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SL(2, ℂ)-REPRESENTATION VARIETIES OF PERIODIC LINKS

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Abstract In this paper, we characterize SL(2, ℂ)-representations of an n-periodic link $\tilde{L}$ in terms of SL(2, ℂ)-representations of its quotient link L and express the SL(2, ℂ)-representation variety $\mathcal{R}(\tilde{L})$ of $\tilde{L}$ as the union of n affine algebraic subsets which have the same dimension Also, we show that the dimension of $\mathcal{R}(\tilde{L})$ is bounded by the dimensions of affine algebraic subsets of the SL(2, ℂ)-representation variety $\mathcal{R}(L)$ of its quotient link L

1. Introduction

Let L be a tame link in the 3-sphere S^3 and let $G = \pi_1(S^3 - L)$ be the fundamental group of the complement $S^3 - L$. Let $\mathrm{R}(G)$ denote the set of all representations of G in the 2×2 special linear group SL(2, ℂ) with entries in the field ℂ of complex numbers Suppose we fix a finite system of generators of G, say $(g_1, \cdots, g_m)$. Then a representation ρ $G \to \mathrm{SL}(2, \mathbb{C})$ is uniquely determined by specifying the m-tuple $(\rho(g_1), \cdots, \rho(g_m))$ We define $\mathcal{R}(G) = \{(\rho(g_1), \cdots, \rho(g_m)) \in \mathrm{SL}(2, \mathbb{C})^m \mid \rho \in \mathrm{R}(G)\}$. Then $\mathcal{R}(G)$ carries with it the structure of an affine algebraic set in $\mathbb{C}^{4m}$ Throughout this paper we shall call it the SL(2, ℂ)*-representation variety of* L and denote it by $\mathcal{R}(L)$ SL(2, ℂ)-representation varieties of knots and links and their applications have

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been studied extensively by many mathematicians. For examples, see [2, 4, 5, 6, 7, 13, 14, 15] and therein.

A link $\tilde{L}$ in S^3 is said to have *period* $n(n \geq 2)$ if there exists an n-periodic homeomorphism ϕ from S^3 onto itself such that $\tilde{L}$ is invariant under ϕ and the fixed point set $\tilde{K}_1$ of the $\mathbb{Z}_n$-action induced by ϕ is homeomorphic to a 1-sphere in S^3 disjoint from $\tilde{L}$. By the positive solution of the Smith Conjecture [9], $\tilde{K}_1$ is unknotted and so the homeomorphism ϕ is conjugate to one point compactification of the $\frac{2\pi}{n}$-rotation about the z-axis in $\mathbb{R}^3$ Hence the quotient map $q : S^3 \to S^3/\mathbb{Z}_n$ is an n-fold cyclic covering branched along the unknot $q(\tilde{K}_1) = K_1$. Set $L = q(\tilde{L})$. Then the link $L_1 = K_1 \cup L$ in the orbit space $S^3/\mathbb{Z}_n \cong S^3$ is called the *quotient link* of $\tilde{L}$. Some authors showed that a certain properties of periodic links can be characterized by their quotient links [3, 5, 8, 11, 12]. In this paper we are interested in studying the SL(2, $\mathbb{C}$)-representation variety $\mathcal{R}(\tilde{L})$ of an n-periodic link $\tilde{L}$ in S^3 in terms of SL(2, $\mathbb{C}$)-representations of its quotient link L_1 in SL(2, $\mathbb{C}$)

The paper is organized as follows In Section 2, we review a few basic terminologies concerning affine algebraic sets. In Section 3, we consider the SL(2, $\mathbb{C}$)-representation variety $\mathcal{R}(L_1)$ of a link $L_1 = K_1 \cup L$ with unknotted component K_1. In Section 4, we show that SL(2, $\mathbb{C}$)-representations of an n-periodic link $\tilde{L}$ are completely determined by the SL(2, $\mathbb{C}$)-representations of its quotient link L_1 and express the SL(2, $\mathbb{C}$)-representation variety $\mathcal{R}(\tilde{L})$ of $\tilde{L}$ as the union of n affine algebraic subsets which have the same dimension. As a consequence, we show that the dimension of $\mathcal{R}(\tilde{L})$ is bounded by the dimensions of algebraic subsets of the SL(2, $\mathbb{C}$)-representation variety $\mathcal{R}(L_1)$ of its quotient link L_1

2. Representation variety of knots and links

Let $\mathbb{C}$ be the field of complex numbers An *(affine) algebraic set* in the affine space $\mathbb{C}^n (n \geq 1)$ is the set of zeros of some finite set of polynomials $f_1, \cdots, f_s$ in $\mathbb{C}[X_1, \cdots, X_n]$ We denote it by $\mathcal{V}(f_1, \cdots, f_s)$

or simply by $\mathcal{V}$, i.e., $\mathcal{V}(f_1, \cdots, f_s) =$

$$\{(a_1, \cdots, a_n) \in \mathbb{C}^n \mid f_i(a_1, \cdots, a_n) = 0, \ \forall\, i = 1, 2, \cdots, s\}.$$

If $\mathcal{U}$ is the ideal of $\mathbb{C}[X_1, \cdots, X_n]$ generated by $f_1, \cdots, f_s$, then the set of all zeros of $f_i's$ is equal to the set of all zeros of every $g \in \mathcal{U}$ and so we will denote $\mathcal{V}(f_1, \cdots, f_s)$ also by $\mathcal{V}(\mathcal{U})$. A non-empty affine algebraic set is said to be *irreducible* if it cannot be expressed as the union of two proper algebraic subsets An irreducible algebraic subset $\mathcal{V} = \mathcal{V}(f_1, \cdots, f_s)$ of $\mathbb{C}^n$ is called an *affine variety* defined by $f_1, \cdots, f_s$ Every affine algebraic set may be written canonically as a finite union of affine varieties, called its *irreducible components.* An affine algebraic set $\mathcal{V}$ has a well-defined (complex) *dimension*, denoted by $\dim(\mathcal{V})$ If $\mathcal{V} \subset \mathbb{C}^m$ and $\mathcal{W} \subset \mathbb{C}^n$ are affine algebraic sets, a map $\phi \; \mathcal{V} \to \mathcal{W}$ is said to be *regular* if it is the restriction of some map from $\mathbb{C}^m$ to $\mathbb{C}^n$ which is defined by n polynomials in m variables[10].

Let $\mathrm{M}(2, \mathbb{C})$ be the set of all 2×2 matrices with entries in $\mathbb{C}$. Throughout this paper, we shall identify $\mathrm{M}(2, \mathbb{C})$ with $\mathbb{C}^4$ by simply writing down the rows of each matrix one after the other and so, for example, $\mathrm{M}(2, \mathbb{C})^m$ is identified with $\mathbb{C}^{4m}$. The general linear group $\mathrm{GL}(2, \mathbb{C})$ is the group of all members of $\mathrm{M}(2, \mathbb{C})$ with nonzero determinant and the special linear group $\mathrm{SL}(2, \mathbb{C})$ is the subgroup of $\mathrm{GL}(2, \mathbb{C})$ with determinant 1

Let G be a finitely presented group A homomorphism $\rho \; G \to \mathrm{SL}(2, \mathbb{C})$ is called a representation of G in $\mathrm{SL}(2, \mathbb{C})$ Two representations ρ and ρ' are *equivalent*, denoted by $\rho \equiv \rho'$, if $\rho' = \Lambda\rho$, where Λ is an inner automorphism of $\mathrm{SL}(2, \mathbb{C})$. Let $\mathrm{R}(G)$ denote the set of all representations of G in $\mathrm{SL}(2, \mathbb{C})$. Then it can be parametrized by points of an affine algebraic subset of $\mathbb{C}^{4m}$ for some positive integer m as follows. Let $\mathcal{P} = < x_1, \cdots, x_m \mid r_j(x_1, \cdots, x_m), j = 1, 2, \cdots, n >$ be a group presentation of G. Define $\mathcal{R}(G, \mathcal{P}) =$

$$\{P = (A_1, \cdots, A_m) \in \mathrm{SL}(2, \mathbb{C})^m \mid R_j(P) - I = O, j = 1, 2, \cdots, n\},$$

where $R_j(P) (j = 1, 2, \cdots, n)$ denotes the matrix $r_j(A_1, \cdots, A_m)$ obtained from the relator $r_j(x_1, \cdots, x_m)$ by substituting A_i for x_i, I denotes the 2×2 identity matrix and O denotes the 2×2 zero matrix. Then $\mathcal{R}(G, \mathcal{P})$ is an affine algebraic subset of $\mathbb{C}^{4m}$. For each point $P = (A_1, \cdots, A_m) \in \mathcal{R}(G, \mathcal{P})$, we define a representation ρ_P .

$G \to \mathrm{SL}(2,\mathbb{C})$ by $\rho_P(x_i) = A_i (1 \le i \le m)$ Then ρ_P becomes a representation of G in $\mathrm{SL}(2,\mathbb{C})$. Conversely, for an arbitrary given representation $\rho : G \to \mathrm{SL}(2,\mathbb{C})$, the point $P = (\rho(x_1), \cdots, \rho(x_m))$ is an element of $\mathcal{R}(G,\mathcal{P})$ such that $\rho_P = \rho$. Therefore there is a natural 1-1 correspondence between the points of $\mathcal{R}(G,\mathcal{P})$ and $\mathrm{R}(G)$. If $\mathcal{Q}$ is an another presentation of G, then there exists a canonical isomorphism ϕ . $\mathcal{R}(G,\mathcal{P}) \to \mathcal{R}(G,\mathcal{Q})$ as affine algebraic sets. We shall identify points in $\mathcal{R}(G,\mathcal{P})$ with the corresponding representations. Although $\mathcal{R}(G;\mathcal{P})$ is not a variety in general, we call $\mathcal{R}(G,\mathcal{P})$ the $\mathrm{SL}(2,\mathbb{C})$-*representation variety of* G *associated to* $\mathcal{P}$.

Now let $L = K_1 \cup \cdots \cup K_\mu$ be an oriented tame link in S^3 of μ components($\mu \ge 1$) and let $G = \pi_1(S^3 - L)$ be the link group of L, i.e., the fundamental group of the complement $S^3 - L$ with a finite presentation $\mathcal{P}$. Then in what follows the variety $\mathcal{R}(G,\mathcal{P})$ is called the $\mathrm{SL}(2,\mathbb{C})$-*representation variety of the link* L associated to $\mathcal{P}$ and denoted by $\mathcal{R}(L,\mathcal{P})$. Note that the isomorphism class $\mathcal{R}(L)$ of $\mathcal{R}(L,\mathcal{P})$ is an invariant of the link type L.

3. Representation variety of a link with one trivial component

Let $L_1 = K_1 \cup K_2 \cup \cdots \cup K_\mu$ be an oriented link in S^3 of μ components($\mu \ge 2$) such that K_1 is unknotted For each $2 \le i \le \mu$, let $\lambda_{1i} = lk(K_1, K_i)$, the linking number of K_1 and K_i. Let $N_i (i = 1, \cdots, \mu)$ be a small open tubular neighborhood of K_i in S^3 whose boundary $\partial N_i = \mathbb{T}_i$ is a torus in S^3. Let (m_i, l_i) be a meridian-longitude pair of $\mathbb{T}_i$ Then $\pi_1(\mathbb{T}_i)$ is a free abelian group generated by m_i and l_i and it has a presentation $\pi_1(\mathbb{T}_i) =< x_i, \xi_i \cdot x_i \xi_i x_i^{-1} \xi_i^{-1} >$, where x_i and ξ_i represent m_i and l_i, respectively This presentation is called a *canonical presentation* of $\pi_1(\mathbb{T}_i)$.

For our simplicity, we assume that $\mu = 2$ and $\lambda_{12} \neq 0$. Applying an isotopy deformation if necessary, we can choose an oriented diagram $D = D_1 \cup D_2$ in $\mathbb{R}^2$ of the link $L_1 = K_1 \cup K_2$ which is of the form as shown in Figure 1, where $D_i (i = 1, 2)$ denotes a diagram representing the component K_i.

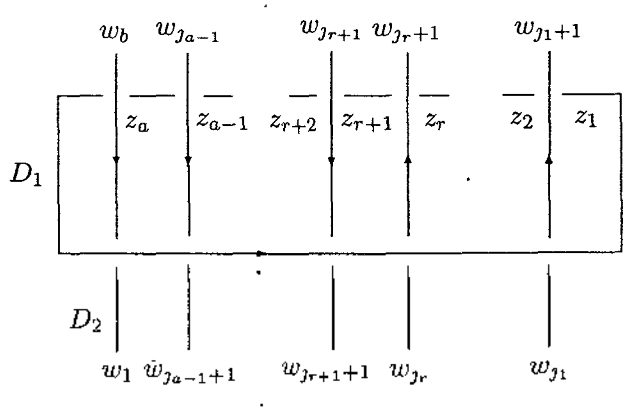

FIGURE 1. $D = D_1 \cup D_2$

Using Wirtinger presentation and Tietz transformations if necessary, we obtain a deficiency one presentation $\mathcal{P}'$ of the group $G = \pi_1(S^3 - L_1)$ which contains a canonical presentation of $\pi_1(\mathbb{T}_1)$, which is of the form(cf. [1])

$$\mathcal{P}' = < z_1, \cdots, z_a, w_1, \cdots, w_b, \xi_1 \mid r', s', \\ r'_{1i}(1 \le i \le a-1), r'_{2j}(1 \le i \le b-1) >,$$

where the generators z_i and w_j correspond to the i-th and j-th branch of the component D_1 and D_2 of D, respectively, and ξ_1 represents a longitude l_1 of D_1 and

$$r' = z_1 \xi_1 z_1^{-1} \xi_1^{-1},$$
$$s' = \xi_1 (w_{j_1+1} w_{j_2+1} \cdots w_{j_r+1} w_{j_{r+1}}^{-1} \cdots w_{j_{a-1}}^{-1} w_b^{-1})^{-1}.$$

The relators r'_{1i} and r'_{2j} correspond to the crossings in D. The relators r'_{1i} correspond to the crossings incident to the component D_1, which have the form(cf. Figure 1)

$$
\begin{aligned}
&r'_{11} = w_{j_1+1}^{-1} z_1 w_{j_1+1} z_2^{-1},\ r'_{12} = w_{j_2+1}^{-1} z_2 w_{j_2+1} z_3^{-1},\\
&\qquad\qquad \vdots\\
&r'_{1r} = w_{j_r+1}^{-1} z_r w_{j_r+1} z_{r+1}^{-1},\ r'_{1r+1} = w_{j_{r+1}} z_{r+1} w_{j_{r+1}}^{-1} z_{r+2}^{-1},\\
&\qquad\qquad \vdots\\
&r'_{1a-2} = w_{j_{a-2}} z_{a-2} w_{j_{a-2}}^{-1} z_{a-1}^{-1},\ r'_{1a-1} = w_{j_{a-1}} z_{a-1} w_{j_{a-1}}^{-1} z_a^{-1},\\
&r'_{2j_1} = w_{j_1} z_1 w_{j_1+1}^{-1} z_1^{-1}, \cdots, r'_{2j_r} = w_{j_r} z_1 w_{j_r+1}^{-1} z_1^{-1},\\
&r'_{2j_{r+1}+1} = z_1 w_{j_{r+1}+1} z_1^{-1} w_{j_{r+1}}^{-1}, \cdots, r'_{2j_{a-1}+1} = z_1 w_{j_{a-1}+1} z_1^{-1} w_{j_{a-1}}^{-1},
\end{aligned}
$$

The relators r'_{2j} correspond to the self crossings of the component D_2, which have the form:

$$
r'_{2q} = (w'_q)^{\epsilon_q} w_q (w'_q)^{-\epsilon_q} w_{q+1}^{-1}, 1 \leq q \leq b-1 \text{ with } q \neq j_1 - 1, \cdots, j_{a-1},
$$

where w'_q is a certain generator $w_j (1 \leq j \leq b)$ and $\epsilon_q = \pm 1$.

We modify the presentation $\mathcal{P}'$ of G as follows. Since $H_1(S^3 - L_1) = G/[G,G]$ is generated by z_1, w_1, we have that $z_i \equiv z_1 \pmod{[G,G]}, i = 2, \cdots, a$, and $w_j \equiv w_1 \pmod{[G,G]}, j = 2, \cdots, b$, and $\xi_1 \equiv w_1^{\lambda_{12}} = w_1^{a-2r} \pmod{[G,G]}$. Introduce new generators $x_1 = z_1, x_i = z_i x_1^{-1} (2 \leq i \leq a), y_1 = w_1, y_j = w_j y_1^{-1} (2 \leq j \leq b)$, and $\ell_1 = \xi_1 y_1^{-\lambda_{12}}$. Using these generators, we obtain a new deficiency one presentation $\mathcal{P}$ of G

$$
\begin{aligned}
\text{(1)}\qquad \mathcal{P} = <x_1, \cdots, x_a, y_1, \cdots, y_b, \ell_1 \mid r, s,&\\
r_{1i} (1 \leq i \leq a-1), r_{2j} (1 \leq j \leq b-1)>,&
\end{aligned}
$$

where r, s, r_{1i} and r_{2j} are obtained from r', s', r'_{1i} and r'_{2j} by rewriting in terms of the new generators x_i, y_j and ℓ_1. Precisely,

$$
(2)\qquad
\begin{aligned}
r &= x_1\ell_1 y_1^{\lambda_{12}} x_1^{-1} y_1^{-\lambda_{12}} \ell_1^{-1},\\
s &= \ell_1 y_1^{\lambda_{12}} (y_{j_1+1} y_1 y_{j_2+1} y_1 \cdots y_{j_r+1} y_{j_{r+1}}^{-1} \cdots y_1^{-1} y_{j_{a-1}}^{-1} y_1^{-1} y_{j_a}^{-1})^{-1},\\
r_{11} &= y_1^{-1} y_{j_1+1}^{-1} x_1 y_{j_1+1} y_1 x_1^{-1} x_2^{-1},\\
r_{12} &= y_1^{-1} y_{j_2+1}^{-1} x_2 x_1 y_{j_2+1} y_1 x_1^{-1} x_3^{-1},\\
&\vdots\\
r_{1r} &= y_1^{-1} y_{j_r+1}^{-1} x_r x_1 y_{j_r+1} y_1 x_1^{-1} x_{r+1}^{-1},\\
r_{1r+1} &= y_{j_{r+1}} y_1 x_{r+1} x_1 y_1^{-1} y_{j_{r+1}}^{-1} x_1^{-1} x_{r+2}^{-1},\\
&\vdots\\
r_{1a-1} &= y_{j_{a-1}} y_1 x_{a-1} x_1 y_1^{-1} y_{j_{a-1}}^{-1} x_1^{-1} x_a^{-1}\\
r_{2j_1} &= y_{j_1} y_1 x_1 y_1^{-1} y_{j_1+1}^{-1} x_1^{-1},\\
&\vdots\\
r_{2j_r} &= y_{j_r} y_1 x_1 y_1^{-1} y_{j_r+1}^{-1} x_1^{-1},\\
r_{2j_{r+1}+1} &= x_1 y_{j_{r+1}+1} y_1 x_1^{-1} y_1^{-1} y_{j_{r+1}}^{-1},\\
&\vdots\\
r_{2j_{a-1}+1} &= x_1 y_{j_{a-1}+1} y_1 x_1^{-1} y_1^{-1} y_{j_{a-1}}^{-1},\\
r_{2q} &= (w_q)^{\epsilon_q} y_q y_1 (w_q)^{-\epsilon_q} y_1^{-1} y_{q+1}^{-1}
\end{aligned}
$$

Now let $\mathcal{R}(L_1, \mathcal{P})$ be the SL(2, ℂ)-representation variety of L_1 associated to the presentation $\mathcal{P}$ in (1)

Let $A_i = \begin{pmatrix} X_{4(i-1)+1} & X_{4(i-1)+2} \\ X_{4(i-1)+3} & X_{4i} \end{pmatrix}, B_j = \begin{pmatrix} X_{4(a+j-1)+1} & X_{4(a+j-1)+2} \\ X_{4(a+j-1)+3} & X_{4(a+j)} \end{pmatrix}$, $C_1 = \begin{pmatrix} X_{4(a+b)+1} & X_{4(a+b)+2} \\ X_{4(a+b)+3} & X_{4(a+b+1)} \end{pmatrix} \in \mathrm{M}(2, \mathbb{C})$ for $i = 1, 2, \cdots, a$, $j = 1, 2, \cdots, b$ A point $P = (A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in \mathrm{M}(2, \mathbb{C})^{a+b+1}$ lies in $\mathcal{R}(L_1, \mathcal{P})$, i.e., the map defined by $x_i \mapsto A_i (1 \le i \le a), y_j \mapsto B_j (1 \le j \le b), \ell_1 \mapsto C_1$ is a representation of G in SL(2, ℂ) if and only if

(3) $\det(A_1) = 1$

(4) $\det(A_i) = 1, \det(B_j) = 1, \det(C_1) = 1, 2 \leq i \leq a, 1 \leq j \leq b,$

(5) $R(P) - I = O, S(P) - I = O, R_{1i}(P) - I = O, 1 \leq i \leq a-1,$

(6) $R_{2j}(P) - I = O, 1 \leq j \leq b-1.$

On the other hand, a presentation $\mathcal{P}_*$ of $G_* = \pi_1(S^3 - K_2)$ is obtained from $\mathcal{P}$ by adding one relator $x_1 = 1$. Let $\mathcal{R}(K_2, \mathcal{P}_*)$ be the SL(2, $\mathbb{C}$)-representation variety of K_2 associated to the presentation $\mathcal{P}_*$.

PROPOSITION 3 1. $\mathcal{R}(K_2, \mathcal{P}_*)$ is an affine algebraic subset of $\mathcal{R}(L_1, \mathcal{P})$.

Proof A point $P = (A_1, \cdots, A_a, B_1, \cdots, B_b, C_1) \in \mathrm{M}(2, \mathbb{C})^{a+b+1}$ lies in $\mathcal{R}(K_2, \mathcal{P}_*)$ if and only if it satisfies the equations (3), (4), (5), (6) and the equation $A_1 = I$, i.e ,

(7)
$$\mathcal{R}(K_2, \mathcal{P}_*) = \{(A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in \mathcal{R}(L_1, \mathcal{P}) \mid A_1 = I\}$$

This implies that $\mathcal{R}(K_2, \mathcal{P}_*)$ is an affine algebraic set defined by the defining polynomials of $\mathcal{R}(L_1, \mathcal{P})$, together with the polynomials $X_1 - 1 = 0, X_2 = 0, X_3 = 0$ and $X_4 - 1 = 0$. □

Let $\mathcal{U}(\mathcal{P})$ be the ideal of $\mathbb{C}[X_1, X_2, X_3, X_4, X_5, \cdots, X_{4(a+b+1)}]$ generated by the polynomials in (3) and (4) and the entries of the left hand side of the matrix equations in (5) and (6). Note that $\mathcal{R}(L_1, \mathcal{P}) = V(\mathcal{U}(\mathcal{P}))$. Let $\pi_4 : \mathrm{M}(2, \mathbb{C})^{a+b+1} \to \mathrm{M}(2, \mathbb{C})^{a+b}$ be the projection map which sends $(A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1)$ to $(A_2, \cdots, A_a, B_1, \cdots, B_b, C_1)$ and let $\mathcal{U}_4(\mathcal{P}) = \mathcal{U}(\mathcal{P}) \cap \mathbb{C}[X_5, \cdots, X_{4(a+b+1)}]$ be the 4-th elimination ideal of $\mathcal{U}(\mathcal{P})$. Then it is well known that the projection $\pi_4(\mathcal{R}(L_1, \mathcal{P}))$ is given by

$$\pi_4(\mathcal{R}(L_1, \mathcal{P})) = \{(A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in V(\mathcal{U}_4(\mathcal{P})) \mid \exists\, A_1 \in \mathrm{M}(2, \mathbb{C}) \text{ s.t. } (A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in \mathcal{R}(L_1, \mathcal{P})\}$$

and $V(\mathcal{U}_4(\mathcal{P})) = \overline{\pi_4(\mathcal{R}(L_1, \mathcal{P}))}$, the Zariski closure of $\pi_4(\mathcal{R}(L_1, \mathcal{P}))$ in $\mathbb{C}^{4(a+b)}$

Let n be an integer ≥ 2 and set $\zeta = \exp(\frac{2\pi\sqrt{-1}}{n})$, a primitive n-th root of 1 Let $\mathcal{V}(L_1, \mathcal{P})$ be the affine algebraic subset of $\mathbb{C}^{4(a+b+1)}$ consisting of all points $P = (A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in \mathrm{M}(2, \mathbb{C})^{a+b+1}$ satisfying all equations in (4), (5) and (6). For each $k = 0, 1, \cdots, n-1$, let $\mathcal{D}_k^n = \{M \in \mathrm{M}(2, \mathbb{C}) \mid M^n = I, \det(M) = \zeta^k\}$. Then we define $\mathcal{V}_k(L_1, \mathcal{P}), 0 \leq k \leq n-1$, to be the subset of $\mathbb{C}^{4(a+b+1)}$ given by

$$\mathcal{V}_k(L_1, \mathcal{P}) = \mathcal{V}(L_1, \mathcal{P}) \cap (\mathcal{D}_k^n \times V(\mathcal{U}_4(\mathcal{P})))$$

and define

$$\mathcal{R}_n(L_1, \mathcal{P}) = \bigcup_{k=0}^{n-1} \mathcal{V}_k(L_1, \mathcal{P}).$$

In particular, $\mathcal{V}_0(L_1, \mathcal{P}) =$

$$\{(A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in \mathcal{R}(L_1, \mathcal{P}) \mid A_1^n = I\} \tag{8}$$

PROPOSITION 3.2. (1) For each $k = 0, 1, \cdots, n-1$, $\mathcal{V}_k(L_1, \mathcal{P})$ is an affine algebraic subset of $\mathbb{C}^{4(a+b+1)}$ and so is $\mathcal{R}_n(L_1, \mathcal{P})$.

(2) If $0 \leq i \neq j \leq n-1$, then $\mathcal{V}_i(L_1, \mathcal{P}) \cap \mathcal{V}_j(L_1, \mathcal{P}) = \emptyset$

(3) For each $k = 1, \cdots, n-1$, $\mathcal{V}_k(L_1, \mathcal{P})$ is isomorphic to $\mathcal{V}_0(L_1, \mathcal{P})$ as affine algebraic sets.

(4) $\mathcal{R}(K_2, \mathcal{P}_*) \subset \mathcal{V}_0(L_1, \mathcal{P}) \subset \mathcal{R}(L_1, \mathcal{P})$ and $\mathcal{R}_n(L_1, \mathcal{P}) \cap \mathcal{R}(L_1, \mathcal{P}) = \mathcal{V}_0(L_1, \mathcal{P})$

Proof. Since $\mathcal{V}(L_1, \mathcal{P}), \mathcal{D}_k^n$ and $V(\mathcal{U}_4(\mathcal{P}))$ are all affine algebraic sets, (1) follows immediately (2) follows from the fact that $\mathcal{D}_i^n \cap \mathcal{D}_j^n = \emptyset$ if $i \neq j$.

(3) We consider the map $\phi : \mathcal{V}_0(L_1, \mathcal{P}) \to \mathcal{V}_k(L_1, \mathcal{P})$ defined by

$$\begin{aligned}&\phi((A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1)) \\ &= (\zeta^{\frac{k}{2}} A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1)\end{aligned}$$

for all $(A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in \mathcal{V}_0(L_1, \mathcal{P})$ By the definition of $\mathcal{V}_0(L_1, \mathcal{P})$, it follows that $P = (A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in V(\mathcal{U}_4(\mathcal{P}))$, $\det(\zeta^{\frac{k}{2}} A_1) = \zeta^k \det(A_1) = \zeta^k$ and $(\zeta^{\frac{k}{2}} A_1)^n = \zeta^{\frac{nk}{2}} A_1^n = A_1^n = I$. Notice that either the relators r, s, r_{1i} and r_{2j} in (2) contain both the generator x_1 and its inverse x_1^{-1} exactly once or they do not contain both x_1 and x_1^{-1} at all. This gives that

$$R(\zeta^{\frac{k}{2}}A_1, P) = R(A_1, P) = I, S(\zeta^{\frac{k}{2}}A_1, P) = S(A_1, P) = I,$$
$$R_{1_i}(\zeta^{\frac{k}{2}}A_1, P) = R_{1_i}(A_1, P) = I, R_{2_j}(\zeta^{\frac{k}{2}}A_1, P) = R_{2_j}(A_1, P) = I.$$

Hence $(\zeta^{\frac{k}{2}}A_1, P) \in \mathcal{V}_k(L_1, \mathcal{P})$. It is clear that ϕ is the restriction of a polynomial map from $\mathbb{C}^{4(a+b+1)}$ to itself. Thus ϕ is a well-defined regular mapping. Now let $\psi : \mathcal{V}_k(L_1, \mathcal{P}) \to \mathcal{V}_0(L_1, \mathcal{P})$ be a map defined by

$$\begin{aligned}&\psi((A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1))\\ &= (\zeta^{-\frac{k}{2}}A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1)\end{aligned}$$

for all $(A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in \mathcal{V}_k(L_1, \mathcal{P})$. By similar argument above, ψ is a regular mapping. It is easy to check that $\psi \circ \phi = id_{\mathcal{V}_0(L_1, \mathcal{P})}$ and $\phi \circ \psi = id_{\mathcal{V}_k(L_1, \mathcal{P})}$. Therefore ϕ is an isomorphism.

(4) It follows from (7) and (8) shows that $\mathcal{R}(K_2, \mathcal{P}_*) \subset \mathcal{V}_0(L_1, \mathcal{P})$. By definition, $\mathcal{V}_0(L_1, \mathcal{P}) \subset \mathcal{R}(L_1, \mathcal{P}) \cap \mathcal{R}_n(L_1, \mathcal{P})$ Now let

$$P = (A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1) \in \mathcal{R}(L_1, \mathcal{P}) \cap \mathcal{R}_n(L_1, \mathcal{P}).$$

Then P represents a representation of G into $\mathrm{SL}(2, \mathbb{C})$ and so $P \in \mathcal{V}(L_1, \mathcal{P})$ and $\det(A_1) = 1$. Since $P \in \mathcal{R}_n(L_1, \mathcal{P}), A_1 \in D_k^n$ for some k. By (2), $\mathcal{R}_n(L_1, \mathcal{P}) = \coprod_{k=0}^{n-1} \mathcal{V}_k(L_1, \mathcal{P})$ and hence $P \in \mathcal{V}_0(L_1, \mathcal{P})$. This completes the proof. □

4. Representation variety of an n-periodic link

Let $L_1 = K_1 \cup K_2$ be an oriented link in S^3 with 2 components such that K_1 is unknotted Let μ be the greatest common divisor of n and λ_{12}. For any integer $n \geq 2$, let $\pi : S^3 \to S^3$ be the n-fold cyclic cover branched along K_1. Then K_2 is covered by μ knots $\tilde{K}_1, \cdots, \tilde{K}_\mu$ in S^3. We give orientations to $\tilde{K}_1, \cdots, \tilde{K}_\mu$ inherited from K_2. Then the oriented link $\tilde{L} = \pi^{-1}(K_2) = \tilde{K}_1 \cup \cdots \cup \tilde{K}_\mu$ is the n-periodic link in S^3 with L as its quotient link. Note that every periodic links arises in this way

Let $\tilde{G} = \pi_1(S^3 - \tilde{L})$ be the link group of $\tilde{L}$. Then from the choice of the generators in the presentation $\mathcal{P}$ of $G = \pi_1(S^3 - L_1)$ as given in (1), the group $\tilde{G}$ has a presentation $\tilde{\mathcal{P}}$ of the form(cf. [11])

$$\tag{9} \begin{aligned} \tilde{\mathcal{P}} = < x_{ik}, y_{jk}, z_k(1 \le i \le a-1, 1 \le j \le b, 1 \le k \le n) \mid r_k, s_k, \\ r^k_{1i}, r^k_{2j}(1 \le i \le a-1, 1 \le j \le b-1, 1 \le k \le n) >, \end{aligned}$$

where

$$\tag{10} \begin{aligned} &x_{ik} = x_1^{k-1} x_{i+1} x_1^{-(k-1)}, y_{jk} = x_1^{k-1} y_j x_1^{-(k-1)}, z_k = x_1^{k-1} \ell_1 x_1^{-(k-1)}, \\ &x_1^n = 1, x_1^k \ne 1 \text{ for all } k = 1, \cdots, n-1, \end{aligned}$$

and

$$\tag{11} \begin{aligned} r_k = x_1^{k-1} r x_1^{-(k-1)}, s_k = x_1^{k-1} s x_1^{-(k-1)}, \\ r^k_{1i} = x_1^{k-1} r_{1i} x_1^{-(k-1)}, r^k_{2j} = x_1^{k-1} r_{2j} x_1^{-(k-1)}, \end{aligned}$$

or equivalently, for each $k = 1, \cdots, n$,

$$\tag{12} \begin{aligned} r_k &= z_{k+1} y_{1k+1}^{\lambda_{12}} y_{1k}^{-\lambda_{12}} z_k^{-1}, \\ s_k &= z_k y_{1k}^{\lambda_{12}} (y_{j_1+1k} y_{1k} y_{j_2+1k} y_{1k} \cdot y_{j_r+1k} y^{-1}_{j_{r+1}k} \cdots \\ &\qquad y_{1k}^{-1} y^{-1}_{j_{a-1}k} y_{1k}^{-1} y^{-1}_{j_a k})^{-1}, \\ r^k_{11} &= y_{1k}^{-1} y^{-1}_{j_1+1k} y_{j_1+1k+1} y_{1k+1} x^{-1}_{1k}, \\ r^k_{12} &= y_{1k}^{-1} y^{-1}_{j_2+1k} x_{1k} y_{j_2+1k+1} y_{1k+1} x^{-1}_{2k}, \\ &\vdots \\ r^k_{1r} &= y_{1k}^{-1} y^{-1}_{j_r+1k} x_{r-1k} y_{j_r+1k+1} y_{1k+1} x^{-1}_{rk}, \\ r^k_{1r+1} &= y_{j_{r+1}k} y_{1k} x_{r1} y^{-1}_{1k+1} y^{-1}_{j_{r+1}k+1} x^{-1}_{r+1k}, \\ &\vdots \\ r^k_{1a-1} &= y_{j_{a-1}k} y_{1k} x_{a-2k} y^{-1}_{1k+1} y^{-1}_{j_{a-1}k+1} x^{-1}_{a-1k} \\ r^k_{2j_1} &= y_{j_1k} y_{1k} y^{-1}_{1k+1} y^{-1}_{j_1+1k+1}, \\ &\qquad . \end{aligned}$$

$$
(13)\qquad
\begin{aligned}
r^k_{2j_r} &= y_{j_r k} y_{1k} y^{-1}_{1k+1} y^{-1}_{j_r+1k+1},\\
r^k_{2j_{r+1}+1} &= y_{j_{r+1}+1k+1} y_{1k+1} y^{-1}_{1k} y^{-1}_{j_{r+1}k},\\
&\ \vdots\\
r^k_{2j_{a-1}+1} &= y_{j_{a-1}+1k+1} y_{1k+1} y^{-1}_{1k} y^{-1}_{j_{a-1}k},\\
r^k_{2q} &= (w_{qk})^{\epsilon_{qk}} y_{qk} y_{1k} (w_{qk})^{-\epsilon_{qk}} y^{-1}_{1k} y^{-1}_{q+1k}.
\end{aligned}
$$

We shall introduce some notations for the following theorem. Let $P_1 = (M_{11}, \cdots, M_{m1}), P_2 = (M_{12}, \cdots, M_{m2}), \cdots, P_n = (M_{1n}, \cdots, M_{mn})$ be n points in $\mathrm{M}(2,\mathbb{C})^m$, where m is an integer ≥ 1 and $M_{ij} \in \mathrm{M}(2,\mathbb{C})$ Then $(P_1, P_2, \cdots, P_n)$ denotes the point $(M_{11}, \cdots, M_{m1}, M_{12}, \cdots, M_{m2}, \cdots, M_{1n}, \cdots, M_{mn})$ in $\mathrm{M}(2,\mathbb{C})^{mn}$. For a matrix $N \in \mathrm{M}(2,\mathbb{C})$ and an integer k, $N^k P_j N^{-k} (1 \leq j \leq n)$ denotes the point $(N^k M_{1j} N^{-k}, \cdots, N^k M_{mj} N^{-k})$ in $\mathrm{M}(2,\mathbb{C})^m$

THEOREM 4 1. Let $L_1 = K_1 \cup K_2$ be an oriented link in S^3 such that K_1 is unknotted and $\lambda_{12} = lk(K_1, K_2) \neq 0$ and let $\mathcal{P}$ be the presentation of $G = \pi_1(S^3 - L_1)$ as given in (1). For any integer $n \geq 2$, let $\tilde{L}$ be an n-periodic link in S^3 with the quotient link L_1 and let $\mathcal{R}(\tilde{L}, \tilde{P})$ be the $\mathrm{SL}(2,\mathbb{C})$-representation variety of $\tilde{L}$ associated to the presentation $\tilde{\mathcal{P}}$ in (9) Then a point $P = (P_1, P_2, \cdots, P_n) \in \mathrm{M}(2,\mathbb{C})^{(a+b)n}$ lies in $\mathcal{R}(\tilde{L}, \tilde{P})$ if and only if $P_1 \in V(\mathcal{U}_4(\mathcal{P}))$ and for each $k = 2, \cdots, n, P_k = M^{k-1} P_1 M^{-(k-1)}$ for some matrix $M \in \mathrm{GL}(2,\mathbb{C})$ such that $(M, P_1) \in \mathcal{R}_n(L_1, \mathcal{P})$.

Proof Let

$$
(14)\qquad
\begin{aligned}
P_1 &= (A_{11}, \cdots, A_{a-11}, B_{11}, \cdots, B_{b1}, C_1),\\
P_2 &= (A_{12}, \cdots, A_{a-12}, B_{12}, \cdots, B_{b2}, C_2),\\
&\ \vdots\\
P_n &= (A_{1n}, \cdots, A_{a-1n}, B_{1n}, \cdots, B_{bn}, C_n).
\end{aligned}
$$

Suppose that $P = (P_1, P_2, \cdots, P_n)$ is a point of $\mathcal{R}(\tilde{L}, \tilde{P})$, i.e., the mapping defined by $x_{ik} \mapsto A_{ik}, y_{jk} \mapsto B_{jk}, z_k \mapsto C_k$ is a representation of $\tilde{G}$ in SL(2, ℂ). Then

$$\det(A_{ik}) = 1, \det(B_{jk}) = 1, \det(C_k) = 1, \tag{15}$$

$$R_k(P) - I = O, S_k(P) - I = O, \tag{16}$$

$$R^k_{1i}(P) - I = O, R^k_{2j}(P) - I = O \tag{17}$$

for all $1 \leq i \leq a-1, 1 \leq j \leq b$ and $1 \leq k \leq n$.

By (10), it follows that for all i, j and k, $A_{ik} = M^{k-1}A_{i1}M^{-(k-1)}$, $B_{jk} = M^{k-1}B_{j1}M^{-(k-1)}$, $C_k = M^{k-1}C_1M^{-(k-1)}$ for some matrix $M \in \mathrm{GL}(2, \mathbb{C})$ such that $M^n = I$, i.e., for each $k = 1, \cdots, n$,

$$P_k = M^{k-1}P_1M^{-(k-1)} = MP_{k-1}M^{-1}. \tag{18}$$

From (12) and (13), it follows that for each $k = 1, \cdots, n$, the relators r_k, s_k, r^k_{1i} and r^k_{2j} in $\tilde{\mathcal{P}}$ consist of the generators $x_{ik}, x_{ik+1}, y_{jk}, y_{jk+1}, z_k$ or z_{k+1}, where $1 \leq i \leq a-1$ and $1 \leq j \leq b$. So all entries of the matrices $R_k(P), S_k(P), R^k_{1i}(P)$ and $R^k_{2j}(P)$ are polynomials with indeterminants which are the entries of the matrices $A_{ik}, A_{ik+1}, B_{jk}, B_{jk+1}, C_k$ and C_{k+1}. Hence we obtain that for each $k = 1, \cdots, n$,

$$\begin{aligned} R_k(P) &= r_k(P_1, P_2, \cdots, P_n) = r_k(P_k, P_{k+1}), \\ S_k(P) &= s_k(P_1, P_2, \cdots, P_n) = s_k(P_k, P_{k+1}), \\ R^k_{1i}(P) &= r^k_{1i}(P_1, P_2, \cdots, P_n) = r^k_{1i}(P_k, P_{k+1}), \\ R^k_{2j}(P) &= r^k_{2j}(P_1, P_2, \cdots, P_n) = r^k_{2j}(P_k, P_{k+1}) \end{aligned} \tag{19}$$

By (10), we have that $x_{i1} = x_{i+1}, y_{j1} = y_j, z_1 = \ell_1$, where x_{i+1}, y_j and ℓ_1 are the generators of the presentation $\mathcal{P}$ in (1) and so it follows from (2), (12) and (13) that

$$\begin{aligned} r_1(P_1, P_2) &= r_1(P_1, MP_1M^{-1}) = r(M, P_1), \\ s_1(P_1, P_2) &= s_1(P_1, MP_1M^{-1}) = s(M, P_1), \\ r^1_{1i}(P_1, P_2) &= r^1_{1i}(P_1, MP_1M^{-1}) = r_{1i}(M, P_1), \\ r^1_{2j}(P_1, P_2) &= r^1_{2j}(P_1, MP_1M^{-1}) = r_{2j}(M, P_1), \end{aligned} \tag{20}$$

where r, s, r_{1i} and r_{2j} are the relators of the presentation $\mathcal{P}$ of $G = \pi_1(S^3 - L_1)$ in (1). By (16), (17) and (19), it follows that $r(M, P_1) = I, s(M, P_1) = I, r_{1i}(M, P_1) = I, r_{2j}(M, P_1) = I$ and hence $P_1 \in V(\mathcal{U}_4(\mathcal{P}))$ and $(M, P_1) \in \mathcal{R}_n(L_1, \mathcal{P})$.

Conversely, let $P = (P_1, MP_1M^{-1}, \cdots, M^{n-1}P_1M^{-(n-1)})$ be a point of $\mathrm{M}(2,\mathbb{C})^{(a+b)n}$ satisfying the conditions Since $P_1 \in V(\mathcal{U}_4(\mathcal{P}))$ and the ideal $\mathcal{U}_4(\mathcal{P})$ contains all polynomials in (4), $P_1 \in \mathrm{SL}(2,\mathbb{C})^{a+b}$ and so $M^{k-1}P_1M^{-(k-1)}$
$\in \mathrm{SL}(2,\mathbb{C})^{a+b}$ for all $k = 2, \cdots, n$ Hence $P \in \mathrm{SL}(2,\mathbb{C})^{(a+b)n}$. Since $(M, P_1) \in \mathcal{R}_n(L_1, \mathcal{P})$, it follows from (19) and (20) that $R_1(P) = I, S_1(P) = I, R^1_{1i}(P) = I, R^1_{2j}(P) = I$. Then by (12), (13) and (19), we obtain that for each $k = 2, \cdots, n$,

$$\begin{aligned} R_k(P) &= r_k(P_k, P_{k+1}) \\ &= r_k(M^{k-1}P_1M^{-(k-1)}, M^{k-1}P_2M^{-(k-1)}) \\ &= M^{k-1}r_1(P_1, P_2)M^{-(k-1)} \\ &= M^{k-1}R_1(P)M^{-(k-1)} \\ &= I. \end{aligned}$$

Similarly, $S_k(P) = I, R^k_{1i}(P) = I$ and $R^k_{2j}(P) = I$ for all i, j and $k = 2, \cdots, n$ Therefore $P \in \mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$. This completes the proof. □

Let $\eta : \tilde{G} \to \mathrm{SL}(2,\mathbb{C})$ be a representation of $\tilde{G}$ in $\mathrm{SL}(2,\mathbb{C})$ and let $\Theta : \tilde{G} \to \tilde{G}$ denote the n-periodic automorphism of $\tilde{G}$ defined by $\Theta(x_{ik}) = x_{ik+1}, \Theta(y_{jk}) = y_{jk+1}$ and $\Theta(z_k) = z_{k+1}$. Then it is immediate that $\eta \circ \Theta$ is also a representation of $\tilde{G}$ in $\mathrm{SL}(2,\mathbb{C})$

THEOREM 4.2. Let $\mathcal{F}(\tilde{L}, \tilde{\mathcal{P}})$ denote the set of all points $P = (P_1, P_2, \cdots, P_n)$ in $\mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$ such that $\eta_P \circ \Theta = \eta_P$, where η_P denotes the representation of $\tilde{G}$ corresponding to the point P. Then

(1) $\mathcal{F}(\tilde{L}, \tilde{\mathcal{P}})$ is an affine algebraic subset of $\mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$
(2) $\mathcal{F}(\tilde{L}, \tilde{\mathcal{P}}) = \{(P_1, P_1, \cdots, P_1) \in R(\tilde{L}, \tilde{\mathcal{P}}) \mid \exists M \in \mathrm{GL}(2,\mathbb{C})$ s.t. $(M, P_1) \in \mathcal{R}_n(L_1, \mathcal{P}), MP_1 = P_1M\}$.

Proof. (1) Let $P = (P_1, P_2, \cdots, P_n)$ be a point of $\mathcal{F}(\tilde{L}, \tilde{\mathcal{P}})$, where $P_1, P_2, \cdots, P_n$ are points of $\mathrm{M}(2,\mathbb{C})^{a+b}$ as given in (14). Since $P = (P_1, P_2, \cdots, P_n)$ lies in $\mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$, P satisfies the matrix equations in (15) and (17). It is clear that $\eta_P \circ \Theta = \eta_P$ if and only if

$$(21) \qquad A_{ik+1} - A_{ik} = O, B_{jk+1} - B_{jk} = O, C_{k+1} - C_k = O$$

for all $1 \leq i \leq a-1, 1 \leq j \leq b$ and $k = 1, 2, \cdots, n$. This shows that P is a zero of the polynomials in (15) and the polynomials which are the entries of the left hand side matrix of the equations in (17) and (21).

(2) Let $P = (P_1, P_2, \cdots, P_n)$ be a point of $\mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$ such that $\eta_P \circ \Theta = \eta_P$, where $P_1, P_2, \cdots, P_n$ are points of $\mathrm{M}(2, \mathbb{C})^{a+b}$ as given in (14). By Theorem 4.1, $P_1 \in \mathcal{V}(\mathcal{U}_4(P))$ and for each $k = 2, 3, \cdots, n$, $P_k = M^{k-1} P_1 M^{-(k-1)}$ for some matrix $M \in \mathrm{GL}(2, \mathbb{C})$ such that $(M, P_1) \in \mathcal{R}_n(L_1, \mathcal{P})$. Since $\eta_P \circ \Theta = \eta_P$, it follows that $A_{ik+1} = A_{ik}, B_{jk+1} = B_{jk}, C_{k+1} = C_k$ and so $MA_{ik}M^{-1} = A_{ik}, MB_{jk}M^{-1} = B_{jk}, MC_kM^{-1} = C_k$ for all $k = 1, 2, \cdots, n-1$. Therefore $P_1 = MP_1M^{-1}$. Conversely, if $P = (P_1, P_2, \cdots, P_n)$ is a point of $\mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$ such that $P_1 = P_2 = \cdots = P_n$, then it is clear that the corresponding representation η_P satisfies that $\eta_P \circ \Theta = \eta_P$. This completes the proof. □

Let n be an integer ≥ 2 and set $\zeta = \exp(\frac{2\pi\sqrt{-1}}{n})$. For each $k = 0, 1, \cdots, n-1$, we define $\mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}})$ to be the subset of $\mathbb{C}^{4n(a+b)}$ given by $\mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}}) =$

$$\{(P, MPM^{-1}, \cdots, M^{n-1}PM^{-(n-1)}) \in \mathcal{R}(\tilde{L}, \tilde{\mathcal{P}}) \mid \det(M) = \zeta^k\}$$

and define $\phi_k : \mathcal{V}_k(L_1, \mathcal{P}) \to \mathrm{M}(2, \mathbb{C})^{n(a+b)} (= \mathbb{C}^{4n(a+b)})$ to be the mapping given by

$$\phi_k((M, P)) = (P, MPM^{-1}, \cdots, M^{n-1}PM^{-(n-1)})$$

for all $(M, P) \in \mathcal{V}_k(L_1, \mathcal{P})$.

LEMMA 4.3. (1) For each $k = 0, 1, \cdots, n-1$, ϕ_k is a regular map from $\mathcal{V}_k(L_1, \mathcal{P})$ onto $\mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}})$, i.e., $\phi_k(\mathcal{V}_k(L_1, \mathcal{P})) = \mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}})$.

(2) $\mathcal{R}(\tilde{L}, \tilde{\mathcal{P}}) = \bigcup_{k=0}^{n-1} \mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}})$

(3) $\phi_0(\mathcal{R}(K_2, \mathcal{P}_*)) \subset \mathcal{F}(\tilde{L}, \tilde{\mathcal{P}}) \subset \bigcap_{k=0}^{n-1} \mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}})$.

Proof. (1) Let $M = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, $P = (M_2, \cdots, M_{a+b}) \in \mathrm{M}(2, \mathbb{C})^{a+b-1}$, where $M_i = \begin{pmatrix} X_{4(i-1)+1} & X_{4(i-1)+2} \\ X_{4(i-1)+3} & X_{4i} \end{pmatrix}$ for each $i = 2, \cdots, a+b$.

Suppose that $(M, P) \in \mathcal{V}_k(L_1, \mathcal{P})$. By Theorem 4.1, $\phi_k((M, P)) = (P, MPM^{-1}, \cdots, M^{n-1}PM^{-(n-1)}) \in \mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$. Since $\det(M) = \zeta^k$, it follows that $\phi_k((M, P)) \in \mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}})$. It is easy to see that $\phi_k(\mathcal{V}_k(L_1, \mathcal{P})) = \mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}})$.

Now since $M^{-1} = \zeta^{-k} \begin{pmatrix} X_4 & -X_2 \\ -X_3 & X_1 \end{pmatrix}$, we have the following equations: for each $1 \le m \le n-1, 2 \le i \le a+b, M^m M_i M^{-m} =$

$$\frac{1}{\zeta^{km}} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}^m \begin{pmatrix} X_{4(i-1)+1} & X_{4(i-1)+2} \\ X_{4(i-1)+3} & X_{4i} \end{pmatrix} \begin{pmatrix} X_4 & -X_2 \\ -X_3 & X_1 \end{pmatrix}^m .$$

This shows that all entries of the matrix $M^m M_i M^{-m}$ are polynomials in $X_1, X_2, X_3, X_4, X_{4(i-1)+1}, X_{4(i-1)+2}, X_{4(i-1)+3}$ and X_{4i}. Therefore each ϕ_k is a regular map.

(2) Let $P = (P_1, \cdots, P_n)$ be a point of $\mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$ By Theorem 4.1, $P_1 \in V(\mathcal{U}_4(\mathcal{P}))$ and there exists a matrix $M \in \mathrm{GL}(2, \mathbb{C})$ such that $M^n = I$ and $P = (P_1, MP_1M^{-1}, \cdots, M^{n-1}P_1M^{-(n-1)})$ Since $M^n = I$, $\det(M)^n = 1$. So $\det(M)$ must be a n-th root of unity, i e., $\det(M) = \zeta^k$ for some $k (0 \le k \le n-1)$ Thus $P \in \mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}})$ for some $k (0 \le k \le n-1)$

(3) Let $P = (A_1, A_2, \cdots, A_a, B_1, \cdots, B_b, C_1)$ be a point of $\mathcal{R}(K_2, \mathcal{P}_*)$. By (7), $A_1 = I$. Set $P_1 = \pi_4(P) = (A_2, \cdots, A_a, B_1, \cdots, B_b, C_1)$. Note that $\phi_0(P) = (P_1, \cdots, P_1)$. By (4) of Proposition 3.2, $P = (I, P_1) \in \mathcal{V}_0(L_1, \mathcal{P}) \subset \mathcal{R}_n(L_1, \mathcal{P})$. By (2) of Theorem 4.2, $\phi_0(P) \in \mathcal{F}(\tilde{L}, \tilde{\mathcal{P}})$

Now let $P = (P_1, \cdots, P_1)$ be a point of $\mathcal{F}(\tilde{L}, \tilde{\mathcal{P}})$. By (2) of Theorem 4.2, there exists a matrix $M \in \mathrm{GL}(2, \mathbb{C})$ such that $(M, P_1) \in \mathcal{V}_j(L_1, \mathcal{P}) \subset \mathcal{R}_n(L_1, \mathcal{P})$ and $MP_1 = P_1M$ for some $0 \le j \le n-1$. For each $k = 0, 1, \cdots, n-1$, let $M_k = \zeta^{\frac{k-j}{2}} M$. Then $\det(M_k) = \zeta^k$. Since $\mathcal{F}(\tilde{L}, \tilde{\mathcal{P}}) \subset \mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$, by Theorem 4 2 $P_1 \in \mathcal{V}(\mathcal{U}_4(\mathcal{P}))$. It follows from (2) that (M_k, P_1) satisfies the matrix equations in (4), (5), and (6) and so $(M_k, P_1) \in \mathcal{V}_k(L_1, \mathcal{P})$ for each k. Note that $M_kP_1 = P_1M_k$ for all k. Now $P = (P_1, \cdots, P_1) = \phi_k(M_k, P_1) \in \phi_k(\mathcal{V}_k(L_1, \mathcal{P})) = \mathcal{R}_k(L_1, \mathcal{P})$ for all $k = 0, 1, \cdots, n-1$. Hence $P \in \bigcap_{k=0}^{n-1} \mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}})$. Therefore $\mathcal{F}(\tilde{L}, \tilde{\mathcal{P}}) \subset \bigcap_{k=0}^{n-1} \mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}})$ This completes the proof.

□

In view of (1) in Lemma 4.3, for each $k = 0, 1, \cdots, n-1$, we obtain an affine algebraic subset $\overline{\phi_k(\mathcal{V}_k(L_1, \mathcal{P}))}$ of $\mathbb{C}^{4n(a+b)}$. In the rest of this paper we denote it by $\tilde{\mathcal{V}}_k$ for simplicity, that is, $\tilde{\mathcal{V}}_k = \overline{\phi_k(\mathcal{V}_k(L_1, \mathcal{P}))} = \overline{R_k(\tilde{L}, \tilde{\mathcal{P}})}, 0 \le k \le n-1$ Then we have the following theorem·

THEOREM 4.4. (1) $\mathcal{R}(\tilde{L}, \tilde{\mathcal{P}}) = \bigcup_{k=0}^{n-1} \tilde{\mathcal{V}}_k$

(2) $\overline{\phi_0(\mathcal{R}(K_2, \mathcal{P}_*))} \subset \mathcal{F}(\tilde{L}, \tilde{\mathcal{P}}) \subset \bigcap_{k=0}^{n-1} \tilde{\mathcal{V}}_k$

Proof. (1) By Lemma 4 3, we obtain that $\mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}}) = \phi_k(\mathcal{V}_k(L_1, \mathcal{P})) \subset \tilde{\mathcal{V}}_k$ and

$$\mathcal{R}(\tilde{L}, \tilde{\mathcal{P}}) = \bigcup_{k=0}^{n-1} \mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}}) \subset \bigcup_{k=0}^{n-1} \tilde{\mathcal{V}}_k.$$

Note that $\mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}}) \subset \mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$ and $\mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$ is an affine algebraic subset of $\mathbb{C}^{4n(a+b)}$. Since $\tilde{\mathcal{V}}_k$ is the smallest algebraic subset of $\mathbb{C}^{4n(a+b)}$ containing $\phi_k(\mathcal{V}_k(L_1, \mathcal{P})) = \mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}})$, we have that $\tilde{\mathcal{V}}_k \subset \mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$ for all $k = 0, 1, \cdots, n-1$. Therefore $\bigcup_{k=0}^{n-1} \tilde{\mathcal{V}}_k \subset \mathcal{R}(\tilde{L}, \tilde{\mathcal{P}})$.

(2) By (3) of Lemma 4.3,

$$\overline{\phi_0(\mathcal{R}(K_2, \mathcal{P}_*))} \subset \mathcal{F}(\tilde{L}, \tilde{\mathcal{P}}) \subset \bigcap_{k=0}^{n-1} \mathcal{R}_k(\tilde{L}, \tilde{\mathcal{P}}) \subset \bigcap_{k=0}^{n-1} \tilde{\mathcal{V}}_k.$$

This completes the proof. □

COROLLARY 4.5. (1)

$$\dim(\overline{\phi_0(\mathcal{R}(K_2, \mathcal{P}_*))}) \le \dim(\mathcal{F}(\tilde{L}, \tilde{\mathcal{P}})) \le \dim(\mathcal{R}(\tilde{L}, \tilde{\mathcal{P}}))$$

(2)

$$\dim(\mathcal{F}(\tilde{L},\tilde{\mathcal{P}})) \leq \dim(\mathcal{R}(\tilde{L},\tilde{\mathcal{P}})) \leq \dim(\mathcal{V}_0(L_1,\mathcal{P})) \\ \leq \dim(\mathcal{R}(L_1,\mathcal{P})).$$

Proof. (1) follows from Theorem 4.2 and Theorem 4.4.

(2) Since $\mathcal{V}_0(L_1,\mathcal{P}) \subset \mathcal{R}(L_1,\mathcal{P})$, $\dim(\mathcal{V}_0(L_1,\mathcal{P})) \leq \dim(\mathcal{R}(L_1,\mathcal{P}))$. By Theorem 4.2, $\dim(\mathcal{F}(\tilde{L},\tilde{\mathcal{P}})) \leq \dim(\mathcal{R}(\tilde{L},\tilde{\mathcal{P}}))$ and, by Theorem 4.4, $\dim(\mathcal{R}(\tilde{L},\tilde{\mathcal{P}})) \leq \max\{\dim(\tilde{\mathcal{V}}_0),\dim(\tilde{\mathcal{V}}_1),\cdots,\dim(\tilde{\mathcal{V}}_{n-1})\}$. Since $\phi_k : \mathcal{V}_k(L_1,\mathcal{P}) \to \tilde{\mathcal{V}}_k$ is a dominating map, $\dim(\tilde{\mathcal{V}}_k) \leq \dim(\mathcal{V}_k(L_1,\mathcal{P}))$ for each $k=0,1,\cdots,n-1$. By (3) of Proposition 3 2, $\dim(\mathcal{V}_0(L_1,\mathcal{P})) = \dim(\mathcal{V}_k(L_1,\mathcal{P}))$ for all $k=1,\cdots,n-1$. Hence $\dim(\tilde{\mathcal{V}}_k) \leq \dim(\mathcal{V}_0(L_1,\mathcal{P}))$ for all $k=0,1,\cdots,n-1$. Therefore $\dim(\mathcal{R}(\tilde{L},\tilde{\mathcal{P}})) \leq \dim(\mathcal{V}_0(L_1,\mathcal{P}))$. This completes the proof. □

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