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THE BERGMAN KERNEL FUNCTION AND ASSOCIATED INVARIANTS NEAR STRONGLY PSEUDOCONVEX BOUNDARY POINTS

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Abstract We study the asymptotic boundary behavior of the Bergman kernel function on the diagonal, the Bergman metric and the holomorphic sectional curvatures of the Bergman metric in bounded strongly pseudoconvex domains

1. Introduction

The asymptotic boundary behavior of the Bergman kernel function on the diagonal, the Bergman metric and the holomorphic sectional curvatures of the Bergman metric have been studied by many experts. In this paper, we obtain those asymptotic boundary behavior for bounded strongly pseudoconvex domains by the scaling method. Originally, those results were found by Hörmander [9], Diederich [4, 5], Kim and Yu [10]

Let G be a bounded domain in $\mathbb{C}^n$ and let $d\mu$ the standard volume form of $\mathbb{C}^n$. Consider the space

$$\mathcal{H}^2(G) := \left\{ f \ G \to \mathbb{C} \,\middle|\, f \text{ is holomorphic}, \int_G |f|^2 \, d\mu < \infty \right\}$$

which is usually called the *Bergman space* Since it is a separable Hilbert space with respect to the L^2 norm, we may choose an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ Then the *Bergman kernel function* $K_G \cdot G \times G \to \mathbb{C}$

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can be represented by

$$K_G(z,\zeta) := \sum_{j=1}^{\infty} \varphi_j(z)\,\overline{\varphi_j(\zeta)}.$$

Then the *Bergman metric* of G is given by

$$ds^2_G(z;\cdot,\cdot) = \sum_{\alpha,\beta=1}^{n} g_{\alpha\overline{\beta}}(z)\,dz^\alpha \otimes d\overline{z}^\beta$$

where

$$g_{\alpha\overline{\beta}}(z) := \frac{\partial^2 \log K(z,\overline{z})}{\partial z^\alpha \partial \overline{z}^\beta}.$$

One of the important features of this metric is that it is one of the invariant metrics, in the sense that every biholomorphic mapping becomes an isometry. It is also obvious that this metric is Kählerian.

The holomorphic (sectional) curvature R at z in the direction ξ is given by

$$R_G(z;\xi) = \frac{R_{\overline{h}jk\overline{l}}(z)\cdot \overline{\xi}^h \xi^j \xi^k \overline{\xi}^l}{[g_{j\overline{k}}(z)\cdot \xi^j \overline{\xi}^k]^2}$$

where

$$R_{\overline{h}jk\overline{l}} = -\frac{\partial^2 g_{j\overline{h}}}{\partial z^k \partial \overline{z}^l} + g^{\nu\overline{\mu}} \frac{\partial g_{j\overline{\mu}}}{\partial z^k} \frac{\partial g_{\nu\overline{h}}}{\partial \overline{z}^l}.$$

Here, we have employed the so-called summation convention. Moreover, $g^{\overline{\mu}\nu}$ represents the $\mu\nu$-th entry of the inverse matrix of $\left(g_{\alpha\overline{\beta}}\right)$

The asymptotic boundary behavior of these quantities was first analyzed by Bergman [2, 3]. There, he investigated the kernel and the metric for rather special domains. The celebrated asymptotic expansion formula of Bergman kernel function for strongly pseudoconvex domains was obtained by Fefferman [6]. For the results on strongly pseudoconvex domains, see Hörmander [9], Diederich [4, 5], Klembeck [11], and others

2. Main results

In this section, we present the main results.

Let a domain G in $\mathbb{C}^n$ have C^2 boundary and let ρ be a C^2 defining function. Let $p \in \partial G$. An n-tuple $w = (w_1, \dots, w_n)$ of complex numbers is called a *complex tangent vector* to ∂G at p if

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p)\, w_j = 0.$$

The collection of all complex vector to ∂G at p is called the *complex tangent space* to ∂G at p and is denoted by $T_p^{\mathbb{C}}(\partial G)$. The quadratic expression

$$L_{\rho,p}(w,\overline{w}) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k, \quad w \in \mathbb{C}^n$$

of ρ at p is called the *complex Hessian* or the *Levi form* of ρ at p Let U be a neighborhood of $\overline{G}$. We say that ∂G is *strongly pseudoconvex* at p if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k > 0, \qquad \forall w \neq 0 \in T_p^{\mathbb{C}}(\partial G).$$

A domain is called *strongly (Levi) pseudoconvex* if all its boundary points are strongly pseudoconvex.

LEMMA 1 (Lee [12]). *Let $G \subseteq \mathbb{C}^n$ be a domain with C^2 boundary Let $p \in \partial G$ Let ϕ and ρ be C^2 defining functions for G. Suppose that $\|\nabla \phi(p)\| = \|\nabla \rho(p)\|$ Then, for every $\xi \in T_p^{\mathbb{C}}(\partial G)$, we have that*

$$L_{\phi,p}(\xi,\overline{\xi}) = L_{\rho,p}(\xi,\overline{\xi}).$$

Lemma 1 implies that the positive definite hermitian form $L_{\partial G,p}$ on the complex tangent space $T_p^{\mathbb{C}}(\partial G)$, defined by $L_{\partial G,p} = \|\nabla \rho(p)\|^{-1} L_{\rho,p}$, is independent of any defining function ρ at $p \in \partial G$; $L_{\partial G,p}$ is called the *Levi form of ∂G at $p \in \partial G$* Some authors ([1], [8], [13]) use the normalization condition $\|\nabla_z \rho(p)\| = 1$. In such a case, we have that

$$L_{\partial G,p}(\xi,\xi) = \frac{1}{2} L_{\rho,p}(\xi,\xi), \quad \xi \in T_p^{\mathbb{C}}(\partial G),$$

since

$$\nabla_z \rho(p) = \frac{1}{2}\left(\frac{\partial \rho}{\partial x_1}(p) - \sqrt{-1}\frac{\partial \rho}{\partial y_1}(p), \dots, \frac{\partial \rho}{\partial x_n}(p) - \sqrt{-1}\frac{\partial \rho}{\partial y_n}(p)\right).$$

DEFINITION 1. By a *stream* approaching p in G we mean a C^2 curve $q\colon (0,\epsilon) \to G$ satisfying $\lim_{t\downarrow 0} q(t) = p$.

Now we have the following results:

THEOREM 1. *Let G be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Let K_G, ds_G^2 and R_G be the Bergman kernel function, the Bergman metric and the holomorphic curvature of the Bergman metric for G, respectively. Let $p \in \partial G$ and $q(t)$ an stream in G, approaching p. For $\xi \in \mathbb{C}^n$, let $\xi_{N,p(t)}$ and $\xi_{T,p(t)}$ be the normal and tangential components of ξ with respect to $T^{\mathbb{C}}_{p(t)}(\partial G)$, respectively. Then we have that*

(1)

$$
\begin{aligned}
K_G(q(t),q(t)) &\sim \frac{n!}{4\pi^n}\cdot\left(\frac{1}{d(q(t))}\right)^{n+1}\cdot\det(L_{\partial G,p(t)}),\\
ds_G^2(q(t);\xi,\overline{\xi}) &\sim (n+1)\left[\left(\frac{\|\xi_{N,p(t)}\|}{2\cdot d(q(t))}\right)^2+\frac{L_{\partial G,p(t)}(\xi_{T,p(t)},\xi_{T,p(t)})}{d(q(t))}\right],\\
R_G(q(t);\xi) &\sim -\frac{4}{n+1}.
\end{aligned}
$$

Here $A(t) \sim B(t)$ means that $\lim_{t\to 0}\frac{B(t)}{A(t)} = 1$

The others of this paper is devoted to the proof of the theorem.

3. Minimum Integrals

In this section, we summarize the minimum integrals Let G be a bounded domain in $\mathbb{C}^n$. Let $z \in G$, and let $\xi = (\xi_1,\ldots,\xi_n) \in T_zG = \mathbb{C}^n$ be a nonzero vector. We consider the minimum integrals:

$$
\begin{aligned}
I_0^G(z) &= \inf\left\{\int_G |f|^2\,d\mu \,:\, f\in\mathcal{H}^2(G), f(z)=1\right\},\\
I_1^G(z;\xi) &= \inf\left\{\int_G |f|^2\,d\mu \quad f\in\mathcal{H}^2(G), f(z)=0, \sum_{j=1}^n \xi_j\frac{\partial f}{\partial z_j}(z)=1\right\},\\
I_2^G(z;\xi) &= \inf\left\{\int_G |f|^2\,d\mu \,.\, f\in\mathcal{H}^2(G), f(z)=0,\right.\\
&\qquad\left.\frac{\partial f}{\partial z_1}(z)=\cdots=\frac{\partial f}{\partial z_n}(z)=0, \sum_{j,k=1}^n \xi_j\xi_k\frac{\partial^2 f}{\partial z_j\partial z_k}(z)=1\right\},
\end{aligned}
$$

We write down some basic properties of the minimum integrals:

(a) Let Ω be a bounded domain in $\mathbb{C}^n$ with $z\in\Omega\subset G$. Then by the definitions of the minimum integrals we can see that

$$
I_0^\Omega(z)\le I_0^G(z), \qquad \text{and} \qquad I_i^\Omega(z;\xi)\le I_i^G(z;\xi), \quad i=1,2.
$$

Then we have the following mild modification from [14], [10]:

PROPOSITION 1. *Let $\{G_j\}_{j=1}^\infty$ be a sequence of bounded domains in $\mathbb{C}^n$ that converges to a convex bounded domain $G\subset\mathbb{C}^n$ in such a way that there exists a common interior point q of G and G_j for all j and such that for every $\epsilon>0$ there exists j_0 satisfying*

$$
(1-\epsilon)(G-q)\subset G_j-q\subset(1+\epsilon)(G-q),
$$

where $G-q$ denotes the affine translation by $-q$ of the set G in $\mathbb{C}^n$. Then for every nonzero vector $\xi\in T_q(G)=\mathbb{C}^n$,

$$
\lim_{j\to\infty} I_0^{G_j}(q)\to I_0^G(q), \qquad \lim_{j\to\infty} I_k^{G_j}(q,\xi)\to I_k^G(q,\xi), \quad k=1,2.
$$

(b) We can study the Bergman kernel, Bergman metric, and its curvature with the minimum integrals·

PROPOSITION 2 (Bergman [2], Fuchs [7]). *Let z, ξ, G be as above. Then*

$$K_G(z,z) = \frac{1}{I_0^G(z)}, \quad ds_G^2(z;\xi,\overline{\xi}) = \frac{I_0^G(z)}{I_1^G(z;\xi)},$$
$$R_G(z;\xi) = 2 - \frac{(I_1^G(z,\xi))^2}{I_0^G(z) I_2^G(z;\xi)}.$$

(c) We may localize the minimum integrals:

PROPOSITION 3 (Kim-Yu [10]). *Let G be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$. Let $p \in \partial G$. Let U be a neighborhood of p. Then, we have*

$$\lim_{z \to p} \frac{I_i^G(z;\xi)}{I_i^{G \cap U}(z;\xi)} = 1, \qquad i = 0, 1, 2.$$

4. The Scaling Method

Let G be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Let p be a boundary point of G. Let $q(t)$ be an approaching stream to p in G. In this section, we demonstrate a construction of a biholomorphic mapping of a local domain $\Omega = G \cap B(p;r)$ onto a perturbation of the unit ball in $\mathbb{C}^n$, where $B(p;r)$ is the ball of radius r centered at p for some positive constant r.

DEFINITION 2. We call an approaching stream $q(t)$ to p is *radial* if $q(t)$ lies in the inward normal real line to the real complex tangent space $T_p(\partial G)$ at p.

PROPOSITION 4. *There are some positive constants C, r and ϵ, and a biholomorphic mapping Ψ of $\Omega = G \cap B(p;r)$ such that for $0 < d(q(t)) < \epsilon$*

$$\|\Psi(q(t))\| = O(d(q(t)))$$

and

$$B\left(0; \sqrt{1 - C\sqrt{d(q(t))}}\right) \subset \Psi(\Omega) \subset B\left(0; \sqrt{1 + C\sqrt{d(q(t))}}\right), \tag{2}$$

where $d(q(t)) = \operatorname{dist}(q(t), \partial G)$. We may choose the constants C, r and ϵ uniformly for every $p \in \partial G$.

Proof. Let ρ be a C^2 defining function of G with $\|\nabla\rho(z)\| = 1$ for $z \in \partial G$. Without loss of generality, we may assume that the stream $q(t)$ is radial.

Using a rotation and a unitary transformation, we may assume that $p = \mathbf{0}$ ($\mathbf{0} = (0, \ldots, 0)$ the origin), $\nabla\rho(\mathbf{0}) = (1, 0, \ldots, 0)$, and

$$\begin{aligned}\rho(z) &= 2\operatorname{Re}\left(\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(\mathbf{0})z_j + \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial z_k}(\mathbf{0})z_j z_k\right) \\ &\quad + \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \overline{z}_k}(\mathbf{0})z_j \overline{z}_k + o(\|z\|^2) \\ &= \operatorname{Re}\left(z_1 + \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial z_k}(\mathbf{0})z_j z_k\right) \\ &\quad + \sum_{j=2}^{n} \frac{\partial^2\rho}{\partial z_j \partial \overline{z}_j}(\mathbf{0})z_j \overline{z}_j + o(|z_1| + \|z'\|^2)\end{aligned}$$

where $z' = (0, z_2, \ldots, z_n)$ and where

$$(0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$$

are the eigenvectors for $(\frac{\partial^2\rho}{\partial z_j \partial \overline{z}_k}(\mathbf{0}))_{j,k=2}^{n}$ And we may assume that

$$q(t) = (-t, 0, \ldots, 0),$$

$0 < t < \epsilon$ for some positive constant ϵ.

Define $\mathcal{V}\colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{V}(z) = \left(z_1 + \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial z_k}(\mathbf{0})z_j z_k, z_2, z_3, \ldots, z_n\right).$$

The inverse function theorem says that $\mathcal{V}$ is biholomorphic on a neighborhood of $\overline{B(\mathbf{0}; r)}$ for some $r > 0$. Then $\mathcal{V}$ maps $q(t)$ to $\mathcal{V}(q(t)) = (-t + bt^2, 0, \ldots, 0)$ where $b = \frac{\partial^2\rho}{\partial z_1 \partial z_1}(\mathbf{0})$. The defining function becomes

$$\rho \circ \mathcal{V}^{-1}(w) = \operatorname{Re} w_1 + \sum_{j=2}^{n} \frac{\partial^2\rho}{\partial w_j \partial \overline{w}_j}(\mathbf{0})w_j \overline{w}_j + o(\|w_1\| + \|w'\|^2).$$

Define $\mathcal{L}\colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{L}(w) = (w_1, a_2 w_2, \cdots, a_n w_n)$$

where $a_j = \sqrt{\frac{\partial^2 \rho}{\partial w_j \partial \overline{w}_j}(\mathbf{0})}$. The map $\mathcal{L}$ fixes $\mathcal{V}(q(t))$, and the defining function becomes

$$\rho \circ \mathcal{V}^{-1} \circ \mathcal{L}^{-1}(u) = \operatorname{Re} u_1 + |u_2|^2 + \cdots + |u_n|^2 + o(|u_1| + \|u'\|^2).$$

Let $\mathcal{S}$ be a linear map from $\mathbb{C}^n$ onto $\mathbb{C}^n$, defined by

$$\mathcal{S}(u) = \left(\frac{1}{t}u_1, \frac{1}{\sqrt{t}}u_2, \ldots, \frac{1}{\sqrt{t}}u_n\right).$$

Then $\mathcal{S}$ maps $(\mathcal{L} \circ \mathcal{V})(q(t))$ to $(-1 + bt, 0, \ldots, 0)$, and the defining function becomes

$$(\rho \circ \mathcal{V}^{-1} \circ \mathcal{L}^{-1} \circ \mathcal{S}^{-1})(v) = t(\operatorname{Re} v_1 + |v_2|^2 + \cdots + |v_n|^2) + o(t(|v| + \|v'\|^2)).$$

We apply the Cayley transformation $\mathcal{T}$ on $(\mathcal{S} \circ \mathcal{L} \circ \mathcal{V})(\Omega)$, $(\Omega = G \cap B(\mathbf{0}; r))$ defined by

$$\mathcal{T}(v) = \left(\frac{1 + v_1}{1 - v_1}, \frac{2v_2}{1 - v_1}, \ldots, \frac{2v_n}{1 - v_1}\right).$$

Then the reference point becomes $(\mathcal{T} \circ \mathcal{S} \circ \mathcal{L} \circ \mathcal{V})(q(t)) = \left(\frac{bt}{2-bt}, \ldots, 0\right)$, and the defining function becomes

$$\begin{aligned}
&(\rho \circ \mathcal{V}^{-1} \circ \mathcal{L}^{-1} \circ \mathcal{S}^{-1} \circ \mathcal{T}^{-1})(\zeta) \\
&= t\left(\frac{|\zeta|^2 - 1}{|\zeta_1 + 1|^2} + \frac{|\zeta_2|^2}{|\zeta_1 + 1|^2} + \cdots + \frac{|\zeta_n|^2}{|\zeta_1 + 1|^2}\right) + o(t(|v_1| + \|v'\|^2)).
\end{aligned}$$

For $z \in \overline{\Omega}$, we have that

$$\begin{aligned}
\|(\mathcal{T} \circ \mathcal{S} \circ \mathcal{L} \circ \mathcal{V})(z)\|^2 &= \left|\frac{1 + \frac{u_1}{t}}{1 - \frac{u_1}{t}}\right|^2 + \left\|\frac{2\frac{u'}{\sqrt{t}}}{1 - \frac{u_1}{t}}\right\|^2 \\
&= \frac{|t + u_1|^2 + 4t\|u'\|^2}{|t - u_1|^2} \\
&= 1 + \frac{4t(\operatorname{Re} u_1 + \|u'\|^2)}{t^2 - 2t(\operatorname{Re} u_1) + |u_1|^2},
\end{aligned}$$

where $u = \mathcal{L} \circ \mathcal{V}(z)$ and $u' = (0, u_2, \ldots, u_n)$.

Let $\partial\Omega = V_1 \cup V_2$ where $V_1 = \partial G \cap \overline{B(\mathbf{0}; r)}$ and $V_2 = \partial B(\mathbf{0}; r) \cap \overline{G}$. For $z \in V_1$, we have that

$$
\begin{aligned}
|\|(\mathcal{T} \circ \mathcal{S} \circ \mathcal{L} \circ \mathcal{V})(z)\|^2 - 1| &= \frac{4t(\operatorname{Re} u_1 + \|u'\|^2)}{t^2 - 2t(\operatorname{Re} u_1) + |u_1|^2} \\
&\leq \frac{4t(C|u_1|^2 + C|u_1| \cdot \|u\| + C\|u\|^3)}{t^2 + |u_1|^2} \\
&\leq \frac{4tC|u_1|^{3/2}}{t^2 + |u_1|^2} \\
&= 4C\sqrt{t}\frac{(|u_1|/t)^{3/2}}{1 + (|u_1|/t)^2} \\
&\leq 4C\sqrt{t}\frac{3^{3/4}}{4} \\
&\leq C\sqrt{t}.
\end{aligned} \tag{3}
$$

Here, for convenience we use the same symbol C to stand for different constants. For $z \in V_2$, we have that

$$
|\|(\mathcal{T} \circ \mathcal{S} \circ \mathcal{L} \circ \mathcal{V})(z)\|^2 - 1| = \frac{4t(\operatorname{Re} u_1 + |u'|^2)}{t^2 - 2t(\operatorname{Re} u_1) + |u_1|^2} \leq \frac{4tC}{\delta^2} \leq Ct, \tag{4}
$$

where δ is the minimum of u_1. By (3) and (4), we have that

$$
|\|(\mathcal{T} \circ \mathcal{S} \circ \mathcal{L} \circ \mathcal{V})(z)\|^2 - 1| \leq C\sqrt{t}, \quad z \in \partial\Omega,
$$

for some constant C. It implies that for some positive constant C, r and ϵ,

$$
B\left(0, \sqrt{1 - C\sqrt{t}}\right) \subset (\mathcal{T} \circ \mathcal{S} \circ \mathcal{L} \circ \mathcal{V})(\Omega) \subset B\left(0; \sqrt{1 + C\sqrt{t}}\right),
$$

where $0 < t < \epsilon$. Since ∂G is compact, we can choose the constants C, r and ϵ uniformly for every $p \in \partial G$. □

5. Proof of the main theorem

Let ρ be a C^2 defining function of G with $\|\nabla\rho(z)\| = 1$ for $z \in \partial G$. Using a rotation and a unitary transformation, we may assume that $p = \mathbf{0}$ ($\mathbf{0} = (0, \ldots, 0)$ the origin), $\nabla\rho(\mathbf{0}) = (1, 0, \ldots, 0)$.

Choose the positive constants C, r and ϵ so that (2) is satisfied for every boundary point Proposition 2 and Proposition 3 implies that

$$(5)\quad K_G(q(t), q(t)) \sim K_\Omega(q(t), q(t), \quad ds^2_G(q(t); \xi, \overline{\xi}) \sim ds^2_\Omega(q(t); \xi, \overline{\xi}),$$

and

$$R_G(q(t); \xi) \sim R_\Omega(q(t); \xi).$$

We note that

$$K_{B(0,1)}(z, w) = \frac{n!}{\pi^n} \cdot \frac{1}{(1 - \sum_{j=1}^n z_j \overline{w}_j)^{n+1}},$$
$$ds^2_{B(0,1)}(z; \xi, \overline{\xi}) = \sum_{j,k=1}^n (n+1) \frac{(1 - |z|^2)\delta_{jk} + \overline{z}_j z_k}{(1 - |z|^2)^2} \xi_j \overline{\xi}_k,$$

and

$$R_{B(0,1)}(z; \xi) = -\frac{4}{n+1}.$$

The Radial Stream Case. Let $q(t)$ be the radial stream in the proof of Proposition 4. Consider the map $\Psi = \mathcal{T} \circ \mathcal{S} \circ \mathcal{L} \circ \mathcal{V}$ in Proposition 4.

First we consider the Bergman kernel function. Since Ψ is a biholomorphism on Ω, we have that

$$(6)\qquad K_\Omega(q(t), q(t)) = K_{\Psi(\Omega)}(\Psi(q(t)), \Psi(q(t))) \cdot |\det J_{\mathbb{C}}|_{q(t)}(\Psi)|^2.$$

By (2), we have that

$$(7)\quad K_{B(0;\sqrt{1+C\sqrt{t}})}(\Psi(q(t)), \Psi(q(t))) \leq K_{\Psi(\Omega)}(\Psi(q(t)), \Psi(q(t)))$$
$$\leq K_{B(0,\sqrt{1-C\sqrt{t}})}(\Psi(q(t)), \Psi(q(t))).$$

Note that

(8)
$$
\begin{aligned}
&K_{B(0,\sqrt{1+C\sqrt{t}})}(\Psi(q(t)),\Psi(q(t)))\\
&=K_B\left(\frac{1}{\sqrt{1+C\sqrt{t}}}\Psi(q(t)),\frac{1}{\sqrt{1+C\sqrt{t}}}\Psi(q(t))\right)\cdot\left(\frac{1}{\sqrt{1+C\sqrt{t}}}\right)^n\\
&=\frac{n!}{\pi^n}\frac{1}{\left(1-\frac{1}{1+C\sqrt{t}}\left|\frac{bt}{1-bt}\right|^2\right)^{n+1}}\cdot\left(\frac{1}{\sqrt{1+C\sqrt{t}}}\right)^n\\
&\sim\frac{n!}{\pi^n},
\end{aligned}
$$

where $b=\frac{\partial^2\rho}{\partial z_1\partial\bar{z}_1}(\mathbf{0})$. Similarly, we have that

$$
K_{B(0,\sqrt{1-C\sqrt{t}})}(\Psi(q(t)),\Psi(q(t)))\sim\frac{n!}{\pi^n}. \tag{9}
$$

By (5), (6), (7), (8), and (9), we have

$$
K_G(q(t),q(t))\sim\frac{n!}{4\pi^n}\cdot\left(\frac{1}{d(q(t))}\right)^{n+1}\cdot\det(L_{\partial G,p}).
$$

We now consider the Bergman metric. Since Ψ is a biholomorphic on Ω, we have that

$$
ds^2_\Omega(q(t);\xi,\bar{\xi})=ds^2_{\Psi(\Omega)}(\Psi(q(t)),d\Psi|_{q(t)}(\xi),\overline{d\Psi|_{q(t)}(\xi)}), \tag{10}
$$

By (2) and Proposition 1, we have that

$$
\begin{aligned}
&ds^2_{\Psi(\Omega)}(\Psi(q(t));d\Psi|_{q(t)}(\xi),\overline{d\Psi|_{q(t)}(\xi)})\\
&\quad\sim ds^2_B(\Psi(q(t));d\Psi|_{q(t)}(\xi),\overline{d\Psi|_{q(t)}(\xi)}),
\end{aligned}
\tag{11}
$$

From Proposition 4, we know that

$$
\begin{aligned}
&ds^2_B(\Psi(q(t));d\Psi|_{q(t)}(\xi),\overline{d\Psi|_{q(t)}(\xi)}),\\
&\quad\sim(n+1)\left[\left(\frac{\|\xi_{N,p(t)}\|}{2\cdot d(q(t))}\right)^2+\frac{L_{\partial G,p(t)}(\xi_{T,p(t)},\xi_{T,p(t)})}{d(q(t))}\right].
\end{aligned}
\tag{12}
$$

By (5), (10), (11), and (12), we have

$$ds^2_G(q(t);\xi,\bar{\xi}) \sim (n+1)\left[\left(\frac{\|\xi_{N,p(t)}\|}{2\cdot d(q(t))}\right)^2 + \frac{L_{\partial G,p(t)}(\xi_{T,p(t)},\xi_{T,p(t)})}{d(q(t))}\right].$$

For the holomorphic curvature of the Bergman metric, by the same reason to the metric, we have

$$R_G(q(t);\xi) \sim R_{B^n}(\Psi(q(t);(d\Psi)(\xi)) = -\frac{4}{n+1}.$$

General Stream Case. Let $q(t)$ be an arbitrary stream approaching p. Let $p(t)$ be the closest boundary point to $q(t)$. We may assume that $t = d(q(t), \partial G)$. Let $\mathcal{A}$ be the unitary map such that

$$\mathcal{A}(p(t)) = \mathbf{0} \quad \text{and} \quad \mathcal{A}(q(t)) = (-t, 0, \ldots, 0).$$

Since we may choose the constants C, r, ϵ uniformly in (2), the identity (1) follows in this case by the same method as above. Therefore, we have the desired results. This completes the proof.

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