East Asian Math. J 19 (2003), No 2, pp 291–303

LIPSCHITZ CLASS, GROWTH OF DERIVATIVE AND UNIFORMLY JOHN DOMAINS

KIWON KIM

ABSTRACT A result of Hardy and Littlewood relates Holder continuity of analytic functions in the unit disk with a bound on the derivative Gehring and Martio extended this result to the class of uniform domains In this paper we obtain a similar result to the class of uniformly John domains in terms of the inner diameter metric We give several properties of a domain with the property Also we show some results on the Holder continuity of conjugate harmonic functions in the above domains

1. Introduction

Suppose that D is a domain in the complex plane $\mathbb{C}$. Let $\mathbb{B}(z,r) = \{w \ |w-z| < r\}$ for $z \in \mathbb{C}$ and $r > 0$ and let $\mathbb{B} = \mathbb{B}(0,1)$ be the unit disk in $\mathbb{C}$. Let $\ell(\gamma)$ denote the euclidean length of a curve γ, and $\mathrm{dist}(A,B)$ denote the euclidian distance from A to B for two sets $A, B \subset \overline{\mathbb{C}}$. Let $\mathrm{dia}(\gamma)$ denote a diameter of γ and let $\alpha \in (0,1]$.

A domain D in $\mathbb{C}$ is said to be *b-uniform* if there exists a constant $b \geq 1$ such that each pair of points z_1 and z_2 in D can be joined by a rectifiable arc γ in D with

$$\ell(\gamma) \leq b|z_1 - z_2|$$

Received November 4, 2003

2000 Mathematics Subject Classification 30C65, 30F45

Key words and phrases the Lipschitz class, internal metric, growth of derivative and uniformly John domains

and with

$$\min(\ell(\gamma_1), \ell(\gamma_2)) \le b \operatorname{dist}(z, \partial D) \tag{1.1}$$

for each $z \in \gamma$, where γ_1 and γ_2 are the components of $\gamma \setminus \{z\}$.

A bounded domain $D \subset \mathbb{C}$ is said to be a *b-John domain* if there exist a point $z_0 \in D$ and a constant $b \geq 1$ such that each point $z_1 \in D$ can be joined to z_0 by an arc γ in D satisfying

$$\ell(\gamma(z_1, z)) \le b \operatorname{dist}(z, \partial D)$$

for each $z \in \gamma$, where $\gamma(z_1, z)$ is the subarc of γ with endpoints z_1, z. We call z_0 a *John center*, b a *John constant* and γ a *b-John arc*. We call a simply connected John domain a *John disk*. A domain D in $\mathbb{C}$ is a b-John disk if and only if there is a constant $b \geq 1$ such that each pair of points $z_1, z_2 \in D$ can be joined by an arc γ in D which satisfies (1.1) [NV]. Thus the class of bounded uniform domains is properly contained in the class of John domains. The converse is not true, for example, $\mathbb{B} \setminus [0, 1)$.

We define the internal metric $\rho_D(x, y)$ by

$$\rho_D(x, y) = \inf \operatorname{dia}(\gamma)$$

for x, $y \in D$. Here infimum is taken over all open arcs which join x and y in D. Obviously $|x - y| \le \rho_D(x, y)$.

We say that D is a *b-uniformly John domain* if there exists a constant $b \geq 1$ such that each pair of points x, $y \in D$ can be joined by an arc $\gamma \subset D$ which satisfies (1.1) and

$$\ell(\gamma) \le b\rho_D(x, y). \tag{1.2}$$

We call b a *uniformly John constant* and γ a *b-uniformly John arc*.

A uniformly John domain is a domain intermediate between a uniform domain and a John domain. By definition the class of uniform domains is properly contained in the class of uniformly John domains and also the class of uniformly John domains is properly contained in the class of John domains. Balogh and Volberg [BV1, BV2] introduced a uniformly John domain in connection with conformal dynamics

Suppose that f is a real or complex valued function defined in D. We say that f is in the *Lipschitz class*, $Lip_\alpha(D)$, $0 < \alpha \leq 1$, if there exists a constant m such that

$$|f(z_1) - f(z_2)| \leq m|z_1 - z_2|^\alpha \tag{1.3}$$

for all z_1 and z_2 in D, and we let $||f||_\alpha$ denote the infimum of the numbers m for which (1.3) holds. f is said to belong to the *local Lipschitz class*, $locLip_\alpha(D)$, if there is a constant m such that (1 3) holds whenever z_1, z_2 lie in any open disk which is contained in D Let $||f||_\alpha^{loc}$ denote the infimum of the numbers m such that (1.3) holds in this situation.

A domain D is called a *Lip_α-extension domain* if there exists a constant a depending on D and α such that $f \in locLip_\alpha(D)$ implies $f \in Lip_\alpha(D)$ with

$$||f||_\alpha \leq a||f||_\alpha^{loc}.$$

Suppose that f is analytic in D. If f is in $Lip_\alpha(D)$, then it is not difficult to show that

$$|f'(z)| \leq m dist(z, \partial D)^{\alpha - 1}$$

in D. Conversely, we have the following well known result of Hardy and Littlewood.

THEOREM 1.1 *([HL]) If D is an open disk and f is analytic in D with*

$$|f'(z)| \leq m\mathrm{dist}(z, \partial D)^{\alpha - 1} \tag{1.4}$$

for all z in D and for every $\alpha \in (0, 1]$, then $f \in Lip_\alpha(D)$ with

$$||f||_\alpha \leq \frac{cm}{\alpha},$$

where c is an absolute constant.

The above theorem leads to the following notion, introduced in [GM1].

DEFINITION 1.2. A proper subdomain D in $\mathbb{C}$ is said to have the *Hardy-Littlewood property of order* α if there exists a constant $c = c(D)$ such that whenever f is analytic in D with (1.4) for all $z \in D$ and for some $\alpha \in (0, 1]$, then $f \in Lip_\alpha(D)$ with

$$||f||_\alpha \leq \frac{cm}{\alpha}.$$

Theorem 1.1 tells that each open disk has the Hardy-Littlewood property of order α. In [GM1, Corollary 2.2] it is proved moreover that uniform domains have the Hardy-Littlewood property and therefore have the Hardy-Littlewood property of order α. Also it is showed that there exist domains having the Hardy-Littlewood property of order α without being uniform [La].

Next in [AHHL] they give a characterization of a domain which has the Hardy-Littlewood property with order α as follows.

THEOREM 1.3. *([AHHL]) A simply connected domain D in $\mathbb{C}$ has the Hardy-Littlewood property with order α if and only if D is a Lip_α-extension domain.*

On the other hands, in [L] it is showed that the Hardy-Littlewood property of order α does not hold for John disks and that John disks hold analogues of the Hardy-Littlewood property which is explained in terms of the inner length metric.

THEOREM 1.4. *([L]) If D is a b-John disk and f is analytic in D and f satisfies the condition (1.4) in D for some $\alpha \in (0, 1]$, then*

$$|f(z_1) - f(z_2)| \leq \frac{cm}{\alpha}\lambda_D(z_1, z_2)^\alpha,$$

for all z_1 and z_2 in D, where c is a constant which depends only on b,

$$\lambda_D(z_1, z_2) = \inf \ell(\beta).$$

Here infimum is taken over all open arcs β in D which join z_1 and z_2.

Suppose that f is a real or complex valued function defined in D. We say that f is in the *Lipschitz class with the inner diameter metric*, $Lip_\alpha^d(D)$, $0 < \alpha \leq 1$, if there exists a constant m_1 such that

$$|f(z_1) - f(z_2)| \leq m_1 \rho_D(z_1, z_2)^\alpha \tag{1.5}$$

for all z_1 and z_2 in D, and we let $||f||_\alpha^d$ denote the infimum of the numbers m_1 for which (1.5) holds.

By definition it is clear that if $f \in Lip_\alpha(D)$, then $f \in Lip_\alpha^d(D)$.

DEFINITION 1 5. A proper subdomain D in $\mathbb{C}$ is said to have the *Hardy-Littlewood property with the inner diameter metric of order* α, if there exists a constant $c = c(D)$ such that whenever f is analytic and satisfies (1.4) in D for some $\alpha \in (0, 1]$, then f is in $Lip_\alpha^d(D)$ with

$$||f||_\alpha^d \leq \frac{c(D)}{\alpha} m.$$

Also a proper subdomain D in $\mathbb{C}$ is said to have the *Hardy-Littlewood property with the inner length metric of order* α, if we replace $\rho_D(z_1, z_2)$ by $\lambda_D(z_1, z_2)$ in definition of the Hardy-Littlewood property with the inner diameter metric of order α.

Clearly the Hardy-Littlewood property of order α implies the Hardy-Littlewood property with the inner diameter metric of order α and it implies the Hardy-Littlewood property with the inner length metric of order α.

In Section 2 we show a corresponding property to Theorem 1.4 for a uniformly John domain and give some properties of a domain which have the Hardy-Littlewood property with the inner diameter metric of order α Also in Section 3 we show some results on the Hölder continuity of conjugate harmonic functions in domains introduced above.

In [K] we showed similar properties for a John disk and for a domain which have the Hardy-Littlewood property with the inner length metric of order α to the theorems in this paper.

Results in this paper and in [K] also show that a uniformly John domain is a domain intermediate between a uniform domain and a John domain

2. Uniformly John domains and the Hardy-Littlewood property with the inner diameter metric of order α

Let

$$|\partial f(z)| = \lim\sup_{|h|\to 0} \frac{|f(z+h)-f(z)|}{|h|}$$

for $z \in D$.

THEOREM 2.1. *If D is a b-uniformly John domain and f is defined and satisfies*

$$|\partial f(z)| \leq m\text{dist}(z, \partial D)^{\alpha-1} \tag{2.1}$$

in D for some $\alpha \in (0,1]$, then

$$|f(z_1) - f(z_2)| \leq \frac{cm}{\alpha}\rho_D(z_1,\, z_2)^\alpha,$$

for all z_1 and z_2 in D, where c is a constant which depends only on b.

Proof. Fix $z_1, z_2 \in D$ and let γ be a uniformly John arc joining z_1, z_2 in D. Next let s denote arclength measured along γ from z_1, let $\ell = \ell(\gamma)$, let $z(s)$ denote the corresponding representation for γ and set $g(s) = f(z(s))$. Then

$$|\partial g(s)| = \lim\sup_{h\to 0} \frac{|g(s+h)-g(s)|}{|h|} \leq |\partial f(z(s))|.$$

By (1.1)

$$\min(s, \ell - s) \leq b\text{dist}(z(s), \partial D).$$

Thus by (2.1),

$$|\partial g(s)| \leq m\text{dist}(z(s), \partial D)^{\alpha-1} \leq (\frac{\min(s,\ell-s)}{b})^{\alpha-1}$$

for $0 < s < \ell$, g is absolutely continuous and

$$|f(z_1) - f(z_2)| = |g(\ell) - g(0)| \leq \int_0^\ell |\partial g(s)|ds \leq 2mb^{1-\alpha} \int_0^{\frac{\ell}{2}} s^{\alpha-1} ds$$
$$\leq \frac{2b^{1-\alpha}m}{\alpha}(\frac{\ell}{2})^\alpha \leq \frac{cm}{\alpha}\rho_D(z_1, z_2)^\alpha,$$

by (1.2) where $c = 2b$.

□

COROLLARY 2.2 *If D is a b-uniformly John domain, then D have the Hardy-Littlewood property with the inner diameter metric of order α with*

$$||f||^d_\alpha \leq \frac{c}{\alpha}m$$

for $c = c(b)$.

Now we show that the converse of Corollary 2.2 is not true.

THEOREM 2.3. *There exists a domain D in $\mathbb{C}$ having the Hardy-Littlewood property with the inner diamter metric of order α which is not a uniformly John domain*

Proof. Let $G_j = \mathbb{B}(z_j, \frac{2^{-j}}{\sqrt{3}})$ where $z_j = |z_j|e^{i\theta_j}$ and

$$|z_j| = 1 - \frac{4^{-j}}{2} + \frac{2^{-j}}{\sqrt{3}}, \quad \theta_j = \frac{3\pi}{2}(1 - 2^{-j}), \quad j = 0, 1, 2, \dots.$$

Next let $D = \mathbb{B} \cup \bigcup_{j=0}^\infty G_j$. Then we know that D is not a John disk [K, Theorem 2 1] and thus it is not uniformly John domain. But by [La, Corollary 7 11] D satisfies the Hardy-Littlewood property of order α and thus it has the Hardy-Littlewood property with the inner diameter metric of order α. □

Now let us recall the distance functions k_α and δ_α on a domain D, introduced in [KW]. For each $\alpha \in (0, 1]$ and for z_1, z_2 in D we define

$$k_\alpha(z_1, z_2) = \inf_\gamma \int_\gamma \operatorname{dist}(x, \partial D)^{\alpha-1} ds,$$

where the infimum is taken over all rectifiable arcs γ joining z_1 to z_2 in D. Furthermore,

$$\delta_\alpha(z_1, z_2) = \sup_f |f(z_1) - f(z_2)|,$$

where the supremum is taken over all analytic functions f on D satisfying

$$|f'(z)| \leq \operatorname{dist}(z, \partial D)^{\alpha-1}$$

for all $z \in D$

The next theorem characterizes a domain which satisfies the Hardy-Littlewood property with the inner diameter metric of order α.

THEOREM 2.4. *A domain D in $\mathbb{C}$ has the Hardy-Littlewood property with the inner diameter metric of order α if and only if there is a constant $M < \infty$ such that for all z_1, $z_2 \in D$ there exists a rectifiable curve γ joining z_1 to z_2 in D with*

$$\int_\gamma \operatorname{dist}(x, \partial D)^{\alpha-1} ds \leq M \rho_D(z_1, z_2)^\alpha. \tag{2.2}$$

To prove Theorem 2.4 we need the following Lemma 2.5 which shows that δ_α is connected to the metric k_α.

LEMMA 2.5. *([KW]) In a simply connected bounded domain $D \subset \mathbb{C}$ we have*

$$\delta_\alpha \leq k_\alpha \leq c_1 \delta_\alpha, \tag{2.3}$$

where $\alpha \in (0, 1]$ and c_1 is an absolute constant.

Proof of Theorem 2.4. Assume that D has the Hardy-Littlewood property with the inner diameter metric of order α. By the definition of δ_α, the second inequality of (2.3) and

$$|f(z_1) - f(z_2)| \leq \frac{c(D)}{\alpha}\rho_D(z_1, z_2)^\alpha$$

for f analytic in D and $|f'(z)| \leq \operatorname{dist}(z, \partial D)^{\alpha-1}$ in D, we obtain

$$k_\alpha(z_1, z_2) \leq c_1\delta_\alpha(z_1, z_2) = c_1 \sup_f |f(z_1) - f(z_2)| \leq \frac{c_1 c(D)}{\alpha}\rho_D(z_1, z_2)^\alpha.$$

Hence there exists a rectifiable curve γ joining z_1 and z_2 in D such that (2.2) is satisfied with $M = \frac{2c_1 c(D)}{\alpha}$.

Conversely, assume that there exists a constant $M < \infty$ such that for all $z_1, z_2 \in D$ there exists a rectifiable curve γ joining z_1 to z_2 in D with (2.2). Then by the first inequality of (2.3) for all f analytic in D with $|f'(z)| \leq \operatorname{dist}(z, \partial D)^{\alpha-1}$ in D, we obtain

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq \sup_f |f(z_1) - f(z_2)| \leq \inf_\gamma \int_\gamma \operatorname{dist}(x, \partial D)^{\alpha-1} ds \\ &\leq M\rho_D(z_1, z_2)^\alpha \end{aligned}$$

□

THEOREM 2.6. *If a domain D in $\mathbb{C}$ is a Lip_α-extension domain, then it has the Hardy-Littlewood property with the inner diameter metric of order α.*

Proof. In [GM2, Theorem 2.2] it is showed that a domain D in $\mathbb{C}$ is a *Lip_α-extension domain* if and only if there is a constant $M < \infty$ such that for all $z_1, z_2 \in D$ there exists a rectifiable curve γ joining z_1 to z_2 in D with (2.1) replaced $\rho_D(z_1, z_2)^\alpha$ by $|z_1 - z_2|^\alpha$. Therefore we get the conclusion.

□

REMARK 2.7. *By Theorem 2.3, Theorem 2.6 and definition of the Hardy-Littlewood property of order α, we observe that the classes of uniformly John domains, Lip_α-extension domain and domains which satisfies the Hardy-Littlewood property of order α are properly contained in the class of domains which satisfies the Hardy-Littlewood property with the inner diameter metric of order α.*

3. The Hölder continuity of conjugate harmonic functions in domains

LEMMA 3.1. *([GM1, Theorem 1.1]) If f is harmonic and in $Lip_\alpha(D)$, then*

$$|\partial f(z)| \leq \frac{4}{\pi}||f||_\alpha \mathrm{dist}(z, \partial D)^{\alpha-1} \tag{3.1}$$

in D.

In [GM1] combining Lemma 3.1 and the fact that an uniform domain has the Hardy-Littlewood property yields the following extension of a result due to Privaloff on the continuity of conjugate harmonic functions in the unit disk.

LEMMA 3.2. *([GM1, Corollary 2.2]) If D is b-uniform and if f is analytic with $Re(f)$ in $Lip_\alpha(D)$, then f is in $Lip_\alpha(D)$ with*

$$||f||_\alpha \leq \frac{c}{\alpha}||Re(f)||_\alpha, \tag{3.2}$$

where c is a constant which depends only on the constant b.

The above result does not hold for a uniformly John domain.

THEOREM 3.3. *There exists an analytic function f on a uniformly John domain such that $Re(f)$ is in $Lip_\alpha(D)$, but f is not in $Lip_\alpha(D)$.*

Proof. Let $D = \mathbb{B} \setminus (-1, 0]$ and define a function f on D by $f(z) = Log z$ which is an analytic branch of $log z$. Then clearly D is a uniformly John domain. Also $f(z) = log|z| + iArg(z)$ and $Re(f) = log|z|$ is in $Lip_\alpha(D)$, but $Arg(z)$ is not in $Lip_\alpha(D)$ [K]. □

To obtain an analogous result of Lemma 3.2 for a uniformly John domain, we need a following analogous result of Lemma 3.1 for $f \in Lip^d_\alpha(D)$. The proof is similar to the proof of Lemma 3.1 [GM1, Theorem 1.1].

THEOREM 3.4. *If f is harmonic and in $Lip^d_\alpha(D)$, then for $z \in D$*

$$|\partial f(z)| \leq \frac{4}{\pi}||f||^d_\alpha \mathrm{dist}(z, \partial D)^{\alpha-1}. \tag{3.3}$$

Proof. For $z \in \mathbb{C}$ and $0 < r < \infty$ let $B(z, r)$ denote the open disk with center z and radius r. If $z \in D$ and $r < \mathrm{dist}(z, \partial D)$, then $\overline{\mathbb{B}}(z, r) \subset D$ and with the Poisson integral formula we obtain

$$\begin{aligned} f(z+h) - f(z) &= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{r^2 - |h|^2}{|re^{i\theta} - h|^2} - 1\right)(f(z + re^{i\theta}) - f(z))d\theta \\ &= \frac{|h|}{\pi}\int_0^{2\pi} \frac{rcos(\theta - \phi) - |h|}{|re^{i\theta} - h|^2}(f(z + re^{i\theta}) - f(z))d\theta \end{aligned}$$

for $|h| < r$ where $h = |h|e^{i\phi}$. Thus by (1.5),

$$\frac{|f(z+h) - f(z)|}{|h|} \leq \frac{1}{\pi}\int_0^{2\pi} \frac{r|cos(\theta - \phi)| + |h|}{(r - |h|)^2} m\rho_D(z + re^{i\theta}, z)^\alpha d\theta.$$

Then since $\rho_D(z + re^{i\theta}, z) = r$, we have

$$|\partial f(z)| \leq \frac{4}{\pi} m r^{\alpha - 1}.$$

Letting $r \to \mathrm{dist}(z, \partial D)$ and $m \to ||f||^d_\alpha$ then yields (3.3).

□

THEOREM 3.5. *If a domain D in $\mathbb{C}$ has the Hardy-Littlewood property with the inner diameter metric of order α and if f is analytic with $Re(f) \in Lip^d_\alpha(D)$, then f is in $Lip^d_\alpha(D)$ with*

$$||f||^d_\alpha \leq \frac{8}{\pi}\frac{c(D)}{\alpha}||Re(f)||^d_\alpha. \tag{3.4}$$

Proof. Let $u = Re(f)$. Then u is harmonic in D,

$$|f'(z)| = |\frac{\partial u}{\partial x}(z) - i\frac{\partial u}{\partial y}(z)| \leq 2|\partial u(z)| \leq \frac{8}{\pi}||u||_\alpha^d \text{dist}(z, \partial D)^{\alpha-1}$$

by the Cauchy-Riemann equations and Theorem 3.4. Then since D satisfies the Hardy-Littlewood property with the inner diameter metric of order α, we obtain that f is in $Lip_\alpha^d(D)$ with (3.4). □

Now Corollary 2.2 and Theorem 3.5 give an analogous result of Lemma 3.2 for a uniformly John domain.

COROLLARY 3.6. *If a domain D in $\mathbb{C}$ is a b-uniformly John domain and if f is analytic with $Re(f)$ in $Lip_\alpha^d(D)$, then f is in $Lip_\alpha^d(D)$ with (3.4) replaced $c(D)$ by $c(b)$.*

REFERENCES

[AHHL] K. Astala, K Hag, P Hag and V Lappalainen, *Lipschitz Classes and the Hardy-Littlewood Property*, Monatsh. Math **115** (1993), 267-279

[BV1] Z. Balogh and A. Volberg, *Boundary Harnack principle for separated semihyperbolic repellers, harmonic measure applications*, Rev. Mat. Iberoamericana **12** (1996), 299-336

[BV2] Z. Balogh and A. Volberg, *Geometric localization, uniformly John property and separated semihyperbolic dynamics*, Ark Mat. **34** (1996), 21-49

[GM1] F. W Gehring and O Martio, *Quasidisks and the Hardy-Littlewood property*, Complex Variables Theory Appl. **2** (1983), 67-78.

[GM2] F W Gehring and O Martio, *Lipschitz classes and quasiconformal mappings*, Ann. Acad. Sci Fenn Math. **10** (1985), 203-219.

[HL] G H Hardy and J E Littlewood, *Some properties of fractional integrals II*, Math Z **34** (1932), 403-439.

[K] K. Kim, *Hardy-Littlewood property with the inner length metric*, Commun Korean Math Soc (to appear)

[KW] R Kaufman and J -M Wu, *Distances and the Hardy-Littlewood property*, Complex Variables Theory Appl. **4** (1984), 1-5

[L] N. Langmeyer, *The quasihyperbolic metric, growth and John domains*, Ann Acad Sci. Fenn. Math. **23** (1998), 205-224

[La] V Lappalainen, *Lip_α-extension domains*, Ann. Acad Sci Fenn Math Diss **56** (1985).

[NV] R. Nakki and J. Vaisala, *John disks*, Exposition. Math. **9** (1991), 3-43.

Department of Mathematics
Silla University
Busan 617-736, Korea
E-mail: kwkim@silla.ac kr