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EXISTENCE OF SOLUTIONS OF NONLINEAR DELAY INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITIONS IN BANACH SPACES

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ABSTRACT In this paper we prove the existence of mild and strong solutions of nonlinear delay integrodifferential equations with nonlocal conditions The results are obtained by using the Schauder fixed point theorem and resolvent operator

1. Introduction

Byszewski([3]) has studied the existence and uniqueness of mild, strong and classical solutions of the following nonlocal Cauchy problem :

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = f(t, u(t)), \quad t \in [0, a] \\ u(t_0) + g(t_1, t_2, \cdots, t_p, u(\cdot)) = u_0 \end{cases}$$

where $0 \leq t_0 < t_1 < \cdots, t_p \leq a, -A$ is the infinitesimal generator of a C_0-semigroup in a Banach space X, $u_0 \in X$ and $f : [0, a] \times X \to X$, $g : [0, a]^p \times X \to X$ are given functions. Subsequently, he has investigated the same type of problem for different type of evolution

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equations in Banach spaces ([4,5]). Ntouyas and Tsamatos ([11]) has established the global existence of semilinear evolution equations with nonlocal conditions. Balachandran and Ilamaran ([2]), Balachandran, Park and Kwun ([1]) have studied the nonlocal Cauchy problem for nonlinear integrodifferential equations of Sobolev type. In Desh ([7]), he consider the following integrodifferential equation

$$\begin{aligned} &\frac{dx(t)}{dt} = A[x(t) + \int_0^t F(t-s)x(s)]ds + f(t, x(t)), t \geq 0 \\ &x(0) = x_0. \end{aligned}$$

These type of equation also occurs in the study of viscoelastic beams and thermo-viscoelasticity.

In this paper we consider the following integrodifferential equation with nonlocal condition of the form :

$$\text{(E)} \quad \begin{cases} \dfrac{dx(t)}{dt} = A[x(t) + \displaystyle\int_0^t F(t-s)x(s)ds] + f(t, x_t) \\ \qquad + \displaystyle\int_0^t g(t, s, x_s, \int_0^s k(s, \tau, x_\tau)d\tau)ds, \quad t \in [0, T] = J \\ x(t) + h(x_{t_1}, x_{t_2}, \cdots, x_{t_p})(t) = \phi(t), \ t \in [-r, 0] \end{cases}$$

where A generates a strongly continuous semigroup in a Banach space X, $F(t)$ is a bounded operator for $t \in J$

$$\begin{aligned} &f : J \times C([-r, 0] : X) \to X, \\ &g : J \times J \times C([-r, 0] : X) \times X \to X, \\ &k : J \times J \times C([-r, 0] : X) \to X, \\ &h : C([-r, 0] : X)^p \to X. \end{aligned}$$

We prove the existence of mild and strong solutions of integrodifferential equations with nonlocal conditions in Banach spaces. The results are obtained by using the Schauder fixed point theorem and the resolvent operator properties.

2. Existence of mild solutions

In this section, we consider the following integrodifferential equation with nonlocal conditions of the form :

$$(2.1)\quad \begin{cases} \dfrac{dx(t)}{dt} = A[x(t) + \displaystyle\int_0^t F(t-s)x(s)ds] + f(t, x_t) \\ \quad + \displaystyle\int_0^t g(t, s, x_s, \int_0^s k(s, \tau, x_\tau)d\tau)ds, \quad t \in [0, T] = J \\ x(t) + h(x_{t_1}, x_{t_2}, \cdots, x_{t_p})(t) = \phi(t) \end{cases}$$

where A generates a strongly continuous semigroup in a Banach space X,

$$\begin{aligned} &f : J \times C([-r, 0] : X) \to X, \\ &g : J \times J \times C([-r, 0] : X) \times X \to X, \\ &k : J \times J \times C([-r, 0] : X) \to X, \\ &h : C([-r, 0] : X)^p \to X \end{aligned}$$

are given functions, $F(t) : Y \to Y$ and for $x(\cdot)$ continuous in Y, $AF(\cdot)x(\cdot) \in L^1(J, X)$, where Y is the Banach space formed from $D(A)$, the domain of A, endowed with the graph norm. For $x \in X$, $F'(t)x$ is continuous in $t \in J$. Then there exists a unique resolvent operator for the equation.

$$(2.2)\qquad \frac{dx(t)}{dt} = A[x(t) + \int_0^t F(t-s)x(s)ds]$$

The resolvent operator $R(t) \in B(X)$ for $t \in J$ satisfies the following conditions ([8]) :

(a) $R(0) = I$ (the identity operator on X),

(b) for all $x \in X$, $R(t)x$ is continuous for $t \in J$,

(c) $R(t) \in B(Y)$, $t \in J$. For $y \in Y$, $R(t)y \in C^1([0,T],X) \cap C([0,T],X)$ and

$$
\begin{aligned}
\frac{d}{dt}R(t)y &= A[R(t)y + \int_0^t F(t-s)R(s)yds] \\
&= R(t)Ay + \int_0^t R(t-s)AF(s)yds, \quad t \in J.
\end{aligned}
\tag{2.3}
$$

In Grimmer ([8]), we have studied the existence and uniqueness of solutions via variation of constants formula and other properties of resolvent operators. In this paper we shall study the existence of mild solution and strong solution of the integrodifferential equation by utilizing the techniques developed by Pazy ([12]) and Byszewski ([3]).

Let $Y = C(J,X)$ and define the sets $X_r = \{x \in X : ||x|| \le r\}$ and $Y_r = \{y \in Y : ||y|| \le r\}$ where the constant r is defined below. Assume the following conditions :

(H1) The resolvent operator $R(t)$ is compact and there exists a constant $M_1 > 0$ such that $||R(t)|| \le M_1$.

(H2) Nonlinear operator $f : J \times C([-r,0] : X) \to X$, $k : J \times J \times C([-r,0] : X) \to X$ are continuous and there exist constants $M_2 > 0$, $M_3 > 0$ such that

$$
\begin{aligned}
&||f(t,x_t)|| \le M_2, \quad t \in J, \quad x_t \in X_r, \\
&||g(t,s,x_s,y(s))|| \le M_3, \quad (t,s) \in J \times J, \quad x_s,\ y \in X_r, \\
&||k(t,s,x_s)|| \le N, \quad (t,s) \in J \times J, \quad x_s \in X_r.
\end{aligned}
$$

(H3) $h : C([-r,0] : X)^p \to X$ is continuous and there exist a constant $H > 0$ such that

$$
||h(x_{t_1},\cdots,x_{t_p})(t)|| \le H, \quad x_{t_i} \in Y_r \ (i = 1,\cdots,p)
$$

and

$$
\begin{aligned}
&h(\alpha x_{t_1} + (1-\alpha)y_{t_1},\cdots,\alpha x_{t_p} + (1-\alpha)y_{t_p})(t) \\
&\qquad = \alpha h(x_{t_1},\cdots,x_{t_p})(t) + (1-\alpha)h(y_{t_1},\cdots,y_{t_p})(t), \\
&\qquad\qquad x_{t_i}, y_{t_i} \in Y_r \ (i = 1,\cdots,p).
\end{aligned}
$$

(H4) The set $\{y(0) : y \in Y_r,\ y(0) = x_0 - h(y_{t_1}, \cdots, y_{t_p})(0)\}$ is precompact in X.

To simplify the notation, let us take

$$Q(t) = \int_0^t k(t, s, x_s)ds.$$

Definition 2.1. A continuous solution $x(t)$ of the integral equation

$$x(t) = x_t(0) = R(t)[\phi(0) - h(x_{t_1}, \cdots, x_{t_p})(0)] + \int_0^t R(t-s)f(s, x_s)ds + \int_0^t R(t-s)\int_0^s g(s, \tau, x_\tau, Q(\tau))d\tau ds$$

is called a mild solution of the problem (2.1).

Theorem 2.1. Assume that the hypotheses $(H1) \sim (H4)$ hold. Then the problem (2.1) has a mild solution on J.

Proof. Let us define the Y_0 in Y by

$$Y_0 = \{\eta \in C([-r, T] : X) : \eta(0) = \phi(0) - h(x_{t_1}, \cdots, x_{t_p})(0), \\ \|\eta - \phi + h(x_{t_1}, \cdots, x_{t_p})\|_C \leq r, \quad 0 \leq t \leq T\}$$

where $r > 0$ and $TM_1(M_2 + TM_3) < \frac{r}{5}$. Clearly, Y_0 is a bounded closed convex subset of Y. For $x_t \in C([0, T], X)$, defined a mapping $\hat{x}_t : [-r, T] \to X$ by

$$\hat{x}_t = \begin{cases} \phi(t) - h(x_{t_1}, \cdots, x_{t_p})(t), & -r \leq t \leq 0 \\ x_t, & 0 \leq t \leq T. \end{cases}$$

Define a mapping $\Psi : Y \to Y_0$ by

$$\Psi x_t(0) = R(t)[\phi(0) - h(x_{t_1}, \cdots, x_{t_p})(0)] + \int_0^t R(t-s)f(s, x_s)ds + \int_0^t R(t-s)\int_0^s g(s, \tau, x_\tau, Q(\tau))d\tau ds$$

For $\Psi\hat{x}_t \in Y_0$, if $-r \leq t+\theta \leq 0$, since

$$\Psi\hat{x}_t(\theta) = \phi(t+\theta) - h(x_{t_1}, \cdots, x_{t_p})(t+\theta)$$

we obtain

$$||\Psi\hat{x}_t - \phi + h(x_{t_1}, \cdots, x_{t_p})||_C \leq r.$$

If $0 < t+\theta \leq T$, then we obtain

$$\begin{aligned}
&||\Psi\hat{x}_t(\theta) - \phi(\theta) + h(x_{t_1}, \cdots, x_{t_p})(\theta)|| \\
&= ||R(t+\theta)[\phi(0) - h(x_{t_1}, \cdots, x_{t_p})(0)] \\
&\quad + \int_0^{t+\theta} R(t+\theta-s)f(s, x_s)ds + \int_0^{t+\theta} R(t+\theta-s) \\
&\quad \int_0^s g(s,\tau,Q(\tau))d\tau ds - \phi(\theta) + h(x_{t_1}, \cdots, x_{t_p})(\theta)|| \\
&\leq \frac{r}{5} + \frac{r}{5} + \frac{r}{5} + \frac{r}{5} + TM_1(M_2 + TM_3) < r
\end{aligned}$$

Since $||\Psi\hat{x}_t - \phi + h(x_{t_1}, \cdots, x_{t_p})||_C \leq r$, Ψ maps Y_0 into Y_0. Further the continuity of Ψ from Y_0 into Y_0 follows from the fact that f, g, k and h are continuous.

We prove that the set $Y_0(t) = \{\Psi x_t : x_t \in Y_0\}$ is precompact in X, for every fixed t, $0 \leq t \leq T$. For $t = 0$, the set $Y_0(0)$ is precompact in X. Let $t > 0$ be fixed. Define, for $0 < \epsilon < t$

$$\begin{aligned}
\Psi_\epsilon x_t(0) = R(t)[\phi(0) - h(x_{t_1}, \cdots, x_{t_p})(0)] \\
+ \int_0^{t-\epsilon} R(t-s)[f(s,x_s) + \int_0^s g(s,\tau,x_\tau,Q(\tau))d\tau]ds.
\end{aligned}$$

Since $R(t)$ is compact for every $t > 0$, the set

$$Y_\epsilon(t) = \{\Psi_\epsilon x_t | x_t \in Y_0\}$$

is precompact in X for every ϵ, $0 < \epsilon < t$. Further, for $x \in Y_0$, we have

$$\begin{aligned}
&||\Psi x_t(\theta) - \Psi_\epsilon x_t(\theta)|| \\
&\leq || \int_{t-\epsilon}^{t} R(t-s)[f(s,x_s) + \int_0^s g(s,\tau,x_\tau,Q(\tau))d\tau]ds|| \\
&\leq M_1(M_2 + TM_3)\epsilon
\end{aligned}$$

which implies that $Y_0(t)$ is totally bounded, that is, $Y_0(t)$ is precompact in X.

We shall show that $\Psi(Y_0) = \{(\Psi x) : x \in Y_0\}$ is an equicontinuous family of functions. For $0 < t < s < T$, we have

$$\begin{aligned}
&||\Psi x_t(\theta) - \Psi x_s(\theta)|| \\
&\quad \leq ||(R(t+\theta) - R(s+\theta))\phi(0)|| \\
&\quad + ||(R(t+\theta) - R(s+\theta))h(x_{t_1}, \cdots, x_{t_p})(0)|| \\
&\quad + || \int_0^{t+\theta} (R(t+\theta-\tau) - R(s+\theta-\tau))[f(\tau,x_\tau) \\
&\quad + \int_0^\tau g(\tau,p,x_p,Q(p))d_p]d\tau|| \\
&\quad + || \int_{t+\theta}^{s+\theta} R(s+\theta-\tau)[f(\tau,x_\tau) + \int_0^\tau g(\tau,p,x_p,Q(p))d_p]d\tau|| \\
&\quad \leq ||R(t+\theta) - R(s+\theta)||(||\phi(0)|| + H) \\
&\quad + (M_2 + TM_3) \int_0^{t+\theta} ||R(t+\theta-\tau) - R(s+\theta-\tau)||d\tau \\
&\quad + |s-t|M_1(M_2 + TM_3)
\end{aligned}$$

The right hand side of the above inequality is independent of $x \in Y_0$ and tends to zero as $s \to t$ It is also clear that $\Psi(Y_0)$ is bounded in Y. Thus by Arzela-Ascoli's theorem, $\Psi(Y_0)$ is precompact. Hence by fixed point theorem, Ψ has a fixed point in Y_0 and any fixed point of Ψ is a mild solution of the nonlocal Cauchy problem (2.1).

3. Existence of the strong solutions

In this section, we prove the strong solution of the equation (2.1).

Definition 3.1. A function $x(t)$ is said to be a strong solution of the problem on J if $x(t)$ is differentiable almost everywhere on J, $x'(t) \in L^1(J, X)$ and satisfies

$$\begin{cases} \dfrac{dx}{dt} = A[x(t) + \displaystyle\int_0^t F(t-s)x(s)ds] + f(t, x_t) \\ \qquad + \displaystyle\int_0^t g(t, s, x_s, \int_0^s k(s, \tau, x_\tau)d\tau)ds, \\ x(t) + h(x_{t_1}, \cdots, x_{t_p})(t) = \phi(t) \qquad \text{a.e. on } J. \end{cases}$$

Further assume that

(H5) X is a reflexive Banach space,

(H6) $f : J \times C([-r, 0], X) \to X$ is a continuous in t on J and there exists a constant $M_4 > 0$ such that

$$||f(t, x_t) - f(s, y_s)|| \leq M_4(|t-s| + ||x_t - y_s||_{C([-r,0]\, X)}),$$
$$t, s \in J, \quad x_t, y_t \in X_r,$$

(H7) $g : J \times J \times C([-r, 0] : X) \times X \to X$ is a continuous in t on J and there exists a constant $M_5 > 0$ such that

$$||g(t, \tau, x_\tau, y) - g(s, \tau, x_\tau, y)|| \leq M_5|t-s|, \quad x_\tau, y \in X_r,$$

(H8)

$$\phi(t) \in C([-r, 0] : D(A)),$$
$$h(x_{t_1}, \cdots, x_{t_p})(t) \in C([-r, 0] : D(A)).$$

Theorem 3.1. Assume that conditions $(H1) \sim (H8)$ hold. If the function $x(t)$ is the unique mild solution of the problem (2.1), then $x(t)$ is the unique strong solution of the problem (2.1).

Proof. We shall show that $x(t)$ is a strong solution of the problem (2.1) on J. For any $t \in J$, we have

$$
\begin{aligned}
&x_{t+\delta} - x_t \\
&= (R(t+\delta) - R(t))[\phi(0) - h(x_{t_1}, \cdots, x_{t_p})(0)] \\
&\quad + \int_0^\delta R(t+\delta-s)[f(s,x_s) + \int_0^s g(s,\tau,x_\tau,Q(\tau))d\tau]ds \\
&\quad + \int_0^t R(t-s)[f(s+\delta, x_{s+\delta}) - f(s,x_s)]ds \\
&\quad + \int_0^t R(t-s)\int_0^s [g(s+\delta,\tau,x_\tau,Q(\tau)) - g(s,\tau,x_\tau,Q(\tau))]d\tau ds \\
&\quad + \int_0^t R(t-s)\int_s^{s+\delta} g(s+\delta,\tau,x_\tau,Q(\tau))d\tau ds.
\end{aligned}
$$

From our assumptions, we have

$$
\begin{aligned}
&||x(t+\delta+\theta) - x(t+\theta)|| \\
&\le ||[R(t+\delta+\theta) - R(t+\theta)](\phi(0) - h(x_{t_1}, \cdots, x_{t_p})(0))|| \\
&\qquad + \delta M_1(M_2 + TM_3) \\
&\qquad + \int_0^{t+\theta} M_1 M_4(\delta + ||x_{s+\delta} - x_s||_C)ds \\
&\qquad + \int_0^{t+\theta} M_1 \int_0^s M_6 \delta ds + \int_0^{t+\theta} M_1 \int_0^{s+\delta} M_3 \delta ds \\
&\le P\delta + M_1 M_4 \int_s^{t+\theta} ||x_{s+\delta} - x_s||_C ds
\end{aligned}
$$

where

$$
\begin{aligned}
P =&(||A\phi|| + ||Ah||)(M_1 + (t+\theta)||F||M_1) + M_1(M_2 + M_3 T) \\
&+ M_1 M_5(t+\theta) + M_1 M_5 T(t+\theta) + M_1 M_3 T.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned} ||x_{t+\delta} - x_t||_C &= \sup_{-r\le\theta\le 0} ||x(t+\delta+\theta) - x(t+\delta)|| \\ &\le P_1\delta + M_1M_4\int_0^T ||x_{s+\delta} - x_s||_C ds \end{aligned}$$

where

$$\begin{aligned} P_1 = &(||A\phi|| + ||Ah||)(M_1 + T||F||M_1) \\ &+ M_1(M_2 + M_3T) + M_1M_4T + M_1M_5T^2 + M_1M_3T. \end{aligned}$$

Using Gronwall's inequality, we get

$$||x_{t+\delta} - x_t||_C \le P_1\delta e^{TM_1M_4}, \quad t \in J.$$

Therefore x_t is Lipschitz continuous on J.

The Lipschitz continuity of x_t on J combined with conditions $(H6)$ and $(H7)$ imply that $t \to f(t, x_t), t \to g(t, s, x_s, Q(s))$ are Lipschitz continuous on J. By Desh ([7]) and Grimmer ([8]) and the definition of strong solution, we see that the linear Cauchy problem

$$\begin{aligned} \frac{dv(t)}{dt} &= A[v(t) + \int_0^t F(t-s)v(s)ds] + f(t, x_t) \\ &\quad + \int_0^t g(t, s, x_s, Q(s))ds, \quad t \in [0, T], \\ v(0) &= \phi(0) - h(x_{t_1}, \cdots, x_{t_p})(0) \end{aligned}$$

has a unique strong solution v satisfying the equation

$$\begin{aligned} v(t) &= R(t)(\phi(0) - h(x_{t_1}, \cdots, x_{t_p})(0)) \\ &\quad + \int_0^t R(t-s)[f(s, x_s) + \int_0^s g(s, \tau, x_\tau, Q(\tau))]ds \\ &= x_t \end{aligned}$$

Consequently, x_t is the unique strong solution of the problem (2.1) on J.

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