \in [c,d];$$

$$(2.3)\qquad |g(s) - g(u)| \le L_2 |s-u|^{r_2} \quad \text{for any} \quad s,u \in [c,d].$$

Then

$$\begin{aligned}
& |f(s)g(s) - f(u)g(u)| \\
= & |f(s)g(s) - f(s)g(u) + f(s)g(u) - f(u)g(u)| \\
\leq & |f(s)||g(s) - g(u)| + |g(u)||f(s) - f(u)| \\
\leq & M_1L_1|s-u|^{r_1} + M_2L_2|s-u|^{r_2} \\
\leq & |s-u|^r\left[M_1L_1|s-u|^{r_1-r} + M_2L_2|s-u|^{r_2-r}\right] \\
\leq & |s-u|^r\left[M_1L_1|d-c|^{r_1-r} + M_2L_2|d-c|^{r_2-r}\right] \\
= & M|s-u|^r,
\end{aligned}$$

where

$$M_1 := \sup_{s\in[c,d]}|f(s)|, \quad M_2 := \sup_{u\in[c,d]}|g(u)|, \quad r = \min(r_1, r_2),$$

and

$$M = M_1L_1|d-c|^{r_1-r} + M_2L_2|d-c|^{r_2-r},$$

proving that fg is locally Hölder continuous on $[a,b]$.

Now, for any $t, \tau \in [a,b]$, we may write that

$$\begin{aligned}
& (f(\tau) - f(t))(g(\tau) - g(t)) \\
= & f(\tau)g(\tau) + f(t)g(t) - f(t)g(\tau) - f(\tau)g(t)
\end{aligned}$$

giving

$$\begin{aligned}
\frac{f(\tau)g(\tau)}{\tau - t} = & f(t)\cdot\frac{g(\tau)}{\tau-t} + g(t)\cdot\frac{f(\tau)}{\tau-t} - \frac{f(t)g(t)}{\tau-t} \\
& + \frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}
\end{aligned}$$

for any $t, \tau \in [a,b]$, $t \neq \tau$.

Consequently,

$$\begin{aligned}
& T(fg)(a,b;t) \\
= & \frac{1}{\pi}PV\int_a^b \frac{f(\tau)g(\tau)}{\tau-t}d\tau \\
= & \frac{1}{\pi}f(t)PV\int_a^b \frac{g(\tau)}{\tau-t}d\tau + g(t)\frac{1}{\pi}PV\int_a^b\frac{f(\tau)}{\tau-t}d\tau \\
& -\frac{1}{\pi}f(t)g(t)PV\int_a^b\frac{d\tau}{\tau-t} + \frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}d\tau \\
= & f(t)T(g)(a,b;t) + g(t)T(f)(a,b;t) \\
& - \frac{f(t)g(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) + \frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}d\tau
\end{aligned}$$

for any $t \in (a,b)$, and the identity (2.1) is proved. □

THEOREM 7. *Assume that f is of L_1-r_1-Hölder type and g is of L_2-r_2-Hölder type on $[a,b]$, where $L_1, L_2 > 0$, $r_1, r_2 \in (0,1]$. Then we have the inequality:*

$$
\begin{aligned}
(2.4)\qquad & \Big| T(fg)(a,b;t) - f(t)\, T(g)(a,b;t) \\
& \qquad - g(t)\, T(f)(a,b;t) + \frac{1}{\pi} f(t)\, g(t) \ln\left(\frac{b-t}{t-a}\right) \Big| \\
\le\ & \frac{L_1 L_2}{\pi (r_1 + r_2)} \left[(b-t)^{r_1+r_2} + (t-a)^{r_1+r_2} \right] \\
\le\ & \frac{L_1 L_2 (b-a)^{r_1+r_2}}{\pi (r_1+r_2)}
\end{aligned}
$$

for any $t \in (a,b)$.

PROOF. Taking the modulus in (2.1), we may write

$$
\begin{aligned}
& \Big| T(fg)(a,b;t) - f(t)\, T(g)(a,b;t) \\
& \qquad - g(t)\, T(f)(a,b;t) + \frac{1}{\pi} f(t)\, g(t) \ln\left(\frac{b-t}{t-a}\right) \Big| \\
\le\ & \frac{1}{\pi} PV \int_a^b \left| \frac{(f(\tau) - f(t))(g(\tau) - g(t))}{\tau - t} \right| d\tau \\
\le\ & \frac{1}{\pi} PV \int_a^b L_1 L_2 |\tau - t|^{r_1+r_2-1} d\tau \\
=\ & \frac{L_1 L_2}{\pi} \left[\frac{(b-t)^{r_1+r_2} + (t-a)^{r_1+r_2}}{r_1 + r_2} \right]
\end{aligned}
$$

and the first part of inequality (2.4) is proved. The second part is obvious. □

The best inequality we can get from (2.4) is embodied in the following corollary.

COROLLARY 2. *With the assumptions in Theorem 7, we have*

$$\begin{aligned}(2.5)\qquad &\Bigg|T(fg)\left(a,b;\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)T(g)\left(a,b;\frac{a+b}{2}\right)\\ &\qquad -g\left(\frac{a+b}{2}\right)T(f)\left(a,b;\frac{a+b}{2}\right)\Bigg|\\ &\leq \frac{L_1L_2(b-a)^{r_1+r_2}}{\pi(r_1+r_2)2^{r_1+r_2-1}}.\end{aligned}$$

The following corollary also holds.

COROLLARY 3. *If f and g are Lipschitzian with the constants K_1 and K_2, then we have the inequality*

$$\begin{aligned}(2.6)\qquad &\Bigg|T(fg)(a,b;t)-f(t)T(g)(a,b;t)\\ &\qquad -g(t)T(f)(a,b;t)+\frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)\Bigg|\end{aligned}$$

$$(2.7)\qquad \leq \frac{K_1K_2}{\pi}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\leq \frac{K_1K_2}{2\pi}(b-a)^2$$

for any $t\in(a,b)$. In particular, for $t=\frac{a+b}{2}$, we have

$$\begin{aligned}(2.8)\qquad &\Bigg|T(fg)\left(a,b;\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)T(g)\left(a,b;\frac{a+b}{2}\right)\\ &\qquad -g\left(\frac{a+b}{2}\right)T(f)\left(a,b;\frac{a+b}{2}\right)\Bigg|\end{aligned}$$

$$(2.9)\qquad \leq \frac{K_1K_2}{4\pi}(b-a)^2.$$

The following theorem also holds.

THEOREM 8. *Assume that f and g are absolutely continuous on $[a,b]$. Then we have the inequality:*

$$\begin{aligned}(2.10)\qquad &\Bigg|T(fg)(a,b;t)-f(t)T(g)(a,b;t)\\ &\qquad -g(t)T(f)(a,b;t)+\frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)\Bigg|\end{aligned}$$

$$\tag{2.11} \leq \frac{1}{\pi} \times \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right] \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\infty} \\ \quad \text{if } f' \in L_\infty[a,b], \ g' \in L_\infty[a,b]; \\ \\ \frac{\delta}{\delta+1}\left[(b-t)^{1+\frac{1}{\delta}} + (t-a)^{1+\frac{1}{\delta}}\right] \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\gamma} \\ \quad \text{if } f' \in L_\infty[a,b], \ g' \in L_\gamma[a,b], \ \gamma > 1, \ \frac{1}{\gamma}+\frac{1}{\delta} = 1; \\ \\ (b-a)\|f'\|_{[a,b],\infty} \|g'\|_{[a,b],1} \\ \quad \text{if } f' \in L_\infty[a,b], \ g' \in L_1[a,b]; \\ \\ \frac{\beta}{\beta+1}\left[(b-t)^{1+\frac{1}{\beta}} + (t-a)^{1+\frac{1}{\beta}}\right] \|f'\|_{[a,b],\alpha} \|g'\|_{[a,b],\infty} \\ \quad \text{if } f' \in L_\alpha[a,b], \ \alpha > 1, \ \frac{1}{\alpha}+\frac{1}{\beta} = 1, \text{ and } g' \in L_\infty[a,b]; \\ \\ \frac{\beta\delta}{\beta+\delta}\left[(b-t)^{\frac{\delta+\beta}{\beta+\delta}} + (t-a)^{\frac{\delta+\beta}{\beta+\delta}}\right] \|f'\|_{[a,b],\alpha} \|g'\|_{[a,b],\gamma} \\ \quad \text{if } f' \in L_\alpha[a,b], \ \alpha > 1, \ \frac{1}{\alpha}+\frac{1}{\beta} = 1, \\ \quad \text{and } g' \in L_\gamma[a,b], \ \gamma > 1, \ \frac{1}{\gamma}+\frac{1}{\delta} = 1; \\ \\ \beta\left[(b-t)^{\frac{1}{\beta}} + (t-a)^{\frac{1}{\beta}}\right] \|f'\|_{[a,b],\alpha} \|g'\|_{[a,b],1} \\ \quad \text{if } f' \in L_\alpha[a,b], \ \alpha > 1, \ \frac{1}{\alpha}+\frac{1}{\beta} = 1, \text{ and } g' \in L_1[a,b]; \\ \\ (b-a)\|f'\|_{[a,b],1} \|g'\|_{[a,b],\infty} \\ \quad \text{if } f' \in L_1[a,b], \ g' \in L_\infty[a,b]; \\ \\ \delta\left[(b-t)^{1+\frac{1}{\delta}} + (t-a)^{1+\frac{1}{\delta}}\right] \|f'\|_{[a,b],1} \|g'\|_{[a,b],\gamma} \\ \quad \text{if } f' \in L_1[a,b], \ g' \in L_\gamma[a,b], \ \gamma > 1, \ \frac{1}{\gamma}+\frac{1}{\delta} = 1; \end{cases}$$

Proof. Since f and g are absolutely continuous on $[a,b]$, we may write that

$$f(\tau) - f(t) = \int_t^\tau f'(u)\,du \quad \text{and} \quad g(\tau) - g(t) = \int_t^\tau g'(u)\,du$$

which implies:

$$
\text{(2.12)} \qquad |f(\tau) - f(t)| \le \begin{cases} \|f'\|_{[\tau,t],\infty} |\tau - t| & \text{if } f' \in L_{\infty}[a,b]; \\ \|f'\|_{[\tau,t],\alpha} |\tau - t|^{\frac{1}{\beta}} & \text{if } f' \in L_{\alpha}[a,b], \\ & \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \|f'\|_{[\tau,t],1} & \end{cases}
$$

and

$$
\text{(2.13)} \qquad |g(\tau) - g(t)| \le \begin{cases} \|g'\|_{[\tau,t],\infty} |\tau - t| & \text{if } g' \in L_{\infty}[a,b]; \\ \|g'\|_{[\tau,t],\gamma} |\tau - t|^{\frac{1}{\delta}} & \text{if } g' \in L_{\gamma}[a,b], \\ & \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \|g'\|_{[\tau,t],1} . & \end{cases}
$$

Using the identity (2.2), we get

$$
\begin{aligned}
\text{(2.14)} \qquad & \Bigg| T(fg)(a,b;t) - f(t)\,T(g)(a,b;t) \\
& \qquad - g(t)\,T(f)(a,b;t) + \frac{1}{\pi} f(t)\,g(t) \ln\left(\frac{b-t}{t-a}\right) \Bigg| \\
\le \; & \frac{1}{\pi} PV \int_a^b \left| \frac{(f(\tau) - f(t))(g(\tau) - g(t))}{\tau - t} \right| d\tau =: I.
\end{aligned}
$$

Then we have, by using (2.12) or (2.13), that

$$I \le \frac{1}{\pi} \times \begin{cases} PV \displaystyle\int_a^b \|f'\|_{[\tau,t],\infty} \|g'\|_{[\tau,t],\infty} |\tau - t| \, d\tau \\ PV \displaystyle\int_a^b \|f'\|_{[\tau,t],\infty} \|g'\|_{[\tau,t],\alpha} |\tau - t|^{\frac{1}{\delta}} \, d\tau \\ PV \displaystyle\int_a^b \|f'\|_{[\tau,t],\infty} \|g'\|_{[\tau,t],1} \, d\tau \\ PV \displaystyle\int_a^b \|f'\|_{[\tau,t],\alpha} \|g'\|_{[\tau,t],\infty} |\tau - t|^{\frac{1}{\beta}} \, d\tau \\ PV \displaystyle\int_a^b \|f'\|_{[\tau,t],\alpha} \|g'\|_{[\tau,t],\gamma} |\tau - t|^{\frac{1}{\beta}+\frac{1}{\delta}-1} \, d\tau \\ PV \displaystyle\int_a^b \|f'\|_{[\tau,t],\alpha} \|g'\|_{[\tau,t],1} |\tau - t|^{\frac{1}{\beta}-1} \, d\tau \\ PV \displaystyle\int_a^b \|f'\|_{[\tau,t],1} \|g'\|_{[\tau,t],\infty} \, d\tau \\ PV \displaystyle\int_a^b \|f'\|_{[\tau,t],1} \|g'\|_{[\tau,t],\gamma} |\tau - t|^{\frac{1}{\delta}-1} \, d\tau \\ PV \displaystyle\int_a^b \|f'\|_{[\tau,t],1} \|g'\|_{[\tau,t],1} |\tau - t|^{-1} \, d\tau. \end{cases} \tag{2.15}$$

However,

$$\begin{aligned} & PV \int_a^b \|f'\|_{[\tau,t],\infty} \|g'\|_{[\tau,t],\infty} |\tau - t| \, d\tau \\ \le \; & \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\infty} \left[\frac{(b-t)^2 + (t-a)^2}{2} \right] \\ = \; & \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\infty} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right], \end{aligned}$$

$$
\begin{aligned}
& PV \int_a^b \left\| f' \right\|_{[\tau,t],\infty} \left\| g' \right\|_{[\tau,t],\alpha} |\tau - t|^{\frac{1}{\delta}} \, d\tau \\
\leq \; & \left\| f' \right\|_{[a,b],\infty} \left\| g' \right\|_{[a,b],\gamma} \left[\frac{(b-t)^{1+\frac{1}{\delta}} + (t-a)^{1+\frac{1}{\delta}}}{\frac{1}{\delta}+1} \right] \\
= \; & \frac{\delta}{\delta+1} \left\| f' \right\|_{[a,b],\infty} \left\| g' \right\|_{[a,b],\gamma} \left[(b-t)^{1+\frac{1}{\delta}} + (t-a)^{1+\frac{1}{\delta}} \right],
\end{aligned}
$$

$$
PV \int_a^b \left\| f' \right\|_{[\tau,t],\infty} \left\| g' \right\|_{[\tau,t],1} d\tau \leq \left\| f' \right\|_{[a,b],\infty} \left\| g' \right\|_{[a,b],1} (b-a),
$$

$$
\begin{aligned}
& PV \int_a^b \left\| f' \right\|_{[\tau,t],\alpha} \left\| g' \right\|_{[\tau,t],\infty} |\tau - t|^{\frac{1}{\beta}} \, d\tau \\
\leq \; & \left\| f' \right\|_{[a,b],\alpha} \left\| g' \right\|_{[a,b],\infty} \cdot \frac{\beta}{\beta+1} \left[(b-t)^{\frac{1}{\beta}+1} + (t-a)^{\frac{1}{\beta}+1} \right],
\end{aligned}
$$

$$
\begin{aligned}
& PV \int_a^b \left\| f' \right\|_{[\tau,t],\alpha} \left\| g' \right\|_{[\tau,t],\gamma} |\tau - t|^{\frac{1}{\beta}+\frac{1}{\delta}-1} \, d\tau \\
\leq \; & \left\| f' \right\|_{[a,b],\alpha} \left\| g' \right\|_{[a,b],\gamma} \frac{1}{\frac{1}{\beta}+\frac{1}{\delta}} \left[(b-t)^{\frac{1}{\beta}+\frac{1}{\delta}} + (t-a)^{\frac{1}{\beta}+\frac{1}{\delta}} \right] \\
= \; & \frac{\beta\delta}{\beta+\delta} \left\| f' \right\|_{[a,b],\alpha} \left\| g' \right\|_{[a,b],\gamma} \left[(b-t)^{\frac{\delta+\beta}{\beta+\delta}} + (t-a)^{\frac{\delta+\beta}{\beta+\delta}} \right],
\end{aligned}
$$

$$
\begin{aligned}
& PV \int_a^b \left\| f' \right\|_{[\tau,t],\alpha} \left\| g' \right\|_{[\tau,t],1} |\tau - t|^{\frac{1}{\beta}-1} \, d\tau \\
\leq \; & \left\| f' \right\|_{[a,b],\alpha} \left\| g' \right\|_{[a,b],1} \beta \left[(b-t)^{\frac{1}{\beta}} + (t-a)^{\frac{1}{\beta}} \right],
\end{aligned}
$$

$$
PV \int_a^b \left\| f' \right\|_{[\tau,t],1} \left\| g' \right\|_{[\tau,t],\infty} d\tau \leq (b-a) \left\| f' \right\|_{[a,b],1} \left\| g' \right\|_{[a,b],\infty}
$$

and

$$\begin{aligned}
& PV\int_a^b \left\|f'\right\|_{[\tau,t],1}\left\|g'\right\|_{[\tau,t],\gamma}\left|\tau-t\right|^{\frac{1}{\delta}-1}d\tau \\
\leq\ & \left\|f'\right\|_{[a,b],1}\left\|g'\right\|_{[a,b],\gamma}\delta\left[(b-t)^{1+\frac{1}{\delta}}+(t-a)^{1+\frac{1}{\delta}}\right].
\end{aligned}$$

For the last inequality we cannot point out a bound as above.

Using (2.14) and (2.15), we deduce the desired inequality (2.10). □

The following lemma also holds.

LEMMA 2. *Let $f:[a,b]\to\mathbb{R}$ be locally Hölder continuous on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ so that g' is absolutely continuous on $[a,b]$. Then we have the identity:*

$$\begin{aligned}
(2.16)\quad & T(fg)(a,b;t) \\
=\ & f(t)\,T(g)(a,b;t)+g(t)\,T(f)(a,b;t)-\frac{1}{\pi}f(t)\,g(t)\ln\left(\frac{b-t}{t-a}\right) \\
& +\frac{1}{\pi}\left[\int_a^b f(\tau)\,d\tau-(b-a)\,f(t)\right]g'(t) \\
& -\frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^\tau (u-\tau)\,g''(u)\,du\right)d\tau
\end{aligned}$$

for any $t\in(a,b)$.

PROOF. We use the following identity:

$$\int_\alpha^\beta \varphi(u)\,du=\varphi(\alpha)(\beta-\alpha)-\int_\alpha^\beta (u-\beta)\,\varphi'(u)\,du$$

which holds for any absolutely continuous function $\varphi:[\alpha,\beta]\to\mathbb{R}$.

Then we have

$$\begin{aligned}
&\frac{1}{\pi} PV \int_a^b [f(\tau) - f(t)] \left[\frac{1}{\tau - t}(g(\tau) - g(t))\right] d\tau \\
=\ & \frac{1}{\pi} PV \int_a^b [f(\tau) - f(t)] \left[\frac{1}{\tau - t}\int_t^\tau g'(u)\,du\right] d\tau \\
=\ & \frac{1}{\pi} PV \int_a^b [f(\tau) - f(t)] \left[g'(t) - \frac{1}{\tau - t}\int_t^\tau (u-\tau) g''(u)\,du\right] d\tau \\
=\ & \frac{1}{\pi}\left[g'(t)\int_a^b f(\tau)\,d\tau - (b-a) f(t) g'(t)\right. \\
& \left. - PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t}\left(\int_t^\tau (u-\tau) g''(u)\,du\right) d\tau\right] \\
=\ & \frac{1}{\pi} g'(t)\int_a^b f(\tau)\,d\tau - \frac{1}{\pi}(b-a) f(t) g'(t) \\
& - \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t}\left(\int_t^\tau (u-\tau) g''(u)\,du\right) d\tau.
\end{aligned}$$

Using (2.1), we deduce (2.16). □

The following theorem holds.

THEOREM 9. *Assume that $f : [a,b] \to \mathbb{R}$ is of H-r-Hölder type and $g : [a,b] \to \mathbb{R}$ is such that g' is absolutely continuous on $[a,b]$. Then we have the inequality:*

$$\begin{aligned}
(2.17)\quad & \Big| T(fg)(a,b;t) - f(t) T(g)(a,b;t) - g(t) T(f)(a,b;t) \\
& + \frac{1}{\pi} f(t) g(t) \ln\left(\frac{b-t}{t-a}\right) - \frac{1}{\pi}\left[\int_a^b f(\tau)\,d\tau - (b-a) f(t)\right] g'(t) \Big|
\end{aligned}$$

$$\leq \frac{H}{\pi}\begin{cases} \frac{1}{2(r+2)}\left[(b-t)^{r+2}+(t-a)^{r+2}\right]\|g''\|_{[a,b],\infty} \\ \quad \text{if } g''\in L_\infty[a,b]; \\ \frac{q}{(rq+q+1)(q+1)^{\frac{1}{q}}}\left[(b-t)^{r+\frac{1}{q}+1}+(t-a)^{r+\frac{1}{q}+1}\right]\|g''\|_{[a,b],p} \\ \quad \text{if } g''\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{(r+1)}\left[(b-t)^{r+1}+(t-a)^{r+1}\right]\|g''\|_{[a,b],1}. \end{cases}$$

PROOF. Using the identity (2.16), we deduce that the left side in (2.17) is upper bounded by

$$\begin{aligned} I \quad : &= \frac{1}{\pi}PV\int_a^b |f(\tau)-f(t)|\frac{1}{|\tau-t|}\left|\int_t^\tau (u-\tau)g''(u)\,du\right| d\tau \\ &\leq \frac{H}{\pi}PV\int_a^b |\tau-t|^{r-1}\left|\int_t^\tau (u-\tau)g''(u)\,du\right| d\tau =: J. \end{aligned}$$

We observe that

$$\left|\int_t^\tau (u-\tau)g''(u)\,du\right| \leq \|g''\|_{[t,\tau],\infty}\frac{(\tau-t)^2}{2}$$

if $g''\in L_\infty[a,b]$,

$$\begin{aligned} \left|\int_t^\tau (u-\tau)g''(u)\,du\right| &\leq \|g''\|_{[t,\tau],p}\left|\int_t^\tau |t-\tau|^q\,d\tau\right|^{\frac{1}{q}} \\ &= \|g''\|_{[t,\tau],p}\frac{|t-\tau|^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \end{aligned}$$

if $g''\in L_p[a,b]$ and, finally,

$$\left|\int_t^\tau (u-\tau)g''(u)\,du\right| \leq |t-\tau|\,\|g''\|_{[t,\tau],1}.$$

Consequently, we have

$$J \le \frac{H}{\pi} \times \begin{cases} PV \displaystyle\int_a^b |\tau - t|^{r-1} \cdot \tfrac{(\tau - t)^2}{2} \cdot \|g''\|_{[t,\tau],\infty} \, d\tau \\[2ex] PV \displaystyle\int_a^b \frac{|\tau - t|^{r-1} \cdot |t-\tau|^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|g''\|_{[t,\tau],p} \, d\tau \\[2ex] PV \displaystyle\int_a^b |\tau - t|^{r-1} \cdot |t-\tau| \, \|g''\|_{[t,\tau],1} \, d\tau \end{cases}$$

$$\le \frac{H}{\pi} \times \begin{cases} \dfrac{1}{2} \|g''\|_{[a,b],\infty} \left[\dfrac{(b-t)^{r+2} + (t-a)^{r+2}}{r+2} \right] \\[2ex] \dfrac{1}{(q+1)^{\frac{1}{q}}} \|g''\|_{[a,b],p} \left[\dfrac{(b-t)^{r+\frac{1}{q}+1} + (t-a)^{r+\frac{1}{q}+1}}{r+\frac{1}{q}+1} \right] \\[2ex] \|g''\|_{[a,b],1} \cdot \left[\dfrac{(b-t)^{r+1} + (t-a)^{r+1}}{r+1} \right], \end{cases}$$

which proves the inequality (2.17). □

The following lemma also holds.

LEMMA 3. *Assume that f and g are as in Lemma 2. Then we have the identity:*

$$\begin{aligned} (2.18)\quad & T(fg)(a,b;t) \\ = \; & f(t)\,T(g)(a,b;t) + g(t)\,T(f)(a,b;t) - \frac{1}{\pi} f(t)\,g(t) \ln\left(\frac{b-t}{t-a}\right) \\ & + \frac{1}{\pi}\left[\int_a^b f(\tau)\,g'(\tau)\,d\tau - [g(b)-g(a)]\,f(t) \right] \\ & - \frac{1}{\pi} PV \int_a^b \frac{f(\tau)-f(t)}{\tau - t} \left(\int_t^\tau (u-t)\,g''(u)\,du \right) d\tau \end{aligned}$$

for any $t \in (a,b)$.

PROOF. In this case, we use the following identity:

$$\int_{\alpha}^{\beta}\varphi(u)\,du=\varphi(\beta)(\beta-\alpha)-\int_{\alpha}^{\beta}(u-\alpha)\varphi'(u)\,du$$

which holds for any absolutely continuous function $\varphi:[\alpha,\beta]\to\mathbb{R}$.

Then, as above, we have

$$\begin{aligned}
&\frac{1}{\pi}PV\int_a^b[f(\tau)-f(t)]\left[\frac{1}{\tau-t}(g(\tau)-g(t))\right]d\tau\\
&=\frac{1}{\pi}PV\int_a^b[f(\tau)-f(t)]\left[\frac{1}{\tau-t}\int_t^{\tau}g'(u)\,du\right]d\tau\\
&=\frac{1}{\pi}PV\int_a^b[f(\tau)-f(t)]\left[g'(t)-\frac{1}{\tau-t}\int_t^{\tau}(u-\tau)g''(u)\,du\right]d\tau\\
&=\frac{1}{\pi}\left[\int_a^b f(\tau)g'(\tau)\,d\tau-f(t)\,PV\int_a^b g'(\tau)\,d\tau\right.\\
&\qquad\left.-PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^{\tau}(u-t)g''(u)\,du\right)d\tau\right]\\
&=\frac{1}{\pi}\left[\int_a^b f(\tau)g'(\tau)\,d\tau-[g(b)-g(a)]f(t)\right.\\
&\qquad\left.-PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^{\tau}(u-t)g''(u)\,du\right)d\tau\right],
\end{aligned}$$

proving the identity (2.18). $\square$

The following result also holds.

THEOREM 10. *With the assumptions in Theorem 9, we have:*

$$\begin{aligned}(2.19)\qquad & \Bigg| T(fg)(a,b;t) - f(t)\,T(g)(a,b;t) - g(t)\,T(f)(a,b;t) \\ & \quad + \frac{1}{\pi} f(t)\,g(t) \ln\left(\frac{b-t}{t-a}\right) \\ & \quad - \frac{1}{\pi}\left[\int_a^b f(\tau)\,g'(\tau)\,d\tau - [g(b)-g(a)]\,f(t)\right]\Bigg| \\ & \le \frac{H}{\pi}\begin{cases} \frac{1}{2(r+2)}\left[(b-t)^{r+2} + (t-a)^{r+2}\right]\|g''\|_{[a,b],\infty} \\ \qquad \text{if } g'' \in L_\infty[a,b]; \\ \frac{q}{(rq+q+1)(q+1)^{\frac{1}{q}}}\left[(b-t)^{r+\frac{1}{q}+1} + (t-a)^{r+\frac{1}{q}+1}\right]\|g''\|_{[a,b],p} \\ \qquad \text{if } g'' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{(r+1)}\left[(b-t)^{r+1} + (t-a)^{r+1}\right]\|g''\|_{[a,b],1}. \end{cases}\end{aligned}$$

PROOF. The proof follows in a similar manner to the one in Theorem 9 by the use of Lemma 3. We omit the details. □

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