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π_2 UNDER TIETZE TRANSFORMATIONS

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ABSTRACT. We study how the second homotopy modules of group presentations are transformed by Tietze transformations and discuss some application

1. Introduction

Let $\mathcal{P} = \langle \mathbf{x} \,;\, \mathbf{r} \rangle$ be a group presentation. G is (isomorphic to) F/N, where F is the free group on $\mathbf{x}$ and N is the normal closure of $\mathbf{r}$ in F. The relation module of G is the abelianization N/N' of N regarded as a left $\mathbb{Z}G$-module, with G- action given by $WN \cdot UN' = WUW^{-1}N'$ ($W \in F$, $U \in N$) Let P be the free left $\mathbb{Z}G$- module with basis $\{t_R \mid R \in \mathbf{r}\}$ and consider the epimorphism

$$\phi : P \longrightarrow N/N', \quad t_R \mapsto rN'.$$

Then there is a short exact sequence

$$0 \longrightarrow \pi_2(\mathcal{P}) \longrightarrow P \longrightarrow N/N' \longrightarrow 0,$$

where $\pi_2(\mathcal{P})$ is the module of Peiffer equivalence classes $\langle \sigma \rangle$ of identity sequences σ over P and $\pi_2(\mathcal{P}) \longrightarrow P$ is the evaluation map $\langle \sigma \rangle \mapsto eval(\sigma)$ Identity sequence over P can be represented geometrically by objects called *spherical pictures* [4]. There are certain operations on spherical pictures called bridge moves, insertions and deletions of floating circles, and insertions and deletions of cancelling pairs. Two spherical pictures are said to be *equivalent* if one can be

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transformed to the other by a finite number of the above operations. More generally, if E is a set of spherical pictures then two pictures are said to be *equivalent* (rel E) if one can be transformed to the other by a finite number of the above operations, and deletions and insertions of E- pictures [4].

Given a presentation $\mathcal{P} = \langle \mathbf{x} \ ; \ \mathbf{r} \rangle$, each Tietze transformation T_1, T_2 transforms it into a presentation $\mathcal{Q}$ in accordance with the following definition.

(T_1) If S is a consequence of $\mathbf{r}$ then let $\mathcal{Q} = \langle \mathbf{x}; \ \mathbf{r}, \ S \rangle$.

(T_2) If y is a symbol not in $\mathbf{x}$ and U is a word on $\mathbf{x}$, then let

$$\mathcal{Q} = \langle \mathbf{x}, \ y; \ \mathbf{r}, y^{-1}U \rangle.$$

To investigate the structure of the second homotopy module of a given group presentation is very important and useful [4]. And also the Tietze transformations are basis and applicable [2]. In this paper we study how the second homotopy modules of group presentations are transformed by Tietze transforms and discuss some application.

2. Main Theorems

LEMMA 2.1. *Suppose $\mathcal{P}_2 = \langle \mathbf{s} \ ; \ \mathbf{r}, \ S \rangle$ is obtained from $\mathcal{P}_1 = \langle \mathbf{x} \ ; \ \mathbf{r} \rangle$ by T_1 where S is a consequence of $\mathbf{r}$. Then*

$$\pi_2(\mathcal{P}_2) \cong \pi_2(\mathcal{P}_1) \oplus \mathbb{Z}G,$$

where G is the group defined by $\mathcal{P}_1$.

Proof. Let X be a generating set for $\pi_2(\mathcal{P}_1)$. Since S is a consequence of $\mathbf{r}$, S is freely equal to a product

$\Pi_{i=1}^{n} W_i R_i^{\epsilon_i} W_i^{-1}$ $(R_i \in \mathbf{r}, \ \epsilon_i = \pm 1, \ W_i$ a word on $\mathbf{x}, \ i = 1, \ 2, ..., n)$.

Then there is a picture $\mathbb{D}_S$ over $\mathcal{P}_1$ which consists of R_i- discs and $\mathbf{x}$-arcs, and $\partial \mathbb{D}_S = S$ Now we can construct a spherical picture $\mathbb{P}_S$ over $\mathcal{P}_2$ of the form dipicted in Figure 1,

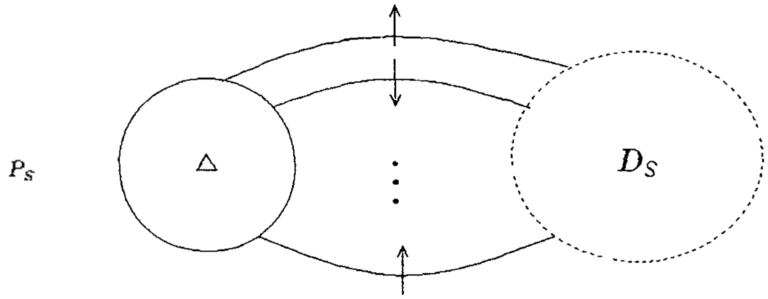

FIGURE 1

where △ is a S- disc

Suppose a reduced spherical picture $\mathbb{P}$ over $\mathcal{P}_2$ has some S-discs. We draw a simple closed curve β such that β encloses only one S-disc. Next we insert an element $\bar{\mathbb{P}}_S$ of the set of all stereographic projections of $\mathbb{P}_s$ and their mirror images in inside β. By bridges moves, the S-disc inside β and the S-discs of $\bar{\mathbb{P}}_S$ made a cancelling pair which can be removed The subpicture of $\mathbb{P}$ which is outside β and $\mathbb{D}_S$ of $\bar{\mathbb{P}}_S$ make another spherical picture $\mathbb{P}'$ over $\mathcal{P}_2$ with one fewer S- discs. We can repeat the above argument with $\mathbb{P}'$ in place of $\mathbb{P}$ and so on We continue the above procedure until we get a picture $\hat{\mathbb{P}}$ without S- discs. Since we can consider as a spherical picture over $\mathcal{P}_1$, $\hat{\mathbb{P}}$ is equivalent (rel X) to the empty picture. So $\pi_2(\mathcal{P}_2)$ is generated by $X \cup \{\mathbb{P}_S\}$

Let $\langle X \rangle$ be the submodule of $\pi_2(\mathcal{P}_2)$ generated by X. Consider the following diagram

$$\begin{array}{ccc} \pi_2(\mathcal{P}_1) & \xrightarrow{\mu_1} & \oplus_{R\in\mathbf{r}}\mathbb{Z}Gt_R \\ & & \downarrow \iota \\ \pi_2(\mathcal{P}_2) & \xrightarrow{\mu_2} & \oplus_{R\in\mathbf{r}}\mathbb{Z}Gt_R \oplus \mathbb{Z}Gt_S \end{array}$$

where μ_1, μ_2 are evaluation maps and ι is an embedding. Since the image of $\langle X \rangle$ under μ_2 lies in $\oplus_{R\in r}\mathbb{Z}Gt_R$ and the image of $\langle \mathbb{P}_S \rangle$ under μ_2 has the form $\xi_s - t_s$ where $\xi_s \in \oplus_{R\in r}\mathbb{Z}Gt_R$, the images of $\langle X \rangle$ and $\langle \mathbb{P}_S \rangle$ under are mutually disjoint. So $\langle X \rangle$ and $\langle \mathbb{P}_S \rangle$ mutually are disjoint in $\pi_2(\mathcal{P}_2)$ because μ_2 is injective. Thus

$$\pi_2(\mathcal{P}_2) \cong \langle X \rangle \oplus \langle \mathbb{P}_S \rangle \cong \pi_2(\mathcal{P}_1) \oplus \mathbb{Z}G.$$

□

In the case T_2, we will consider a more general situation.

Let $\mathcal{P} = \langle \mathbf{y} \; ; \; \mathbf{t} \rangle$ be a group presentation defining a group $G = F/N$. Let $\mathcal{P}_0 = \langle \mathbf{y_0} \; ; \; \mathbf{t_0} \rangle$ be a *full* subpresentation of $\mathcal{P}$ (i.e., $\mathbf{y_0}$ is a subset of $\mathbf{y}$ and $\mathbf{t_0}$ consists of all relators involving $\mathbf{y_0}$). Let $G_0 = F_0/N_0$ be the group defined by $\mathcal{P}_0$. We say that $\mathcal{P}_0$ is an *injective* subpresentation of $\mathcal{P}$ if the natural map $G_0 \longrightarrow G$ is injective. Let X_0 be the set of all spherical pictures over $\mathcal{P}_0$. If $\mathbb{P}$ is a spherical picture over $\mathcal{P}_0$ then the element of $\pi_2(\mathcal{P}_0)$ represented by $\mathbb{P}$ will be denoted by $\langle \mathbb{P} \rangle_0$. Of course, $\mathbb{P}$ also represents an element of $\pi_2(\mathcal{P})$, which will be denoted by $\langle \mathbb{P} \rangle$.

THEOREM 2.2. *If $\mathcal{P}_0$ is an injective subpresentation of $\mathcal{P}$ then the submodule of $\pi_2(\mathcal{P})$ generated by X_0 is isomorphic to $\mathbb{Z}G \otimes_{G_0} \pi_2(\mathcal{P}_0)$ under the map*

$$\langle \mathbb{P} \rangle \longrightarrow 1 \otimes \langle \mathbb{P} \rangle_0 \quad (\mathbb{P} \in X_0).$$

Proof From [4], we get the standard injections

$$\mu_2 \cdot \pi_2(\mathcal{P}) \longrightarrow (\oplus_{T \in t_0} \mathbb{Z}G t_T) \oplus (\oplus_{S \in t/t_0} \mathbb{Z}G t_S)$$
$$\mu_2^0 \quad \pi_2(\mathcal{P}_0) \longrightarrow \oplus_{T \in t_0} \mathbb{Z}G_0 \bar{t_T}$$

If we apply $\mathbb{Z}G \otimes_{G_0} -$, then we get an embedding

$$1 \otimes \mu_2^0 \; : \; \mathbb{Z}G \otimes_{G_0} \pi_2(\mathcal{P}_0) \longrightarrow \oplus_{T \in t_0} \mathbb{Z}G t_T$$

where t_T is identified with $1 \otimes \bar{t_T}$.

Let $\langle X_0 \rangle$ be the submodule of $\pi_2(\mathcal{P})$ generated by X_0. Then

$$1 \otimes \mu_2^0(\mathbb{Z}G \otimes_{G_0} \pi_2(\mathcal{P})) = \mu_2(\langle X_0 \rangle).$$

Since $1 \otimes \mu_2^0$ and μ_2 are injective, we get

$$\langle X_0 \rangle \cong \mathbb{Z}G \otimes_{G_0} \pi_2(\mathcal{P})$$

where the isomorphism is the composition of μ_2 and $(1 \otimes \mu_2^0)^{-1}$.

□

COROLLARY 2.3. *Suppose that* $\mathcal{P}_2 = \langle \mathbf{x},\ \mathbf{y}\ ;\ \mathbf{r},\ y^{-1}U_y \rangle$ *is obtained from* $\mathcal{P} = \langle \mathbf{x}\ ;\ \mathbf{r} \rangle$ *by an operation* T_2, *where* y *is a symbol not in* $\mathbf{x}$ *and* U_y $(y \in \mathbf{y})$ *is a word on* $\mathbf{x}$. *Then*

$$\pi_2(\mathcal{P}_2) \cong \pi_2(\mathcal{P}_1).$$

Proof. Since every reduced spherical picture over $\mathcal{P}_2$ has no $y^{-1}U_y$-discs $(y \in \mathbf{y})$, $\langle X_1 \rangle = \pi_2(\mathcal{P}_2)$ where X_1 is the set all spherical pictures over $\mathcal{P}_1$. By Theorem 2 2, $\pi_2(\mathcal{P}_2) \cong \mathbb{Z}G_2 \otimes_{G_1} \pi_2(\mathcal{P}_1)$. But $\mathbb{Z}G_2 \otimes_{G_1} \pi_2(\mathcal{P}_1) \cong \pi_2(\mathcal{P}_1)$ because $G_1 \cong G_2$. So we get the result.

□

THEOREM 2.4. *Let a group* G *be defined by the following two finite presentations*

$$\mathcal{P}_1 = \langle a_1,\ \dots,\ a_n\ ,\ R_1,\ \dots,\ R_m \rangle$$
$$\mathcal{P}_2 = \langle b_1,\ \dots,\ b_k\ ;\ S_1,\ \dots,\ S_p \rangle$$

where $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are disjoint. Then*

$$\pi_2(\mathcal{P}_1) \oplus (\oplus_{p+n}\mathbb{Z}G) \cong \pi_2(\mathcal{P}_2) \oplus (\oplus_{m+k}\mathbb{Z}G).$$

Proof. The first part of our proof is taken from [2, Theorem 1.5]. Let $\mathbf{a} = \{a_1, \dots, a_n\}$, $\mathbf{r} = \{R_1, \dots, R_m\}$, $\mathbf{b} = \{b_1, \dots, b_k\}$, $\mathbf{s} = \{S_1, \dots, S_p\}$. Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are presentations under the functions $a_i \mapsto g_i (i = 1, \dots, n)$ and $b_j \mapsto h_j (j = 1, \dots, k)$ respectively. Since $h_j \in G$, we can express h_j in terms of $g_1, \dots, g_n$ So we get

$$h_1 = B_1(g_i), \dots, h_k = B_k(g_i)$$

By applying T_2 k-times, we get the presentation

$$\mathcal{P}_3 = \langle \mathbf{a},\ \mathbf{b}\ ,\ \mathbf{r},\ b_1 = B_1(a_i), \dots, b_k = B_k(a_i) \rangle.$$

We note that each S_r $(r = 1, \dots, p)$ is a consequence of relators of $\mathcal{P}_3$. Thus by applying T_1 p-times we get

$$\mathcal{P}_4 = \langle \mathbf{a},\ \mathbf{b}\ ;\ \mathbf{r},\ \mathbf{s},\ b_1 = B_1(a_i), \dots,\ b_k = B_k(a_i) \rangle.$$

Expressing $g_1, \dots, g_n$ in terms of $h_1, \dots, h_k$, we get $g_1 = A_1(h_j), \dots, g_n = A_n(h_j)$. So we get $a_1 = A_1(b_j), \dots, a_n = A_n(b_j)$. By applying T_1 n-times, we get the presentation

$$\mathcal{P}^* = \langle \mathbf{a},\ \mathbf{b}\ ;\ \mathbf{r},\ \mathbf{s},\ b_1 = B_1(a_i), \ldots,\ b_k = B_k(a_i),$$
$$a_1 = A_1(b_j), \ldots,\ a_n = A_n(b_j)\rangle.$$

Similarly, we can get $\mathcal{P}^*$ from $\mathcal{P}_2$ by applying T_2 n-times and then T_1 $(m+k)$-times. Therefore by Proposition 2.1 and Corollary 2.3, we get our result. □

Now we consider the relation module case. There are already alternative proofs, for example [3], but our result gives us the rank of the free module explicitly

THEOREM 2.5. (i) If $\mathcal{P}_1$ and $\mathcal{P}_2$ are the same as in Lemma 2.1, then

$$M(\mathcal{P}_1) = M(\mathcal{P}_2).$$

(ii) If $\mathcal{P}_1$ and $\mathcal{P}_2$ are the same as in Corollary 2.3, then

$$M(\mathcal{P}_2) \cong M(\mathcal{P}_1) \oplus (\oplus_{|\mathbf{y}|}\mathbb{Z}G_2).$$

Proof. (i) It is clear because the normal closures of $\mathbf{r}$ and $\mathbf{r} \cup \mathbf{s}$ in the free group on $\mathbf{x}$ are the same.

(ii) Consider the following diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_2(\mathcal{P}_1) & \xrightarrow{\mu_2^{(1)}} & \oplus_{R\in\mathbf{r}}\mathbb{Z}G_1\bar{t_R} & \longrightarrow & M(\mathcal{P}_1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\phi} & & & & \\
0 & \longrightarrow & \pi_2(\mathcal{P}_2) & \xrightarrow{\mu_2^{(2)}} & (\oplus_{R\in\mathbf{r}}\mathbb{Z}G_2\bar{t_R}) \oplus (\oplus_{y\in\mathbf{y}}\mathbb{Z}G_2 t_y) & \longrightarrow & M(\mathcal{P}_2) & \longrightarrow & 0
\end{array}$$

where ϕ is an embedding given by $\bar{t_R} \mapsto t_R$ (since $G_1 \cong G_2$) and α is the isomorphism in Corollary 2.3. Then we have

$$\begin{aligned}
M(\mathcal{P}_1) &\cong \text{coker}\ \ \mu_2^{(1)} \\
M(\mathcal{P}_2) &\cong \text{coker}\ \ \mu_2^{(2)} \\
&\cong (\oplus_{R\in r}\mathbb{Z}G_2 t_R/\text{Im}\ \ \mu_2^{(2)}) \oplus (\oplus_{y\in\mathbf{y}}\mathbb{Z}G_2 t_y).
\end{aligned}$$

Since Im $\mu_2^{(1)} \cong$ Im $\mu_2^{(2)}$ and $G_1 \cong G_2$, we have an induced isomorphism

$$\oplus_{R\in r}\mathbb{Z}G_1\bar{t_R}/\text{Im}\quad \mu_2^{(1)} \longrightarrow \oplus_{R\in r}\mathbb{Z}G_2t_R/\text{Im}\quad \mu_2^{(2)}.$$

So,

$$\begin{aligned} M(\mathcal{P}_2) &\cong (\oplus_{R\in r}\mathbb{Z}G_1\bar{t_R}/\text{Im}\quad \mu_2^{(1)}) \oplus (\oplus_{y\in \mathbf{y}}\mathbb{Z}G_2t_y) \\ &\cong M(\mathcal{P}_1) \oplus (\oplus_{y\in \mathbf{y}}\mathbb{Z}G_2t_y) \end{aligned}$$

□

COROLLARY 2.6. *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are the same as in Theorem 2.4, then*

$$M(\mathcal{P}_1) \oplus (\oplus_k\mathbb{Z}G) \cong M(\mathcal{P}_2) \cong (\oplus_n\mathbb{Z}G)$$

Proof. By Theorem 2.5 and the proof of Theorem 2 4 □

Remark. *Theorem 2.4 is not cancellative.*

We consider two presentations of the cyclic group of order 6.

$$\begin{aligned} \mathcal{P} &= \langle t,\ t^6\rangle \\ \mathcal{P}' &= \langle a,\ b;\ a^2,\ b^2,\ ab = ba\rangle \end{aligned}$$

Then we can get

$$\mathcal{P}^* = \langle t,\ a,\ b\ ;\ t^6,\ a = t^3,\ b = t^2,\ a^2,\ b^3,\ ab = ba,\ t = ab^{-1}\rangle$$

from $\mathcal{P}$ and $\mathcal{P}'$ by two T_2's, three T_1's and one T_2, three T_1's, respectively. Thus we get

$$\begin{aligned} \pi_2(\mathcal{P}) \oplus (\mathbb{Z}G)^4 &\cong \pi_2(\mathcal{P}') \oplus (\mathbb{Z}G)^3, \\ M(\mathcal{P}) \oplus (\mathbb{Z}G)^2 &\cong M(\mathcal{P}') \oplus \mathbb{Z}G. \end{aligned}$$

If the result of Theorem 2.4 was cancellative, then we would have

$$\pi_2(\mathcal{P}) \oplus \mathbb{Z}G \cong \pi_2(\mathcal{P}').$$

In particular, $\pi_2(\mathcal{P}')$ would be generated by two elements, because $\pi_2(\mathcal{P})$ is generated by only one picture. By Theorem [1] we can get a generating set of $\pi_2(\mathcal{P}')$ which consists of four free elements Therefore it is not cancellative.

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