East Asian Math. J. **19** (2003), No. 1, pp. 63–71

ON DUDEK'S PROBLEMS ON THE SKEW OPERATION IN POLYADIC GROUPS

FEDIR M. SOKHATSKY

ABSTRACT In his invited lecture presented during the First Conference of the Mathematical Society of the Republic of Moldova (Chişinău, August 2001) W. A. Dudek posed several open problems on the role and properties of the skew element in the theory of n-ary groups In this note we give the partial solutions of some of these problems

1. Preliminaries

The non-empty set G together with an $(n+1)$-ary operation $f : G^{n+1} \to G$ is called an $(n+1)$-*ary groupoid* and is denoted by (G, f).

According to the general convention used in the theory of such groupoids the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. In the case $j < i$ this symbol is empty If $x_{i+1} = x_{i+2} = \ldots = x_{i+t} = x$, then instead of x_{i+1}^{i+t} we write $\overset{(t)}{x}$. In this convention $f(x_0, x_1, \ldots, x_n) = f(x_0^n)$ and

$$f(x_0, \ldots, x_i, \underbrace{x, \ldots, x}_{t}, x_{i+s+1}, \ldots, x_n) = f(x_0^i, \overset{(t)}{x}, x_{i+t+1}^n).$$

If $m = kn + 1$, then the m-ary operation g of the form

$$g(x_0^{kn+1}) = \underbrace{f(f(\ldots, f(f}_{k}(x_0^n), x_{n+1}^{2n}), \ldots), x_{(k-1)n+1}^{kn+1})$$

is denoted by $f_{(k)}$.

Received December 10, 2002.
2000 Mathematics Subject Classification 20N15
Key words and phrases. polydic group, polyagroup, n-ary group, skew element

An $(n+1)$-ary groupoid $(Q; f)$ is said to be *polyagroup of the kind* (s, n), where $s|n$ if and only if it is a quasigroup and for all i, j such that $i \equiv j \pmod{s}$ the following the *s-associativity law* hold

$$f(x_0^{i-1}, f(x_i^{i+n}), x_{i+n+1}^{2n}) = f(x_0^{j-1}, f(x_j^{j+n}), x_{j+n+1}^{2n}), \tag{1}$$

THEOREM 1. [6] *In an arbitrary polyagroup $(Q; f)$ be the kind (s, n), where $s < n$, for any $u \in Q$ there exist exactly one triple of operations $(+, \varphi, a)$ of the arity 2, 1, 0, such that $(Q; +)$ is a group, u is its neutral element, φ is an automorphism of $(Q; +)$ and*

$$\varphi^n(x) + a = a + x, \quad \varphi^s(a) = a, \tag{2}$$

$$f(x_0^n) = x_0 + \varphi(x_1) + \varphi^2(x_2) + \cdots + \varphi^n(x_n) + a. \tag{3}$$

And conversely, if an automorphism φ and an element a of a group $(Q; +)$ satisfy (2), then (Q, f) defined by (3) is a polyagroup of the kind (s, n).

The algebra $(Q; +, \varphi, a)$ is called the *decomposition algebra* of the polyagroup $(Q; f)$

It is clear that in a polyagroup $(Q; f)$ for any $x \in Q$ there exists only one element $\overline{x}$ (called *skew* to x) such that $f(\overline{x}, x, \ldots, x) = x$. So, the map $^{-}\colon x \to \overline{x}$ is a unary operation on Q, which is called the *skew operation* of $(Q; f)$.

From Theorem 1 follows that $\overline{x} + \varphi x + \varphi^2 x + \cdots + \varphi^n x + a = x$ and $\varphi^n x + a = a + x$. So

$$\overline{x} = -a - \varphi^{n-1} x - \cdots - \varphi x. \tag{4}$$

Moreover, it is not difficult to see that

$$\begin{aligned} x + y &= f(x, \overset{(s-1)}{u}, \overline{u}, \overset{(n-s-1)}{u}, y), \\ -x &= f(u, \overset{(n-s-1)}{x}, \overline{x}, \overset{(s-1)}{x}, u), \\ \varphi(x) &= f(u, x, \overset{(n-2)}{u}, \overline{u}), \\ a &= f(u, u, \ldots, u). \end{aligned} \tag{5}$$

According Post [5] (see also [1]) the *$(n+1)$-ary power* of an element a is defined as $a^{[m]} = f_{(m)}(\overset{(mn+1)}{a})$ for every natural m, where $a^{[0]} = a$.

The $(n+1)$-*ary order* of a is defined as a minimal $m>0$ (if it exists) such that $a^{[m]}=a$ and is denoted by $\mathrm{ord}_n(a)$.

LEMMA 2. *In any $(n+1)$-group (Q,f) the following conditions are equivalent:*

1) $x^{[m]}=x$,

2) $\mathrm{ord}_n(x)$ *is a divisor of* m;

3) *in* $(Q;+,\varphi,a)$ *we have* $m(\varphi x+\varphi^2x+\cdots+\varphi^n x+a)=0$

COROLLARY 3. *The identity $x^{[m]}=x$ holds in an $(n+1)$-group $(Q;f)$ iff in $(Q;+,\varphi,a)$ we have $ma=0$ and $x+\varphi x+\varphi^2 x+\cdots+\varphi^{mn-1}x=0$ for all $x\in G$*

2. On the first Dudek's problem

W. A. Dudek posed in [2] (see also [4]) the following:

Problem 1. *Describe the class of all $(n+1)$-ary groups for which the skew operation is an endomorphism.*

The partial answer for this problem is given in [4] Namely, in [4] are given conditions under which the skew operation is an endomorphism of a given $(n+1)$-ary group. We give the analogous answer for polyagroups.

THEOREM 4. *In an $(n+1)$-ary polyagroup $(Q;f)$ the following conditions are equivalent.*

1) *the skew operation is an endomorphism,*

2) *in $(Q,+,\varphi,a)$ the mapping $\theta=\varepsilon+\varphi+\varphi^2+\cdots+\varphi^{n-2}$ is an anti-*

endomorphism of $(Q;+)$ and $a+\theta(x)=\theta(x)+a$ holds for all $x\in Q$,

3) *For all $x,y,u\in Q$ we have*

$$f(\overset{n}{u},f(\overset{n-1}{x},u,u))=f(f(\overset{n-1}{x},u,u),\overset{n}{u}),\tag{6}$$

$$f(f(x,\overset{(n-1)}{u},y),\ldots,f(x,\overset{(n-1)}{u},y),\overset{(2)}{u})=\tag{7}$$
$$f(\overset{(n-1)}{y},f(u,f(x,\overset{(n)}{u}),\ldots,f(x,\overset{(n)}{u}),x,u),u).$$

Proof. 1) $\Rightarrow$ 2). If the skew operation is an endomorphism of $(Q;f)$, then $\overline{x} = c + \theta_0 x$ for some $x \in Q$ and an endomorphism θ_0 of the group $(Q;+)$. So, $c + \theta_0 x = -a - \varphi^{n-1}x - \cdots - \varphi x$ for all $x \in Q$. If $x = 0$, then $c = -a$. Thus $\theta_0 = I\varphi\theta$, where $I(x) = -x$ and $\theta = \varepsilon + \varphi + \varphi^2 + \cdots + \varphi^{n-2}$. Since θ_0 and φ are endomorphisms, θ is an anti-endomorphism.

It is easy to see that φ commutes with θ_0. Moreover, $\theta_0\varphi = \varphi I_c^{-1}\theta_0$ by (4), so $I_c^{-1}\theta = \theta$. Hence $a + \theta(x) = \theta(x) + a$ for all $x \in Q$.

2) $\Rightarrow$ 1). Let $\theta_0 = I\varphi\theta$ and $c = -a$. then $\overline{x} = L_c\theta x$ by (4). Since a commutes with all elements of the form $\theta(x)$, then $I_c^{-1}\theta = \theta$ and $\theta_0\varphi = \varphi I_c^{-1}\theta_0$. Moreover,

$$\theta_0 a - a = I\varphi(\varepsilon + \varphi + \cdots + \varphi^{n-2})a - a = -\varphi a - \varphi^2 a - \cdots - \varphi^{n-1}a - a.$$

Since $-c = a$ and $\varphi^n a = a$, then $\theta a - a = \varphi c + \varphi^2 c + \cdots + \varphi^n c$, which means that the skew operation of $(Q;f)$ is an endomorphism.

2) $\Rightarrow$ 3). Note that

$$\begin{aligned} \theta(x) + a &= x + \varphi x + \cdots + \varphi^{n-2}x + \varphi^{n-1}u + \varphi^n u + a \\ &\overset{(3)}{=} f(\overset{(n-1)}{x}, u, u). \end{aligned} \tag{8}$$

Add a to the both side of (8) and using just obtained equality, we obtain

$$a + f(\overset{(n-1)}{x}, u, u) = f(\overset{(n-1)}{x}, u, u) + a,$$

which, by (5), gives

$$\begin{aligned} &f(\overset{(n+1)}{u}, \overset{(s-1)}{u}, \overline{u}, \overset{(n-s-1)}{u}, f(\overset{(n-1)}{x}, u, u)) \\ &= f(f(\overset{(n-1)}{x}, u, u), \overset{(s-1)}{u}, \overline{u}, \overset{(n-s-1)}{u}, \overset{(n+1)}{u}), \end{aligned}$$

i.e.

$$\begin{aligned} &f(\overset{(s)}{u}, f(\overset{(n)}{u}, \overline{u}), \overset{(n-s-1)}{u}, f(\overset{(n-1)}{x}, u, u)) \\ &= f(f(\overset{(n-1)}{x}, u, u), \overset{(s-1)}{u}, f(\overline{u}, \overset{(n)}{u}), \overset{(n-s)}{u})). \end{aligned}$$

This proves (6).

Now consider the identity $\theta(x+y)=\theta(y)+\theta(x)$. Add to the both sides a and applying (8) we obtain

$$f(x+y,\dots,x+y,u,u) \qquad (9)$$
$$=f(\overset{(n-1)}{y},u,u)+f(\overset{(n-1)}{x},u,u)+(-a).$$

Computing the right side of (9) we get

$$f(\overset{(n-1)}{y},u,u)+f(\overset{(n-1)}{x},u,u)+(-a)$$

$$\overset{(5)}{=} f(f(\overset{(n-1)}{y},u,u),\overset{(s-1)}{u},\overline{u},\overset{(n-s-1)}{u},\overline{f}(f(\overset{(n-1)}{x},u,u),\overset{(s-1)}{u},\overline{u},\overset{(n-s-1)}{u},\overline{u}))$$

$$\overset{(1)}{=} f(\overset{(n-1)}{y},u,f(\overset{(s)}{u},\overline{u},\overset{(n-s-1)}{u},f(\overset{(n-1)}{x},u,f(\overset{(s)}{u},\overline{u},\overset{(n-s-1)}{u},\overline{u}))))$$

$$=f(\overset{(n-1)}{y},u,f(\overset{(n-1)}{x},u,\overline{u}))$$

Putting $\lambda x=f(x,\overset{(n)}{u})$ and replacing $+$ by f we obtain

$$x+y=\lambda\lambda^{-1}x+y\overset{(5)}{=} f(f(\lambda^{-1}(x),\overset{(n)}{u}),\overset{(s-1)}{u},\overline{u},\overset{(n-s-1)}{u},y)$$

$$=f(\lambda^{-1}(x),\overset{(n-1)}{u},f(\overset{(s)}{u},\overline{u},\overset{(n-s-1)}{u},y))=f(\lambda^{-1}(x),\overset{(n-1)}{u},y)$$

So, (9) can be written as follows

$$f(f(\lambda^{-1}(x),\overset{(n-1)}{u},y),\dots,f(\lambda^{-1}(x),\overset{(n-1)}{u},y),u,u)$$
$$=f(\overset{(n-1)}{y},u,f(\overset{(n-1)}{x},u,\overline{u})).$$

Now, replacing x by λx we get

$$f(f(x,\overset{(n-1)}{u},y),\dots,f(x,\overset{(n-1)}{u},y),u,u)$$

$$=f(\overset{(n-1)}{y},u,f(f(x,\overset{(n)}{u}),\ \dots,f(x,\overset{(n)}{u}),f(x,\overset{(n)}{u}),u,\overline{u}))$$

$$=f(\overset{(n-1)}{y},f(u,f(x,\overset{(n)}{u}),\dots,f(x,\overset{(n)}{u}),x,u),f(\overset{(n)}{u},\overline{u})),$$

which implies (7).

3) $\Rightarrow$ 2). This implication is obvious $\square$

COROLLARY 5. *In any medial polyagroup the skew operation is an endomorphism.*

COROLLARY 6. *In a ternary polyagroup the skew operation is its endomorphism iff the polyagroup is medial.*

3. On the fifth Dudek's problem

The *i-th skew element* of x is denoted by $\overline{x}^{(i)}$ and defined as the skew element to $\overline{x}^{(i-1)}$, i.e. $\overline{x}^{(i)} = \overline{\overline{x}^{(i-1)}}$. In other words: $\overline{x}^{(0)} = x$, $\overline{x}^{(1)} = \overline{x}$, $\overline{x}^{(2)} = \overline{\overline{x}}$, $\overline{x}^{(3)} = \overline{\overline{\overline{x}}}$, ...

The i-th skew element to x can be described by the *left* and *right* shift of x, i.e. by the mappings of the form

$$\lambda_a(u) = f(a, \dots, a, u), \qquad \rho_a(u) = f(u, a, \dots, a).$$

Namely, the following Lemma is true.

LEMMA 7. *In any $(n+1)$-group $(Q; f)$ for all $x \in Q$ and $i = 0, 1, 2, \dots$ we have*

$$\lambda_{\overline{x}^{(i)}} = \lambda_x^{(1-n)^i}, \qquad \rho_{\overline{x}^{(i)}} = \rho_x^{(1-n)^i}, \qquad \overline{x}^{(i)} = \lambda_x^{\frac{(1-n)^i-1}{n}}(x) = x^{\left[\frac{(1-n)^i-1}{n}\right]}.$$

Proof. The first formula for $i = 0$ is obvious. For $i = 1$, we have

$$\lambda_{\overline{x}}\lambda_x^{n-1}(u) = f(f(\lambda_x^{n-2}(u), \overset{(n)}{x}), \overset{(n)}{\overline{x}}) = f(\lambda_x^{n-2}(u), f(\overset{(n)}{x}, \overline{x}), \overset{(n-1)}{\overline{x}}) =$$

$$= f(\lambda_x^{n-2}(u), x, \overset{(n-1)}{\overline{x}}) = \dots = f(u, \overset{(n-1)}{x}, \overline{x}) = u.$$

Thus $\lambda_{\overline{x}} = \lambda_x^{1-n}$ for all $x \in Q$

If $\lambda_{\overline{x}^{(i)}} = \lambda_x^{(1-n)^i}$ is true for some i, then it is true for $i+1$ Indeed

$$\lambda_{\overline{x}^{(i+1)}} = \lambda_{\overline{x}^{(i)}}^{1-n} = (\lambda_x^{(1-n)^i})^{1-n} = \lambda_x^{(1-n)^{i+1}}.$$

This proves that the first formula is true for each natural i

The second formula is dual to the first. To the proof of third observe that, according to the definition of the i-th skew element, we have

$$f(\overline{x}^{(i-1)}, \dots, \overline{x}^{(i-1)}, \overline{x}^{(i)}) = \overline{x}^{(i-1)},$$

which, by the previous part of our proof, implies

$$\overline{x}^{(i)} = \lambda^{-1}_{\overline{x}^{(i-1)}}(\overline{x}^{(i-1)}) = \lambda_x^{-(1-n)^i}(\overline{x}^{(i-1)})$$

for all $i = 1, 2, \ldots$. Therefore

$$\begin{aligned} \overline{x}^{(i)} &= \lambda_x^{-(1-n)^{i-1}} \circ \lambda_x^{-(1-n)^{i-2}} \circ \cdots \circ \lambda_x^{-(1-n)} \circ \lambda_x^{-1}(x) \\ &= \lambda_x^{-(1-n)^{i-1}-(1-n)^{i-2}-\ldots-(1-n)-1}(x) = \lambda_x^{\frac{(1-n)^i-1}{n}}(x). \end{aligned}$$

This completes the proof. □

In ternary ($n = 3$) groups we have $\overline{\overline{x}} = x$ for all x. In some $(n+1)$-ary groups we have also $\overline{x} = x$ for all x. But there are $(n+1)$-ary groups in which $\overline{x} = \overline{y}$ for all x, y. Such groups are derived from a binary group of the exponent $t | n-1$ (see [1]) and may be characterized as $(n+1)$-ary groups in which there exists an element a such that $f(a, x, \ldots, x, a) = f(a, y, \ldots, y, a)$ holds for all x, y (see [4]).

In connection with this fact W. A. Dudek posed in [4] the following:

Problem 5. *Describe the class of all $(n+1)$-groups in which $\overline{x}^{(i)} = \overline{y}^{(i)}$ for all elements x, y and some fixed $i > 1$.*

For $i = 1$ this problem is solved in [3] We give the partial solution for $i > 1$.

Let $\overline{x}^{(i)} = \overline{y}^{(i)} = b$ for all x, y and some fixed $b \in Q$ and $i > 0$ Then $x^{[\frac{(1-n)^i-1}{n}]} = b$. This implies

$$\left(x^{[\frac{(1-n)^i-1}{n}]}\right)^{[\frac{(1-n)^i-1}{n}]} = b^{[\frac{(1-n)^i-1}{n}]} = b.$$

Hence,

$$x^{\left([\frac{(1-n)^i-1}{n}]\right)^2} = x^{[\frac{(1-n)^i-1}{n}]}$$

i e

$$x^{[\left([\frac{(1-n)^i-1}{n}]\right)^2 - \frac{(1-n)^i-1}{n}]} = x$$

It means the following Proposition is true.

PROPOSITION 8. *If an $(n+1)$-group satisfies the identity $\overline{x}^{(i)} = \overline{y}^{(i)}$, then it is torsion and its exponent is a divisor of the number*

$$\left(\frac{(1-n)^i - 1}{n}\right)^2 - \frac{(1-n)^i - 1}{n}.$$

In this case in the corresponding decomposition algebra $(Q;+,\varphi,a)$ of $(Q;f)$ we have

$$b = \overline{a}^{(i)} = a^{\left[\frac{(1-n)^i-1}{n}\right]} = ma,$$

where $ma = a + a + \cdots + a$ (m times) and $m = \frac{(1-n)^i-1}{n}$.

Using the above method we can solve also the following

Problem 9. *Describe the variety of $(n+1)$-groups defining by the identity $\overline{x}^{(i)} = x$.*

firstly posed in [2] and repeating in [4].

Indeed, using Lemma 2 and Lemma 7 we can prove the following result which is a partial answer to the above problem.

THEOREM 9. *In any $(n+1)$-group $(Q;f)$, where $n > 2$, for any fixed element $x \in Q$ the following conditions are equivalent*

1) $\overline{x}^{(i)} = x$,

2) $x^{[\frac{(1-n)^i-1}{n}]} = x$,

3) $\mathrm{ord}_n(x)$ *is a divisor of* $\frac{(1-n)^i-1}{n}$;

4) $\frac{(1-n)^i-1}{n}(\varphi x + \varphi^2 x + \cdots + \varphi^n x + a) = 0$ *holds in* $(Q;+,\varphi,a)$.

COROLLARY 10. *For an $(n+1)$-group $(Q;f)$, where $n > 2$, the following conditions are equivalent:*

1) $\overline{x}^{(i)} = x$ *for all* $x \in Q$;

2) $x^{[\frac{(1-n)^i-1}{n}]} = x$ *for all* $x \in Q$;

3) *a decomposition algebra $(Q;+,\varphi,a)$ satisfies the equality*

$$\frac{(1-n)^i - 1}{n} a = 0$$

and the identity $x + \varphi x + \varphi^2 x + \cdots + \varphi^{(1-n)^i-2}(x) = 0$.

REFERENCES

[1] I. M. Dudek, W. A. Dudek, *On skew elements in n-groups,* Demonstratio Math. **14** (1981), 827 – 833.

[2] W. A Dudek, *Medial n-groups and skew elements*, Proceedings of the V Universal Algebra Symposium "Universal and Applied Algebra", Turawa 1988, World Scientific, Singapore 1989, 55 – 80.

[3] W. A Dudek, *On n-ary group with only one skew element,* Radovi Matematički (Sarajevo), **6** (1990), 171 – 175.

[4] W. A. Dudek, *On some old and new problems in n-ary groups,* Quasigroups and Related Systems **8** (2001), 15 – 36.

[5] E. L. Post, *Polyadic groups*, Trans Amer. Math. Soc. **48** (1940), 208 – 350

[6] F. Sokhatsky, *On the associativity of multiplace operations,* (in Russian), Discrete Math. **4** (1992), 66 – 84.

[7] F. Sokhatsky, *On the associativity of multiplace operations*, Quasigroups and Related Systems **4** (1997), 51 – 56.

[8] F Sokhatsky, *Invertible elements in associates and semigroups* 1, Quasigroups and Related Systems **5** (1998), 53 – –68.

[9] F Sokhatsky, O. Yurevych, *Invertible elements in associates and semigroups.* 2, Quasigroups and Related Systems **6** (1999), 61 – 70.

Department of Mathematics
Pedagogical University
21 100 Vinnitsa
Ukraine
E-mail: `fedir@vinnitsa.com`