East Asian Math J 19 (2003), No 2, pp 261–271

TWO CLASSES OF THE GENERALIZED RANDERS METRIC

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ABSTRACT We deal with two metrics of Randers type, which are characterized by the solution of certain differential equations respectively Furthermore, we will give the condition for a Finsler space with such a metric to be a locally Minkowski space or a conformally flat space, respectively

1. Introduction

On a differentiable manifold M we shall consider a Finsler metric $L(\alpha, \beta)$ which is a positively homogeneous function of degree one of a Riemannian metric α and 1-form $\beta = b_i(x)y^i$. The structure $(M, L(\alpha, \beta))$ is called a Finsler space with (α, β)-metric. Kikuchi [2] has given the condition that a Randers space with $L = \alpha + \beta$ be a locally Minkowski space Recently a Finsler space with a special (α, β)-metric of Randers type has been studied by some authors ([5], [6]).

The purpose of the present paper is to consider two special (α, β)-metrics which are given by the solution of certain differential equations, and to give the condition that a Finsler space with the metric is a locally Minkowski space. Moreover, in the last section we study the conformally flatness of Finsler space with such a metric.

Received October 20, 2003

2000 Mathematics Subject Classification 53B40

Key words and phrases Berwald connection, locally Minkowski, conformally flat

This work is supported by the Dongil Culture and Scholarship Foundation research grants in 2003

2. Berwald connection and locally Minkowski space

For the Berwald connection $B\Gamma = (G_j{}^i{}_k(x,y), G^i_k(x,y), 0)$, the h-covariant derivative of a vector $X^i(x,y)$ is given by

$$X^i{}_{|j} = \delta_j X^i + G_r{}^i{}_j X^r,$$

where $\delta_j = \partial_j - G^r_j \dot{\partial}_r$, $\partial_j = \partial/\partial x^j$ and $\dot{\partial}_r = \partial/\partial y^r$. Moreover the h-curvature tensor H^2 of $B\Gamma$ is given by

$$H_h{}^i{}_{jk} = \mathcal{U}_{(jk)}(\delta_k G_h{}^i{}_j + G_h{}^r{}_j G_r{}^i{}_k), \tag{2.1}$$

where the symbol $\mathcal{U}_{(jk)}\{\cdots\}$ denotes the interchange of j, k and subtraction.

A Finsler space is called a *Berwald* space, if the connection coefficient $G_j{}^i{}_k$ of $B\Gamma$ is a function of position x^i alone in any coordinate system. If a Finsler space has a covering of coordinate neighborhoods in which g_{ij} does not depend on x, then it is called *locally Minkowski* ([1],[2]). It is well known that a Finsler space is a locally Minkowski, if and only if it is a Berwald space and h-curvature tensor of $B\Gamma$ vanishes.

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space with a fundamental metric function $L(\alpha,\beta)$. Throughout the paper our discussion is restricted to such a domain of M^n that the β does not vanish. Now we consider the function $F(\alpha,\beta)$ of two variables, and denote by the subscripts α, β of F the partial derivatives of F with respect to α, β respectively, that is,

$$F_\alpha = \partial F/\partial\alpha,\ F_\beta = \partial F/\partial\beta,\ F_{\alpha\beta} = \partial^2 F/\partial\alpha\partial\beta,\ \cdots .$$

If we put $F = L^2/2$, then the Cartan tensor $C_{ijk} = \dot{\partial}_k g_{ij}/2$ is given by

$$2C_{ijk} = (F_{\alpha\beta}/\alpha)(K_{ij}p_k + K_{jk}p_i + K_{ki}p_j) + F_{\beta\beta\beta}\ p_i p_j p_k, \tag{2.2}$$

where $K_{ij} = a_{ij} - y_i y_j/\alpha^2$, $y_i = a_{ij}y^j$ and $p_i = b_i - (\beta/\alpha^2)y_i$.

A Finsler space is called C-reducible if the Cartan tensor can be written in the form:

$$C_{ijk} = (h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)/(n+1), \ n \geq 3$$

where $h_{ij} = g_{ij} - l_i l_j, l_i = \dot{\partial}_i L$. According to [1], a C-reducible Finsler space induced to a Randers space and Kropina space.

Let $\gamma_j{}^i{}_k(x)$ be Christoffel symbols of the Riemannian metric α and $G_j{}^i{}_k$ be connection coefficients of $B\Gamma$ of $L(\alpha, \beta)$. To find the Berwald connection $B\Gamma$, we put $2G^i (= G^i{}_0) = \gamma_0{}^i{}_0 + 2B^i$, where the subscript 0 means a contraction by y^i. Then we have

$$G^i{}_j = \gamma_0{}^i{}_j + B^i{}_j,$$
$$G_j{}^i{}_k = \gamma_j{}^i{}_k + B_j{}^i{}_k,$$

where $B^i{}_j = \dot{\partial}_j B^i$ and $B_j{}^i{}_k = \dot{\partial}_j B^i{}_k$. Furthermore, the previous paper [4] gives the equation:

$$L_\alpha B_j{}^k{}_i y^j y_k = \alpha L_\beta (b_{j,i} - B_j{}^k{}_i b_k) y^j, \tag{2.3}$$

where (,) denotes the covariant differentiation with respect to the Riemannian connection $\gamma_j{}^i{}_k(x)$. It is obvious that a Finsler space with $L(\alpha, \beta)$ is a Berwald space if and only if $B_j{}^k{}_j$ given by (2.3) is a function of x alone.

We denote by R_{hijk} a Riemannian curvature tensor with respect to the $\gamma_j{}^i{}_k$ Then h-curvature tensor H^2 of (2.1) is given by [3]

$$H_h{}^i{}_{jk} = R_h{}^i{}_{jk} + \mathcal{U}_{(jk)}(B_h{}^i{}_{j,k} - B_0{}^r{}_k \partial_r B_h{}^i{}_j + B_h{}^r{}_j B_r{}^i{}_k). \tag{2.4}$$

From (2 4), consequently we have

THEOREM 2.1. *([3]) A $F^n = (M^n, L(\alpha, \beta))$ is a locally Minkowski if and only if $B_j{}^k{}_i$ is a function of x alone and $R_h{}^i{}_{jk}$ of the Riemannian metric α is written as:*

$$R_h{}^i{}_{jk} = -\mathcal{U}_{(jk)}(B_h{}^i{}_{j,k} + B_h{}^r{}_j B_r{}^i{}_k) \tag{2.5}$$

If we put $P_{i00} = B_j{}^k{}_i y^j y_k$ and $Q_{i0} = (b_{j,i} - B_j{}^k{}_i b_k)y^j$ in (2.3), we have

$$L_\alpha P_{i00} = \alpha L_\beta Q_{i0}. \tag{2.6}$$

If (2.6) gives $P_{i00} = Q_{i0} = 0$ necessarily, then from (2.3) we have $B_j{}^k{}_i = 0$ and $b_{j,i} = 0$, and (2.5) shows $R_h{}^i{}_{jk} = 0$. On the other hand, if $\alpha^2 \equiv 0\ (mod\beta)$, that is, $a_{ij}(x)y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. Hence in this paper, we assume that $b^2 \neq 0$ and $n \geq 3$.

3. Two classes of generalized Randers metric

Let the function $F(\alpha, \beta)$ be a positively homogeneous of degree 2 in α and β. From the homogeneity of F we obtain

$$\alpha F_{\alpha\alpha\alpha}+\beta F_{\alpha\alpha\beta} = 0,\ \alpha F_{\alpha\beta\alpha}+\beta F_{\alpha\beta\beta} = 0,\ \alpha F_{\beta\beta\alpha}+\beta F_{\beta\beta\beta} = 0, \tag{3.1}$$

which are rewritten in the form

$$\alpha F_{\sigma_1\sigma_2\alpha} + \beta F_{\sigma_1\sigma_2\beta} = 0,\ \sigma_1, \sigma_2 \in \{\alpha, \beta\}$$

If $F_{\beta\beta\beta} = 0$, from (3.1) we have $F_{\beta\beta\alpha} = F_{\alpha\beta\alpha} = F_{\alpha\alpha\alpha} = 0$. Thus we can see that $F_{\beta\beta\beta} = 0$ is equivalent to $F_{\sigma_1\sigma_2\sigma_3} = 0,\ \sigma_1, \sigma_2, \sigma_3 \in \{\alpha, \beta\}$

Let us find the solution of $F_{\beta\beta\beta} = 0$. Integrating this equation by β we get $F = f_1(\alpha)\beta^2 + f_2(\alpha)\beta + f_3(\alpha)$, where $f_i(\alpha)$, $i \in \{1, 2, 3\}$ is differentiable function. On the other side, paying attention to the homogeneity of F we find $L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$ by virtue of $F = L^2/2$. This is the same metric as [5]. If $c_1 = c_2 = c_3 = 1$, then $L = \alpha + \beta$, that is, L is a Randers metric. From (2.2) we can find a simple form of C_{ijk}. Therefore we have

PROPOSITION 3.1. *Let $F(\alpha, \beta)$ be a positively homogeneous function of degree 2 in α and β. Then the followings are equivalent to* *each other:*

$$a)\ C_{ijk} = K_{ij}B_k + K_{jk}B_i + K_{ki}B_j,$$
$$b)\ F_{\sigma_1\sigma_2\sigma_3} = 0,\ \sigma_1, \sigma_2, \sigma_3 \in \{\alpha, \beta\},$$
$$c)\ L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2,\ c_1, c_2, c_3 \neq 0, 1.$$

It is natural to generalize this result in the following way. First, let us consider a differential equation $F_{\underbrace{\beta\beta\cdots\beta}_{m+1}} = 0$, where a function F is a positively homogeneous of degree m in α and β. Integrating this equation by β continuously and paying attention to the homogeneity of F, we get

$$(3.2)\quad F(\alpha, \beta) = c_0\alpha^m + c_1\alpha^{m-1}\beta + \cdots + c_m\beta^k = \sum_{k=0}^{m} c_k\alpha^{m-k}\beta^k,$$

where $c_0, c_1, \ldots, c_m$ are constants.

By the similar way in (3.1), for $\sigma_1, \sigma_2, \ldots, \sigma_m \in \{\alpha, \beta\}$ if we assume that a function F is a positively homogeneous of degree m in α and β, then a function $F_{\sigma_1\sigma_2\cdots\sigma_m}$ is positively homogeneous of degree 0 in α and β. Thus we have

$$(3.3)\qquad \alpha F_{\sigma_1\sigma_2\cdots\sigma_m\alpha} + \beta F_{\sigma_1\sigma_2\cdots\sigma_m\beta} = 0,\ \sigma_1, \sigma_2, \ldots, \sigma_m \in \{\alpha, \beta\}.$$

Therefore, from (3.3) if $F_{\underbrace{\beta\beta\cdots\beta}_{m+1}} = 0$, then we obtain

$$(3.4)\qquad F_{\sigma_1\sigma_2\cdots\sigma_{m+1}} = 0,\ \sigma_1, \sigma_2, \ldots, \sigma_{m+1} \in \{\alpha, \beta\}.$$

It is noted that the solution of the equation (3.4) is given by (3.2). In a Finsler space, since an (α, β)-metric $L(\alpha, \beta)$ is a positively homogeneous of degree 1 in α and β, it is possible to give an (α, β)-metric by putting $F = L^m$. Summarizing up the above, we have

THEOREM 3.1. *Let $F(\alpha,\beta)$ be a positively homogeneous function of degree m in α and β. The followings are equivalent to each other:*

$$\text{(3.5)} \qquad \begin{aligned} &a)\ F_{\sigma_1\sigma_2\cdots\sigma_{m+1}} = 0,\ \sigma_1,\sigma_2,\ldots,\sigma_{m+1} \in \{\alpha,\beta\}, \\ &b)\ L(\alpha,\beta) = \Big(\sum_{k=0}^{m} c_k \alpha^{m-k}\beta^k \Big)^{1/m},\ F = L^m, \end{aligned}$$

where $c_0, c_1, \cdots, c_m$ are constants.

REMARK. *Theorem 3.1 means that a general solution of the differential equation $F_{\sigma_1\sigma_2\cdots\sigma_{m+1}} = 0,\ \sigma_1,\sigma_2,\ldots,\sigma_{m+1} \in \{\alpha,\beta\}$, does not depend on the choice of the subscript variables α and β.*

Secondly, let us find another class of the generalized Randers metric type. We consider $F(\alpha,\beta)$, which is a positively homogeneous function of degree m in α and β. Paying attention to the homogeneity of F, we see that the solution of $F_{\alpha\beta} = 0$ is $F = c_1\alpha^m + c_2\beta^m$, where $c_1,\ c_2$ are constants. Thus we have

THEOREM 3.2. *Let $F(\alpha,\beta)$ be a positively homogeneous function of degree m in α and β. Then the followings are equivalent to each other:*

$$\text{(3.6)} \qquad \begin{aligned} &a)\ F_{\alpha\beta} = 0, \\ &b)\ L(\alpha,\beta) = (c_1\alpha^m + c_2\beta^m)^{1/m},\ F = L^m, \end{aligned}$$

where $c_1,\ c_2$ are constants.

4. Berwald space and locally Minkowski space

We first deal with a Finsler space with the metric (3.5). Let $F^n = (M^n, L)$ be an n-dimensional Finsler space (≥ 3) whose metric function is given by (3.5). Then the partial derivatives with respect to α and β of a metric (3.5) are given by

$$\begin{aligned} mL^{m-1}L_\alpha &= mc_0\alpha^{n-1} + (m-1)c_1\alpha^{m-2}\beta + \cdots + c_{m-1}\alpha^{m-1} \\ &= \sum_{k=0}^{m-1}(m-k)c_k\alpha^{m-1-k}\beta^k, \\ mL^{m-1}L_\beta &= c_1\alpha^{n-1} + 2c_2\alpha^{m-2}\beta + \cdots + mc_m\beta^{m-1} \\ &= \sum_{k=1}^{m} kc_k\alpha^{m-k}\beta^{k-1}. \end{aligned}$$

From these equations and (2.6), we obtain

$$\alpha(AP_{i00} - BQ_{i0}) + CP_{i00} - DQ_{i0} = 0, \tag{4.1}$$

where

$$\begin{aligned} A &= \sum_{r=0}^{s} 2rc_{m-2r}\alpha^{2r-2}\beta^{m-2r},\ s \le \frac{m}{2} \\ B &= \sum_{r=0}^{s}(m-2r)c_{m-2r}\alpha^{2r}\beta^{m-2r-1},\ s = \frac{m-1}{2} \\ C &= \sum_{r=0}^{s}(2r+1)c_{m-2r-1}\alpha^{2r}\beta^{m-2r-1},\ s = \frac{m-1}{2} \\ D &= \sum_{r=0}^{s}(m-2r-1)c_{m-2r-1}\alpha^{2(r+1)}\beta^{m-2(r+1)},\ s \le \frac{m-1}{2}, \end{aligned}$$

where s is a positive integer.

Assume that the Finsler space is a Berwald space, that is, $B_j{}^k{}_i$ is a function of position only. Since α is irrational in y^i, from (4.1) we get

$$\begin{aligned} A\ P_{i00} - B\ Q_{i0} &= 0, \\ C\ P_{i00} - D\ Q_{i0} &= 0. \end{aligned} \tag{4.2}$$

If we consider a determinant ω:

$$\omega = \begin{vmatrix} A & -B \\ C & -D \end{vmatrix},$$

then $\omega \neq 0$. Thus, from (4.2) we have $P_{i00} = 0$ and $Q_{i0} = 0$, from which we conclude $B_j{}^k{}_i = 0$ and $b_{j,i} = 0$.

Conversely if $b_{j,i} = 0$, then the space with an (α, β)-metric is a Berwald space. Thus we have

THEOREM 4.1. *Let F^n be an n-dimensional Finsler space ($n \geq 3$) with the metric (3.5). It is a Berwald space if and only if $b_{j,i} = 0$, and then $B\Gamma = (\gamma_j{}^k{}_i, \gamma_0{}^k{}_i, 0)$.*

In the case $B_j{}^k{}_i = 0$, from (2.5) we obtain $R_h{}^i{}_{jk} = 0$. Summarizing up the above results and using Theorem 2.1, we have

THEOREM 4.2. *Let F^n be an n-dimensional Finsler space ($n \geq 3$) with the metric (3.5). It is a locally Minkowski space if and only if $R_h{}^i{}_{jk} = 0$ and $b_{j,i} = 0$.*

REMARK. *If $m = 1$ in (3.5), then $L = c_0\alpha + c_1\beta$ (Randers type). In this case, the equation (4.1) yields $A = 0$, $B = c_1$, $C = c_0$ and $D = 0$, which imply $c_0 P_{i00} - c_1 \alpha Q_{i0} = 0$. Since α is irrational in y, we have $P_{i00} = Q_{i0} = 0$. It is noted that the space $(M^n, L = c_0\alpha + c_1\beta)$ is locally Minkowski if and only if $R_h{}^i{}_{jk} = 0$ and $b_{j,i} = 0$. This is the same result as Theorem 2.2 of [2].*

Next, we consider the metric (3.6). If $m = 2$, we get $L^2 = c_1\alpha^2 + c_2\beta^2$, which means that $L(\alpha, \beta)$ is a Riemannian metric. Hence we shall treat the non-Riemannian space afterward and assume that $m \neq 2$. The partial derivatives with respect to α and β of a metric (3.6) are given by

$$L_\alpha = (\alpha/L)^{m-1}, \; L_\beta = (\beta/L)^{m-1}. \tag{4.3}$$

Substituting (4.3) into (2.6), we obtain

$$\alpha^{m-1} P_{i00} = \alpha \beta^{m-1} Q_{i0}. \tag{4.4}$$

Now we shall divide our consideration in two cases of which m is even or odd.

(I) Case of $m = 2h+2$, (h is a positive integer) When $m = 2h+2$, we have from (4.4)

$$\alpha^{2h} P_{ioo} = \beta^{2h+1} Q_{i0}. \tag{4.5}$$

Since $\alpha \not\equiv 0 \pmod{\beta}$, from (4.5) we have $B_j{}^i{}_k = 0$ and $b_{i,j} = 0$.

(II) Case of $m = 2h - 1$, (h is a positive integer) From (2.6) and (4.3), we find

$$\alpha^{2h} P_{i00} = \alpha \psi Q_{i0}, \ \psi = \alpha^2 \beta^{2h-2}. \tag{4.6}$$

The terms $\alpha^{2h} P_{i00}$ and ψQ_{i0} of (4.6) are rational in y^i, while α is irrational in y^i. Thus we have, from (4.6), $P_{i00} = Q_{i0} = 0$, which implies $B_j{}^k{}_i = 0$ and $b_{j,i} = 0$. Summarizing case (I) and case (II), we have

THEOREM 4.3. *Let F^n be an n-dimensional Finsler space ($n \geq 3$) with the metric (3.6). It is a locally Minkowski space if and only if $R_h{}^i{}_{jk} = 0$ and $b_{j;i} = 0$*

5. Conformal flatness

Let $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ be two Finsler spaces on the same underlying manifold M^n. If we have a function $\sigma(x)$ in each coordinate neighborhoods of M^n such that $\bar{L}(x,y) = e^\sigma L(x,y)$, then F^n is called *conformal* to $\bar{F}^n$ and the change $L \to \bar{L}$ of metric is called *conformal.* For $\sigma(x)$, a conformal change ([1]) of (α,β)-metric is expressed as $(\alpha,\beta) \to (\bar{\alpha},\bar{\beta})$, where $\bar{\alpha} = e^\sigma \alpha$ and $\bar{\beta} = e^\sigma \beta$. A Finsler space is called *conformally flat,* if it is conformal to a locally Minkowski space. In the previous papers [2], [3] and [5], the authors dealt with conformally flat spaces. For an (α,β)-metric, a conformally invariant symmetric linear connection $M_j{}^i{}_k$ is defined by [1]

$$M_j{}^i{}_k = \gamma_j{}^i{}_k + \delta^i_j M_k + \delta^i_k M_j - M^i a_{jk},$$

where $M_j = \{b_{j,k}b^k - b^k{}_{,k}b_j/(n-1)\}/b^2$ and $M^i = a^{ij}M_j$. We denote by $\overset{m}{\nabla}$ and $M_h{}^i{}_{jk}$ the covariant differentiation with respect to $M_j{}^i{}_k$ and the curvature tensor of this connection respectively. A Finsler space with an (α, β)-metric is called flat-parallel, if $\widehat{R}_h{}^i{}_{jk} = 0$ and $b_{i,j} = 0$.

THEOREM 5.1. *([3]) A Finsler space with (α, β)-metric is conformal to a flat-parallel Minkowski space if and only if the condition*

$$M_h{}^i{}_{jk} = 0, \overset{m}{\nabla}_j M_i = \overset{m}{\nabla}_i M_j, \overset{m}{\nabla}_j b_i = -b_i M_j \tag{5.1}$$

is satisfied.

In an (α, β)-metric, a conformal change preserves the type of metric invariant. From Theorem 4.2 (resp. Theorem 4.3), we can see that F^n with the metric (3.5) (resp (3.6)) is flat-parallel. Thus these conditions are also applicable to the metric (3.5)(resp. (3.6)). Consequently, from Theorem 4.2, Theorem 4.3 and Theorem 5.1 we have

THEOREM 5.2. *Let F^n be an n-dimensional Finsler space ($n \geq 3$) with the metric (3.5) (resp. (3.6)) It is conformally flat if and only if the condition (5.1) is satisfied.*

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