East Asian Math J 19 (2003), No 2, pp 233–240

M-SYSTEM AND N-SYSTEM IN PO-SEMIGROUPS

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ABSTRACT Xie and Wu introduced an m-system in a po-semigroup. Kehayopulu gave characterizations of weakly prime ideals of po-semigroups and Lee and Kwon add two characterizations for weakly prime ideals. In this paper, we give a characterization of weakly prime ideals and a characterization of weakly semi-prime ideals in po-semigroups using m-system and n-system, respectively

Recently, Xie and Wu introduced an m-system in a po-semigroup ([9]). The definition of m-system in a po-semigroup is an extended form of the concept of m-system of semigroups without order (see the book of Petrich ([8])). Now we introduce the definition of n-system in a po-semigroup. Also, the definition of n-system in a po-semigroup is an extended form of the concept of n-system of semigroups without order Giri and Wazalwar studied the properties of m-system and n-system in semigroups ([1])

Kehayopulu([2, 3]) introduced the concepts of weakly prime ideals of ordered semigroups and gave the characterizations of weakly prime (weakly semi-prime) ideals of ordered semigroups analogous to the characterizations of weakly prime ideals of rings considered by Mc-Coy([5, 6]) and Steinfeld ([8]). And Lee and Kwon([4]) gave two characterizations of weakly prime ideals of ordered semigroups.

Received August 8, 2003 Revised December 2, 2003

2000 Mathematics Subject Classification 06F05

Key words and phrases po-semigroup, ideal, left(right) ideal, weakly prime, weakly semi-prime m-system, n-system, m-radical

In this paper, we give a new characterization of weakly prime ideals and a new characterization of weakly semi-prime ideals in a *po*-semigroup using an m-system and n-system, respectively.

A *po*-semigroup(: ordered semigroup) is an ordered set $(S, \leq)$ at the same time a semigroup such that: $a \leq b \Longrightarrow ca \leq cb$ and $ac \leq bc$ for all $a, b, c \in S$.

Let S be a *po*-semigroup and A a nonempty subset of S. A is called a *left*(resp. *right*) *ideal* of S if (1) $SA \subseteq A$(resp. $AS \subseteq A$), (2) $a \in A, b \leq a$ for $b \in S \Longrightarrow b \in A$. A is called an *ideal* of S if it is a right and left ideal of S([2, 3, 4]).

An ideal T of a *po*-semigroup S is *weakly prime* if and only if for each $a, b \in S$ such that $(aSb] \subseteq T$, we have $a \in T$ or $b \in T$. T is a *weakly semi-prime* if and only if for each $a \in S$ such that $(aSa] \subseteq T$, we have $a \in T$([2, 3]).

For any subset H of S, let $(H]$ denote the set of all element of X which are less than or equal to some $h \in H$, i.e.,

$$(H] := \{t \in S | t \leq h \text{ for some } h \in H\}$$

DEFINITION 1. Let M be a non-empty subset of a *po*-semigroup S. M is called an *m-system* if for every $a, b \in M$ there exists $x \in S$ such that $(axb] \cap M \neq \emptyset$, and M is called a *strong m-system* if for every $a, b \in M$ there exists $x \in S$ such that $axb \in M$.

Let M be a non-empty subset of a *po*-semigroup S. N is called an *n-system* if for every $a \in M$ there exists $x \in S$ such that $(axa] \cap N \neq \emptyset$, and N is called a *strong n-system* if for every $a \in M$ there exists $x \in S$ such that $axa \in N$.

The concepts of "m-system" and "strong m-system" in a semigroup (without order) are coincide

REMARK. *If M is a strong m-system, then $(M]$ is also Indeed: If $x, y \in (M]$, then there exist $m_1, m_2 \in M$ such that $x \leq m_1$ and $y \leq m_2$. Since M is a strong m-system, $m_1 z m_2 \in M$ for some $z \in S$. Thus $xzy \leq m_1 z m_2 \in M$, and so $xzy \in (M]$. Therefore $(M]$ is a strong m-system.*

We denote by $I(a)$(resp. $L(a), R(a)$) the ideal(resp. left ideal, right ideal) of S generated by a. One can easily prove that:

$$I(a) = (a \cup Sa \cup aS \cup SaS], \quad L(a) = (a \cup Sa], \quad R(a) = (a \cup aS].$$

We note the following lemma:

LEMMA 1. *([2, 3, 4]) Let S be a po-semigroup. Then we have*
1) *$A \subseteq (A]$ for all $A \subseteq S$.*
2) *$(A] \subseteq (B]$ for $A \subseteq B \subseteq S$.*
3) *$(A](B] \subseteq (AB]$ for all $A, B \subseteq S$*
4) *$((A]] \subseteq (A]$ for all $A \subseteq S$.*
5) *For every left ideal(resp. right ideal, ideal) T of S, $(T] = T$.*
6) *If A, B are ideals of S, then $(AB], A \cap B$ are ideals of S.*
7) *For $a \in S$, $(SaS]$ is an ideal of S.*

THEOREM A. *([2, 4]) Let S be a po-semigroup and T an ideal of S. The following are equivalent.*
(1) *T is weakly prime.*
(2) *If $a, b \in S$ such that $(aSb] \subseteq T$, then $a \in T$ or $b \in T$.*

From Theorem A, we have the following theorem 1.

THEOREM 1. *A proper ideal A of a po-semigroup S is weakly prime if and only if $S \setminus A$ is an m-system.*

Proof. Suppose that a proper ideal A is weakly prime and $a, b \in S \setminus A$ Then $a \notin A$ and $b \notin A$. By (2) of Theorem A, $(aSb] \nsubseteq A$. Now we show that $axb \in S \setminus A$ for some $x \in S$. Suppose that $axb \in A$ for all $x \in S$. Then $aSb \subseteq A$. Thus $(aSb] \subseteq (A] = A$ by (5) of Lemma 1. It is impossible. Hence $axb \in S \setminus A$ for some $x \in S$, and so $(axb] \cap (S \setminus A) \neq \emptyset$. Therefore $S \setminus A$ is an m-system.

Conversely, suppose that $S \setminus A$ is an m-system. Then for $a, b \in S \setminus A$ there exists $x \in S$ such that $(axb] \cap (S \setminus A) \neq \emptyset$. Thus there exists y in S such that $ayb \in (axb] \cap (S \setminus A)$, and so $ayb \notin A$. Hence $aSb \nsubseteq A$, and so $(aSb] \nsubseteq A$. This is the contrapositive form of (2) of Theorem A. Therefore A is weakly prime by (1) of Theorem A. □

THEOREM B. *([2]) If A is an ideal in a po-semigroup of S then the following are equivalent.*

(1) *A is weakly semi-prime.*

(2) *If $(aSa] \subseteq A$, then $a \in A$*

From Theorem B we have the following theorem 2.

THEOREM 2. *A proper ideal A of a po-semigroup S is weakly semi-prime if and only if $S \setminus A$ is an n-system.*

Proof. Suppose that A is weakly semi-prime of S and $a \in S \setminus A$. Then $a \notin A$. By (2) of Theorem B, $(aSa] \nsubseteq A$. Now we show that $axa \in S \setminus A$ for some $x \in S$ Suppose that $axa \in A$ for all $x \in S$. Then $aSa \subseteq A$. Thus $(aSa] \subseteq (A] = A$ by (5) of Lemma 1. It is impossible. Hence $axa \in S \setminus A$ for some $x \in S$. Therefore $(axa] \cap (S \setminus A) \neq \emptyset$, and so $S \setminus A$ is an n-system.

Conversely, suppose that $S \setminus A$ is an n-system. Then for $a \in S \setminus A$, there exists $x \in S$ such that $(axa] \cap (S \setminus A) \neq \emptyset$. Thus there exists $y \in S$ such that $aya \in (axa] \cap (S \setminus A)$, and so $aya \notin A$. Hence $aSa \nsubseteq A$, and so $(aSa] \nsubseteq A$. This is the contrapositive form of (2) of Theorem B. Therefore A is weakly semi-prime by (1) of Theorem B. □

THEOREM 3 *If $(N]$ is a strong n-system containing a in a po-semigroup S, then there exists a strong m-system $(M]$ containing a such that $(M] \subseteq (N]$.*

Proof. Let $(N]$ be a strong n-system containing a. Then there exists $x \in S$ such that $axa \in (N]$. Thus $aSa \cap (N] \neq \emptyset$. We can take $a_1 \in aSa \cap (N]$. Since $a_1 \in (N]$ and $(N]$ is an n-system, there exists $x_1 \in S$ such that $a_1x_1a_1 \in (N]$. Also we can take $a_2 \in a_1Sa_1 \cap (N]$. Since $a_2 \in (N]$ and $(N]$ is an n-system, there exists $x_2 \in S$ such that $a_2x_1a_2 \in (N]$. Continuing this process, we can take $a_{i+1} \in (a_iSa_i] \cap (N]$. Now we construct M as follows;

$$M := \{a, a_1, a_2, \cdots a_i, a_{i+1}, \cdots\}.$$

Then $(M]$ is a strong m-system. Indeed: Let $b_i, b_j \in (M]$ and $i < j$. Then there exist $a_i, a_j \in M$ such that $b_i \leq a_i$ and $b_j \leq a_j$. We have

$$a_{j+1} \in a_j S a_j \subseteq (a_{j-1} S a_{j-1}) S a_j \subseteq a_{j-1} S a_j$$

$$\subseteq (a_{j-2} S a_{j-2}) S a_j \subseteq a_{j-2} S a_j \subseteq \cdots \subseteq a_i S a_j.$$

Thus $a_{j+1} = a_i x a_j$ for some $x \in S$. Therefore $b_i x b_j \leq a_i x a_j = a_{j+1} \in M$. It follows that $(M]$ is a strong m-system.

Finally, we note that $a \in M \subseteq (N]$ Therefore $(M] \subseteq ((N]] = (N]$. □

A semigroup S is a po-semigroup with the partial order $\Delta := \{(a, a) \mid \forall a \in S\}$. Hence $(A] = A$ for a subset A of a semigroup S. Since the concept m-system and strong m-system in a semigroup (without order) are coincide. Therefore we have the following corollaries.

COROLLARY 1. *([1]) A proper ideal A of a semigroup S is weakly prime if and only if $S \setminus A$ is an m-system.*

COROLLARY 2. *([1]) A proper ideal A of a semigroup S is weakly semi-prime if and only if $S \setminus A$ is an n-system.*

COROLLARY 3. *([1]) If N is an n-system in a semigroup S and containins an element a of S, then there exists an m-system M of S such that $a \in M$ and $M \subseteq N$.*

Now we give a new concept in a po-semigroup.

DEFINITION 2. The *m-radical* of an ideal A in a po-semigroup S is the set consisting of those elements $r \in S$ with the property that every strong m-system M in S which contains r meets A(that is, it has nonempty intersection with A). It is denoted by $\sqrt{A}$.

$\sqrt{A} = \{r \in S \mid M \cap A \neq \emptyset$ for every strong m-system M containing $x\}$ (cf. [1]).

We give the following lemma.

LEMMA 2. *Let A be an ideal of a po-semigroup S and $x \in S$. If $x \in \sqrt{A}$, then $x^n \in A$ for a positive integer n.*

Proof. If $x \in \sqrt{A}$, then $M \cap A \neq \emptyset$ for every strong m-system M containing x. Consider $B = \{x^i \mid i = 1, 2, \cdots\}$. Then for any x^i and x^j in B, $x^i x x^j = x^{i+1+j} \in B$. Thus B is a strong m-system containing x, and so $B \cap A \neq \emptyset$. Hence $x^n \in A$ for some positive integer n. □

THEOREM 4. *If A is an ideal in a po-semigroup S, then $\sqrt{A} = \bigcap_\alpha P_\alpha$ for all weakly prime ideal P_α containing A.*

Proof. Let P_α be a weakly prime ideal containing A and $x \in \sqrt{A}$. Then by Lemma 2, $x^n \in A \subseteq P_\alpha$ for some positive integer n and for all α. Since each P_α is weakly prime, $x \in P_\alpha$ for all α. Therefore $\sqrt{A} \subseteq \bigcap_\alpha P_\alpha$ for all weakly prime ideals P_α containing A

Now we show that $\bigcap_\alpha P_\alpha \subseteq \sqrt{A}$ for all weakly prime ideals P_α containing A. Suppose $x \notin \sqrt{A}$. Then there exists a strong m-system M containing x such that $M \cap A = \emptyset$. Consider the set $\mathcal{B}$ of all ideals I of S such that $A \subseteq I$ and $M \cap I = \emptyset$. Since $A \subseteq A$ and $M \cap A = \emptyset$, we get $A \in \mathcal{B}$, and so $\mathcal{B}$ is non-empty. Then $(\mathcal{B}, \subseteq)$ is an ordered set Let $\mathcal{C}$ be a chain in $\mathcal{B}$. Then the set $\bigcup_{c \in \mathcal{C}} C$ is an ideal of S and is an upper bound of $\mathcal{C}$ in $\mathcal{B}$. By Zorn's Lemma, there exists a maximal ideal P such that $A \subseteq P$ and $M \cap P = \emptyset$ Since $x \in M$, we note that $x \notin P$. Now we claim that P is weakly prime. If $a, b \notin P$, then $P \subsetneq P \cup I(a)$ and $P \subsetneq P \cup I(b)$ Since $P \cup I(a)$ and $P \cup I(b)$ are ideals, $M \cap (P \cup I(a)) \neq \emptyset$ and $M \cap (P \cup I(b)) \neq \emptyset$ by the maximality of P. Hence there exist $m_1 \in M \cap (P \cup I(a))$ and $m_2 \in M \cap (P \cup I(b))$. Since M is an m-system, $m_1 z m_2 \in M$ for some z in S. Moreover $m_1 z m_2 \in I(a)SI(b) \subseteq I(a)I(b)$. If $I(a)I(b) \subseteq P$, then

$$\begin{aligned} m_1 z m_2 &\in (P \cup I(a))S(P \cup I(b)) \\ &= PSP \cup I(a)SP \cup PSI(b) \cup I(a)SI(b) \\ &\subseteq P. \end{aligned}$$

Thus $m_1 z m_2 \in M \cap P$, and so $M \cap P \neq \emptyset$. It is impossible. Hence $I(a)I(b) \nsubseteq P$ for $a \notin P$ and $b \notin P$ This is the contrapositive form (3) of Theorem A. Therefore P is a weakly prime ideal by (1) of Theorem A. □

THEOREM 5. *If A and B are any two ideals in a po-semigroup S, then:*

(1) $A \subseteq B \implies \sqrt{A} \subseteq \sqrt{B}$.

(2) $\sqrt{\sqrt{A}} = \sqrt{A}$.

(3) $\sqrt{AB} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.

Proof. (1) Let $x \in \sqrt{A}$ Then for every strong m-system M containing x, $M \cap A \neq \emptyset$. Since $A \subseteq B$, $M \cap B \neq \emptyset$ Therefore $x \in \sqrt{B}$

(2) Since $A \subseteq \sqrt{A}$, $\sqrt{A} \subseteq \sqrt{\sqrt{A}}$ by (1).

For the reverse inclusion, suppose that $x \in \sqrt{\sqrt{A}}$. Then for every strong m-system M containing x, $M \cap \sqrt{A} \neq \emptyset$. Thus there exists $y \in M \cap \sqrt{A}$. Since $y \in \sqrt{A}$, for every strong m-system M' containing y such that $M' \cap A \neq \emptyset$. Since M is a strong m-system containing y, $M \cap A \neq \emptyset$. Therefore $x \in \sqrt{A}$. It follows that $\sqrt{\sqrt{A}} = \sqrt{A}$.

(3) Since A and B are ideals in S, we have $AB \subseteq AS \subseteq A$ and $AB \subseteq SB \subseteq B$. Thus $AB \subseteq A \cap B$, and so $\sqrt{AB} \subseteq \sqrt{A \cap B}$ by (1) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we have $\sqrt{A \cap B} \subseteq \sqrt{A}$ and $\sqrt{A \cap B} \subseteq \sqrt{B}$. Hence $\sqrt{A \cap B} \subseteq \sqrt{A} \cap \sqrt{B}$ Therefore $\sqrt{AB} \subseteq \sqrt{A \cap B} \subseteq \sqrt{A} \cap \sqrt{B}$

For the reverse inclusion, suppose that $x \in \sqrt{A} \cap \sqrt{B}$ Then for every strong m-system M containing x, $M \cap A \neq \emptyset$ and $M \cap B \neq \emptyset$. Now let $y \in M \cap A$ and $z \in M \cap B$. Since M is a strong m-system, $ytz \in M$ for some $t \in S$ Also $ytz \in ASB \subseteq AB$. Thus $M \cap AB \neq \emptyset$, and so $x \in \sqrt{AB}$. It follows that $\sqrt{AB} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$. □

THEOREM 6 *Let $\{P_\alpha\}$ be a family of weakly prime ideals in a po-semigroup S which are totally ordered by the set inclusion. Then $\bigcap_\alpha P_\alpha$ is an weakly prime ideal.*

Proof Let I and J be ideals of S. Assume that $IJ \subseteq \bigcap_\alpha P_\alpha$ and $I \nsubseteq \bigcap_\alpha P_\alpha$. Then for some α, $I \nsubseteq P_\alpha$ and $J \subseteq P_\alpha$ since P_α is weakly prime. Thus $J \subseteq P_\beta$ for all $\beta \geq \alpha$. Suppose that there exists $\gamma < \alpha$ such that $J \nsubseteq P_\gamma$. Then $I \subseteq P_\gamma$ and so $I \subseteq P_\alpha$. This is impossible. Thus $J \subseteq P_\beta$ for all β. Hence $\bigcap_\alpha P_\alpha$ is weakly prime. □

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