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CHARACTERISTIC FUZZY GROUPS

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Abstract. In this paper, we define the notion of a characteristic fuzzy subgroup which is the analogue of a characteristic group in the ordinary group theory and derive various new ones from a given characteristic fuzzy subgroup.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh, and the notion of fuzzy subgroups was introduced by Rosenfeld [7], who showed how some basic notions of group theory could be extended in an elementary manner to fuzzy groups. Since then the theory of fuzzy subgroups has been developed further by many mathematicians such as P. S. Das [1], K. C. Gupta and B. K. Sarma [2, 3], N. P. Mukherjee and P. Bhattacharya [5], S. Ray and Liu and so on. K. C. Gupta and B. K. Sarma have done the research on the fuzzy subgroup of a group which are invariant under various operator domains [2]. Recently, Sidky and Mishref have continued to investigate characteristic fuzzy subgroups and proved some of their properties. In this paper we obtain new characteristic fuzzy subgroups from a given characteristic fuzzy subgroup.

Throughout this paper, G denotes a group with the identity e and N stands for the set of all positive integers. As usual, a fuzzy set λ of a set X is a map from X to $[0, 1]$. For two fuzzy sets λ and μ of X, $\lambda \subseteq \mu$ means that $\lambda(x) \leq \mu(x)$ for all $x \in X$.

2. Preliminaries

We recall first the following basic definitions and results to be used

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in the sequel.

DEFINITION 1. Let $\lambda : X \to [0,1]$ be a fuzzy subset of X and let $t \in [0,1]$. The set

$$\lambda_t = \{x \in X | \lambda(x) \geq t\}$$

is called a *level subset* of λ.

Let f be a map of X into Y, and λ and μ be fuzzy subsets of X and Y, respectively. The image $f(\lambda)$ of λ is the fuzzy subset of Y defined by for $y \in Y$,

$$f(\lambda)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \lambda(x), & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise.} \end{cases}$$

The inverse image $f^{-1}(\mu)$ of μ under f is the fuzzy subset of X defined by for $x \in X$,

$$f^{-1}(\mu)(x) = \mu\left(f(x)\right).$$

DEFINITION 2. Let G be a group. A fuzzy subset $\lambda : G \to [0,1]$ is called a *fuzzy subgroup* of G if

(1) $\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\}$ for every $x, y \in G$.
(2) $\lambda(x) \geq \lambda(x^{-1})$ for every $x \in G$.

It is easy to see that if λ is a fuzzy subgroup of G, then for every $x \in G$, $\lambda(x) = \lambda(x^{-1})$ and $\lambda(e) \geq \lambda(x)$. It is well known that a fuzzy subset λ of G is a fuzzy subgroup of G if and only if each level subset $\lambda_t (t \in [0,1])$ is a subgroup of G. See [1]

DEFINITION 3. For an arbitrary fuzzy subset λ of G, we denote the least fuzzy subgroup of G containing λ by $[\lambda]$, which is given by

$$[\lambda](x) = \sup_{x = t_1 t_2 \ldots t_n} \min\{\sigma(t_1), \sigma(t_2), \ldots, \sigma(t_n)\}, x \in G,$$

where $\sigma(y) = \max\{\lambda(y), \lambda(y^{-1})\}, y \in G$. This fuzzy subgroup $[\lambda]$ is called *fuzzy subgroup generated by* λ.

See [3] or [6].

DEFINITION 4. For an arbitrary fuzzy subsets λ and μ of G, the *sup-min product* is the fuzzy subset of G defined by

$$\lambda \circ \mu(x) = \sup_{x=ab} \min\{\lambda(a), \mu(b)\}, \quad x \in G.$$

For a group G, an *operator domain* on G is a nonempty class $\mathfrak{D}$ of endomorphisms of G. In the ordinary group theory we call a subset A of G admissible under $\mathfrak{D}$, or simply $\mathfrak{D}$-admissible if $f(A) \subseteq A$ for every $f \in \mathfrak{D}$. A subgroup H of G is said to be a *characteristic (normal) subgroup* of G if it is admissible under $Aut(G)(Inn(G))$, that is, $f(H) \subseteq H$ for every automorphism (inner automorphism) f of G, respectively. As a counterpart of a characteristic subgroup of a group G, we can define the characteristic fuzzy subgroup of a fuzzy subgroup λ in the fuzzy setting as follows;

DEFINITION 5. Let $\mathfrak{D}$ be an operator domain of a group G. A fuzzy subset λ of a group G is *admissible* under $\mathfrak{D}$ (or simply $\mathfrak{D}$-*admissible*) if, for every $f \in \mathfrak{D}$, one of the following two equivalent conditions is satisfied:

(1) $f(\lambda) \subseteq \lambda$
(2) $\lambda \subseteq f^{-1}(\lambda)$.

In particular, a fuzzy subgroup λ of a group G is *characteristic (normal)* if it is admissible under $Aut(G)(Inn(G))$ respectively (See [2]).

PROPOSITION 6. *Let $\mathfrak{D}$ be an operator domain on G. Then a fuzzy subset λ of G is $\mathfrak{D}$-admissible if and only if each level subset λ_t is $\mathfrak{D}$-admissible for all $t \in (0,1]$.*

PROOF. Assume that a fuzzy subset λ of G is $\mathfrak{D}$-admissible and let $x \in \lambda_t$, $t \in (0,1]$, $f \in \mathfrak{D}$. Then $\lambda(x) \geq t$, so we have $\lambda(f(x)) \geq \lambda(x) \geq t$, which implies that $f(\lambda_t) \subset \lambda_t$ for all $t \in (0,1]$.

Conversely, let level subset λ_t be $\mathfrak{D}$-admissible for all $t \in (0,1]$. Assume that $\lambda(f(x)) < \lambda(x)$ for some $f \in \mathfrak{D}$ and $x \in G$. Choose $s \in (0,1)$ such that $\lambda(f(x)) < s < \lambda(x)$. Then we have $x \in \lambda_s$, but $f(x) \notin \lambda_s$. This contradicts to the fact $f(\lambda_t) \subseteq \lambda_t$ for all $t \in (0,1]$. □

In a group G, the *commutator* of two elements a and b of G is the element $[a,b] = a^{-1}b^{-1}ab$ of G. If $A, B \subseteq G$, then the *commutator* *subgroup* of A and B is the subgroup $[A,B]$ of G generated by the set $\{[a,b] | a \in A, b \in B\}$. K. C. Gupta and B. K. Sarma had defined the *commutator fuzzy subgroup* as follows: See [3].

DEFINITION 7. Let λ, μ be fuzzy subsets of G. The *commutator* of λ and μ is the fuzzy subgroup $[\lambda, \mu]$ of G generated by (λ, μ), where

$$(\lambda,\mu)(x) = \begin{cases} \sup\limits_{x=[a,b]} \min\{\lambda(a), \mu(b)\}, & \text{if x is a commutator in } G \\ 0, & \text{otherwise} \end{cases}$$

for $x \in G$.

DEFINITION 8. Let λ be a fuzzy subgroup of G and let x be a fixed element of G. The fuzzy subgroup λ_x^* of G, defined by $\lambda_x^*(g) = \lambda(x^{-1}gx)$ for all $g \in G$, is called *a fuzzy conjugate subgroup* of G determined by λ and x. The *core* of λ, denoted by $Core_G\lambda$, is the intersection of all fuzzy conjugate subgroups of G determined by λ, that is, $Core_G\lambda = \bigcap\limits_{x\in G} \lambda_x^*$.

3. Results

LEMMA 9. *If λ is $\mathfrak{D}$-admissible subset of a group G, then the fuzzy subgroup $[\lambda]$ generated by λ is $\mathfrak{D}$-admissible.*

PROOF. See Theorem 3.8 of [2]. □

THEOREM 10. *Let λ be a fuzzy subgroup of G. We define recursively $\lambda_{(0)} = \lambda$, and $\lambda_{(n)} = [\lambda_{(n-1)}, \lambda]$ for $n \in N$. Then for all $n \in N$,*

(1) $\lambda_{(n)} \subseteq \lambda_{(n-1)}$
(2) *If λ is a characteristic fuzzy subgroup of G, so is $\lambda_{(n)}$ for all $n \in N$.*

PROOF. (1) Let $x \in G$. If x is not a commutator in G, we have

$$(\lambda, \lambda)(x) = 0 \leq \lambda(x).$$

If x is a commutator in G,

$$\begin{aligned}(\lambda,\lambda)(x) &= \sup_{x=[a,b]} \min\{\lambda(a),\lambda(b)\} \\ &= \sup_{x=[a,b]} \min\{\lambda(a^{-1}),\lambda(b^{-1}),\lambda(a),\lambda(b)\} \\ &\leq \sup_{x=[a,b]} \lambda(a^{-1}b^{-1}ab) = \lambda(x).\end{aligned}$$

This means that $(\lambda,\lambda) \subseteq \lambda$. Since $[\lambda,\lambda]$ is the smallest fuzzy subgroup containing (λ,λ), we have

$$\lambda_{(1)} = [\lambda,\lambda] \subseteq \lambda = \lambda_{(0)},$$

and the theorem is proved for $n = 1$. Now we suppose that $\lambda_{(n)} \subseteq \lambda_{(n-1)}$ for $n = 1,2,3,\ldots,k$. Since $\lambda_{(k)} \subseteq \lambda_{(k-1)}$, it follows easily that $(\lambda_{(k)},\lambda) \subseteq (\lambda_{(k-1)},\lambda)$. Therefore we have

$$\lambda_{(k+1)} = [\lambda_{(k)},\lambda] \subseteq [\lambda_{(k-1)},\lambda] = \lambda_{(k)}.$$

Hence by induction the result holds for all $n \geq 1$.
(2) Since λ is a characteristic fuzzy subgroup of G, we get $\lambda(x) \leq f^{-1}(\lambda)(x) = \lambda(f(x))$ for all $f \in Aut(G)$ and $x \in G$. If x is not a commutator in G, it is trivial that $(\lambda,\lambda)(x) \subseteq f^{-1}((\lambda,\lambda))(x)$. If x is a commutator in G, we have, for all $f \in Aut(G)$ and $x \in G$,

$$\begin{aligned}(\lambda,\lambda)(x) &= \sup_{x=[a,b]} \min\{\lambda(a),\lambda(b)\} \\ &\leq \sup_{x=[a,b]} \min\{\lambda(f(a)),\lambda(f(b))\} \\ &= \sup_{f(x)=[f(a),f(b)]} \min\{\lambda(f(a)),\lambda(f(b))\} \\ &\leq \sup_{f(x)=[c,d]} \min\{\lambda(c),\lambda(d)\} \\ &= (\lambda,\lambda)(f(x)) = f^{-1}((\lambda,\lambda))(x).\end{aligned}$$

Hence $(\lambda,\lambda) \subseteq f^{-1}((\lambda,\lambda))$ for all $f \in Aut(G)$, that is, (λ,λ) is $Aut(G)$-admissible. Therefore it follows from Lemma 9 that $\lambda_{(1)} = [\lambda,\lambda]$ is a characteristic fuzzy subgroup of G. Now we suppose that $\lambda_{(n)}$ is a characteristic fuzzy subgroup of G for $n = 1,2,\ldots,k$. Similarly if x is a commutator in G, for every $f \in Aut(G)$ and $x \in G$,

$$\begin{aligned}(\lambda_{(k)}, \lambda)(x) &= \sup_{x=[a,b]} \min\{\lambda_{(k)}(a), \lambda(b)\} \\ &\leq \sup_{x=[a,b]} \min\{\lambda_{(k)}(f(a)), \lambda(f(b))\} \\ &= \sup_{f(x)=[f(a),f(b)]} \min\{\lambda_{(k)}(f(a)), \lambda(f(b))\} \\ &\leq \sup_{f(x)=[c,d]} \min\{\lambda_{(k)}(c), \lambda(d)\} \\ &= (\lambda_{(k)}, \lambda)(f(x)) = f^{-1}((\lambda_{(k)}, \lambda))(x).\end{aligned}$$

Hence $(\lambda_{(k)}, \lambda) \subseteq f^{-1}((\lambda_{(k)}, \lambda))$ for all $f \in Aut(G)$, that is, $(\lambda_{(k)}, \lambda)$ is $Aut(G)$-admissible. Therefore it follows from Lemma 9 that $\lambda_{(k+1)} = [\lambda_{(k)}, \lambda]$ is a characteristic fuzzy subgroup of G. Hence by induction the proof is completed. □

DEFINITION 11. A fuzzy subgroup λ of G is said to be *of an isolated tip* if $\lambda^{-1}(\lambda(e)) = \{e\}$. In this case, $\lambda(x) = \lambda(e) \Rightarrow x = e$.

DEFINITION 12. Let μ be a fuzzy subgroup of a group G. For any $x \in G$, a map $\hat{\mu}_x : G \longrightarrow [0,1]$ defined by $\hat{\mu}_x(g) = \mu(gx^{-1})$ for all $g \in G$ is called the *fuzzy coset* of G determined by x and μ.

Let μ be a fuzzy normal subgroup of a group G and $\mathcal{F}$ the set of all the fuzzy cosets of μ. Then $\mathcal{F}$ is a group under the composition

$$\hat{\mu}_x \hat{\mu}_y = \hat{\mu}_{xy}$$

for every $x, y \in G$. Define a map $\bar{\mu} : \mathcal{F} \longrightarrow [0,1]$ by

$$\bar{\mu}(\hat{\mu}_x) = \mu(x)$$

for every $x \in G$. Then it is easily proved that $\bar{\mu}$ is a fuzzy subgroup of $\mathcal{F}$ (See Theorem 4.5 of [5].)

THEOREM 13. *With the same notations as above, let μ be a fuzzy normal subgroup of G of an isolated tip. Then μ is a characteristic fuzzy subgroup of G if and only if $\bar{\mu}$ is a characteristic fuzzy subgroup of $\mathcal{F}$.*

PROOF. For $f \in Aut(G)$, we define a map $\phi_f : \mathcal{F} \to \mathcal{F}$ by $\phi_f(\hat{\mu}_x) = \hat{\mu}_{f(x)}$. Then we have

$$\begin{aligned}\phi_f(\hat{\mu}_x\hat{\mu}_y) &= \phi_f(\hat{\mu}_{xy}) = \hat{\mu}_{f(xy)} = \hat{\mu}_{f(x)f(y)} \\ &= \hat{\mu}_{f(x)}\hat{\mu}_{f(y)} = \phi_f(\hat{\mu}_x)\phi_f(\hat{\mu}_y).\end{aligned}$$

It follows from the fact μ is of an isolated tip that

$$\begin{aligned}Ker(\phi_f) &= \{\hat{\mu}_x \in \mathcal{F} | \phi_f(\hat{\mu}_x) = \hat{\mu}_{f(x)} = \hat{\mu}_e\} \\ &= \{\hat{\mu}_e\}.\end{aligned}$$

These facts yield ϕ_f is an automorphism of a group $\mathcal{F}$. On the other hand, let $g \in Aut(\mathcal{F})$. Then, for every $x \in G$, there exists uniquely $g_x \in G$ such that $g(\hat{\mu}_x) = \hat{\mu}_{g_x}$. Hence we can define, for every $g \in Aut(\mathcal{F})$, a map $\psi_g : G \to G$ by $\psi_g(x) = g_x$. Since

$$\begin{aligned}\hat{\mu}_{g_{xy}} &= g(\hat{\mu}_{xy}) = g(\hat{\mu}_x\hat{\mu}_y) \\ &= g(\hat{\mu}_x)g(\hat{\mu}_y) = \hat{\mu}_{g_x}\hat{\mu}_{g_y} = \hat{\mu}_{g_x g_y},\end{aligned}$$

$g_{xy} = g_x g_y$ follows from the fact μ is of an isolated tip. Hence we have

$$\psi_g(xy) = g_{xy} = g_x g_y = \psi_g(x)\psi_g(y),$$

and

$$\begin{aligned}Ker(\psi_g) &= \{x \in G | g(\hat{\mu}_x) = \hat{\mu}_{g_x} = \hat{\mu}_e\} \\ &= \{x \in G | \hat{\mu}_x = \hat{\mu}_e\} \text{ since } g \text{ is a monomorphism.} \\ &= \{e\} \text{ since } \mu \text{ is of an isolated tip}\end{aligned}$$

and ψ_g is clearly surjective. Hence we can deduce an automorphism ψ_g of a group G from an automorphism g of a group $\mathcal{F}$. Now we suppose that μ is a characteristic fuzzy subgroup of G. Then we have, for every $g \in Aut(\mathcal{F})$ and $\hat{\mu}_x \in \mathcal{F}$,

$$\begin{aligned}\bar{\mu}(\hat{\mu}_x) &= \mu(x) \leq \mu(\psi_g(x)) \text{ since } \mu \text{ is characteristic} \\ &= \mu(g_x), \text{ where } g(\hat{\mu}_x) = \hat{\mu}_{g_x} \\ &= \bar{\mu}(\hat{\mu}_{g_x}) = \bar{\mu}(g(\hat{\mu}_x)),\end{aligned}$$

which implies that $\bar{\mu}$ is a characteristic fuzzy subgroup of a group $\mathcal{F}$. Conversely, we suppose that $\bar{\mu}$ is a characteristic fuzzy subgroup of $\mathcal{F}$. Then we have, for every $f \in Aut(G)$ and $x \in G$,

$$\begin{aligned} \mu(x) &= \bar{\mu}(\hat{\mu}_x) \\ &\leq \bar{\mu}(\phi_f(\hat{\mu}_x)) \text{ since } \bar{\mu} \text{ is characteristic} \\ &= \bar{\mu}(\hat{\mu}_{f(x)}) = \mu(f(x)) \end{aligned}$$

which proves that μ is a characteristic fuzzy subgroup of a group G. □

THEOREM 14. *Let λ be a fuzzy characteristic subgroup of a group G, then so is the core $Core_G\lambda$ of λ.*

PROOF. Let $g \in G$ and $t \in [0,1]$.

$$\begin{aligned} g \in \left(\underset{x\in G}{\cap}\lambda_x^*\right)_t &\Leftrightarrow \inf\left\{\lambda(x^{-1}gx) | x \in G\right\} = \left(\underset{x\in G}{\cap}\lambda_x^*\right)(g) \geq t \\ &\Leftrightarrow \lambda(x^{-1}gx) \geq t \text{ for all } x \in G \\ &\Leftrightarrow x^{-1}gx \in \lambda_t \text{ for all } x \in G \\ &\Leftrightarrow g \in x\lambda_t x^{-1} \text{ for all } x \in G \\ &\Leftrightarrow g \in \underset{x\in G}{\cap} x\lambda_t x^{-1}. \end{aligned}$$

Therefore we have $\left(\underset{x\in G}{\cap}\lambda_x^*\right)_t = \underset{x\in G}{\cap} x\lambda_t x^{-1}$ for every $t \in [0,1]$. Since λ is fuzzy characteristic, λ_t is a characteristic subgroup of G by Proposition 6, and so a normal subgroup of G. Hence we get

$$f(x\lambda_t x^{-1}) = f(\lambda_t) \subseteq \lambda_t = x\lambda_t x^{-1}$$

for every $f \in Aut(G)$, which implies that $x\lambda_t x^{-1}$ is a characteristic subgroup of G for every $x \in G$. Since the intersection of a family of characteristic subgroups of a group is characteristic, $\left(\underset{x\in G}{\cap}\lambda_x^*\right)_t = \underset{x\in G}{\cap} x\lambda_t x^{-1}$ is a characteristic subgroup of a group G for all $t \in [0,1]$. Therefore, again by Proposition 6, $\left(\underset{x\in G}{\cap}\lambda_x^*\right) = Core_G\lambda$ is a characteristic fuzzy subgroup. □

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