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NON-DIFFERENTIABLE POINTS OF A SELF-SIMILAR CANTOR FUNCTION

IN-SOO BAEK AND YOUNG-HA KIM

ABSTRACT We study the properties of non-diffenrentiable points of a self-similar Cantor function from which we conjecture a generalization of Darst's result that the Hausdorff dimension of the non-diffenrentiable points of the Cantor function is $(\frac{\ln 2}{\ln 3})^2$

1. Introduction

Darst([2],[3]) and Eidswick([5]) studied the characterizations of the set of non-differentiable points of the Cantor function. In particular, Darst showed that the Hausdorff dimension of the set at which the Cantor function is not differentiable is $(\frac{\ln 2}{\ln 3})^2$. The Cantor function is constructed on the basis of the classical Cantor set which is a symmetric self-similar Cantor set Recently we([1],[8]) studied generalized forms of such classical Cantor set which contain a non-symmetric similar Cantor set.
In this paper, we generalize the theorems which are essential to compute the Hausdorff dimension of the set of non-differentiable points of the symmetric self-similar Cantor function. Using our more generalized results, we guess the Hausdorff dimension of the set of non-differentiable points of a non-symmetric self-similar Cantor function.

In fact, the main purpose of this paper is to support our conjecture that Hausdorff dimension of the set $\mathcal{N}^+$ (the set of non-end

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points of the self-similar Cantor set $C_{a,b}$ at which the corresponding self-similar Cantor function $f_{a,b}$ does not have a right-side derivative, finite or infinite) is $(\frac{ln2}{ln\frac{1}{C_p}})^2$ where $C_p = a^p b^{(1-p)}$ with $p = a^s$ and $a^s + b^s = 1$. That is, we prove some fundamental theorems to support such conjecture.

2. Preliminaries

2.1. Construction of a self-similar Cantor set $C_{a,b}$([7],[8])

From now on, we assume that a and b satisfying the inequalities $0 < b \leq a < \frac{1}{2}$.
Now, we generate a self-similar Cantor set $C_{a,b}$ in $[0, 1]$ by recursively removing middle segments of relative length $1 - (a + b)$:

$$C^0_{a,b} = [0, 1],$$
$$C^1_{a,b} = [0, a] \cup [1 - b, 1],$$
$$C^2_{a,b} = [0, a^2] \cup [a - ab, a] \cup [1 - b, 1 - b + ba] \cup [1 - b^2, 1],$$
$$\vdots$$

and $C_{a,b} = \cap_{n\geq 1} C^n_{a,b}$.

2.2. Ternary representation of points in $C_{a,b}$; location codes

Recalling the ternary representation of points in Cantor set, we see that each point t in $C_{a,b}$ has a corresponding ternary representation $\{t\} = (t_1, \cdots, t_{z(n)}, \cdots)$, where $t_{z(n)} = 0$ or 2, which locates its position in $C_{a,b}$(0 corresponds to 'left side' and 2 corresponds to 'right side'): the ternary representation of t is also called a location code for t. Code spaces are discussed at length in [4], where they are called string spaces. From now on, this ternary representation will be used for any expansion without confusion.

2.3. Construction of a self-similar Cantor function $f_{a,b}$

We easily see that there is a unique continuous non-decreasing function
$f_{a,b}$ satisfying

$f_{a,b}(x_n) = 0.\frac{t_1}{2}\frac{t_2}{2}\frac{t_3}{2}\ldots\frac{t_n}{2}$ for $x_n = 0.t_1t_2t_3\ldots t_n$. Obviously

$f_{a,b}(x) = \frac{1}{2}$ for $x \in [a, 1-b]$,
$f_{a,b}(x) = \frac{1}{4}$ for $x \in [a^2, a-ab]$,
$f_{a,b}(x) = \frac{3}{4}$ for $x \in [1-b+ba, 1-b^2]$, etc.

2.4. The upper right derivative of $f_{a,b}$ at a non-right-end point x of $C_{a,b}$

The following statement, which is necessary to support our main theorem, verifies that the upper·right derivative of $f_{a,b}$ at a non-right-end point x of $C_{a,b}$ is infinite: The ternary representation of a non-right-end point of $C_{a,b}$ has infinitely many zero entries. Let $x = 0.t_1t_2t_3\ldots$ and $x_n = 0.t_1t_2t_3\ldots t_n$ then,

$f(x_n) = 0.\frac{t_1}{2}\frac{t_2}{2}\frac{t_3}{2}\ldots\frac{t_n}{2}$ and

$\lim_{n\to\infty}\frac{f(x)-f(x_n)}{x-x_n} = \frac{\frac{1}{2^{n+2}}(t_{n+1}+\frac{t_{n+2}}{2}\quad)}{\frac{1}{3^{n+1}}(t_{n+1}+\frac{t_{n+2}}{3}\quad)}$

By L'Hospital's rule, $\lim_{n\to\infty}\frac{f(x)-f(x_n)}{x-x_n} = \infty$. Thus, the upper right derivative of $f_{a,b}$ is infinite at a non-right-end point x.

2.5. Composition of $S_{a,b}$ of non-differentiable points

Fix a, b in $(0, \frac{1}{2})$, $S_{a,b}$ is composed of three sets of points described below, that is, let $S_{a,b}$ be the set of all non-differentiable points of $f_{a,b}$, let K be the set of all endpoints of $C_{a,b}$ and let $\mathcal{N}^+(\mathcal{N}^-)$ be the set of non-end points of the self-similar Cantor set at which the corresponding self-similar Cantor function $f_{a,b}$ does not have a right-side(left-side) derivative, finite or infinite We have seen that the upper right-side(left-side) derivative of $f_{a,b}$ at a non-right-end(non-left-end)point of $C_{a,b}$ is infinite. Thus, $\mathcal{N}^+(\mathcal{N}^-)$ is the set of non-end points of $C_{a,b}$ at which the lower right(left) derivative of $f_{a,b}$ is finite Hence

$$S_{a,b} = \mathcal{N}^+ \cup \mathcal{N}^- \cup K.$$

2.6. Hausdorff dimension

For $U \subset \mathbb{R}$, we will denote $|U|$ by the diameter of U throughout this paper. We recall the s-dimensional Hausdorff measure([6],[9]) of F :

$$H^s(F) = \lim_{\delta\to 0} H^s_\delta(F),$$

where $H^s_\delta(F) = \inf\{\sum_{n=1}^{\infty} \mid U_n \mid^s : \{U_n\}_{n=1}^{\infty}$ is a δ-cover of $F\}$, and the Hausdorff dimension([6]) of F :

$$\dim_H(F) = \sup\{s > 0 : H^s(F) = \infty\}(= \inf\{s > 0 : H^s(F) = 0\}).$$

3. Main Result

We will establish two facts, Theorems 3.5 and 3.7 below, relating elements in $\mathcal{N}^+(\mathcal{N}^-)$ to long strings of zeroes(twos) in ternary representation. Let $z(n)(t(n))$ denote the position of the nth zero (two) in $\{t\}$.

DEFINITION 3.1. Suppose that F^* and F are subsets of $\mathcal{N}^+$. Let k is the position of the nth zero in $\{t\}$ and let $n_0(t|k)$ denotes the number of times the digit 0 occurs in the first k places of our base 3-expansion of t We define

$$F^*(p) = \{t \in C_{a,b} : \overline{\lim_{k\to\infty}} \frac{n_0(t|k)}{k} \le p\},$$

and

$$F(p) = \{\ t \in C_{a,b} : \lim_{k\to\infty} \frac{n_0(t|k)}{k} = p\ \}.$$

REMARK 3.2. *Let t be a non-end point of $C_{a,b}$ $(a > b)$ and $z(n)$ denote the position of the nth zero in $\{t\}$. If $\overline{\lim}_{n\to\infty}\{\frac{z(n+1)}{z(n)}\} < \frac{\ln\frac{1}{C_p}}{\ln 2}$, then*

(i) there is a positive integer M such that $\frac{z(n+1)}{z(n)} < \frac{\ln\frac{1}{C_p}}{\ln 2}$ for all $n \ge M$, where $C_p = a^p b^{1-p}$ $(0 \le p \le 1)$.

(ii) there is a $\beta > 0$ such that $z(n+1) < \{\frac{p\ln a + (1-p-\beta)\ln b}{\ln\frac{1}{2}}\}z(n)$ for all $n \ge M$.

Consider $N_1 \ge M$,

(iii) $(\frac{1}{2})^{z(n+1)} \ge \{a^p b^{(1-p-\beta)}\}^{z(n)}$ for all $n \ge N_1$.

REMARK 3.3. *Let t be a non-end point of $C_{a,b}$ $(a > b)$. Let $z(n)$ denote the level at which $\{t\}$ and $\{x\}$ split. Then* $\frac{f(x)-f(t)}{x-t} \geq \{\frac{a^p b^{(1-p-\beta)}}{a^{\frac{n}{z(n)}} b^{(1-\frac{n}{z(n)})}}\}^{z(n)} \frac{1}{ab}$. *Here,*
$a^n b^{z(n)-n} = \{ a^{\frac{n}{z(n)}} b^{(1-\frac{n}{z(n)})}\}^{z(n)}$, $x - t \leq (a^n b^{z(n)-n}) \frac{1}{ab}$. *Now, fix $\varepsilon > 0$ such that $\beta - \varepsilon > 0$ where β is in Remark 3.2. Then*

For N_1 in Remark 3.2,

$$\begin{aligned} \frac{f(x)-f(t)}{x-t} &\geq (a^{p-\frac{n}{z(n)}+\beta} b^{\frac{n}{z(n)}-p-\beta})^{z(n)} \frac{1}{ab} \\ &= \{(\tfrac{a}{b})^{p-\frac{n}{z(n)}+\beta}\}^{z(n)} \frac{1}{ab} \quad \text{for all } n \geq N_1 \\ &\geq \{(\tfrac{a}{b})^{\beta-\varepsilon}\}^{z(n)} \frac{1}{ab} \to \infty \quad \text{as } n \to \infty. \end{aligned}$$

Using Remarks 3 2 and 3.3, we get the following theorem.

THEOREM 3.4. *Let t be a non-end point of $C_{a,b}$. Let $z(n)$ denote the position of the nth zero in $\{t\}$; then*

$$\text{if} \quad t \in \mathcal{N}^+ \cap F^*(p), \quad \text{then} \quad \overline{\lim_{n\to\infty}} \frac{z(n+1)}{z(n)} \geq \frac{\ln \frac{1}{C_p}}{\ln 2},$$

$$\text{and in particular, if} \quad t \in \mathcal{N}^+ \cap F(p), \quad \text{then} \quad \overline{\lim_{n\to\infty}} \frac{z(n+1)}{z(n)} \geq \frac{\ln \frac{1}{C_p}}{\ln 2}.$$

Proof. It is sufficient to show that the lower-right derivative of $f_{a,b}$ is infinite at a non-end point t of $C_{a,b}$ when $\overline{\lim}_{n\to\infty} \frac{z(n+1)}{z(n)} < \frac{\ln \frac{1}{C_p}}{\ln 2}$ since $\overline{\lim}_{x\downarrow t} \frac{f(x)-f(t)}{x-t} = \infty$.
Suppose that $\overline{\lim}_{n\to\infty} \frac{z(n+1)}{z(n)} < \frac{\ln \frac{1}{C_p}}{\ln 2}$ and $t \in F^*(p) \cap \mathcal{N}^+$. By Remark 3.2, there is $\beta > 0$ such that $\frac{z(n+1)}{z(n)} < \frac{-\ln a^p b^{1-p-\beta}}{\ln 2} < \frac{-\ln a^p b^{1-p}}{\ln 2}$. Since $t \in F^*(p) \cap \mathcal{N}^+$,
$\frac{n}{z(n)} < p + \varepsilon$ for all $n \geq N_2$ for some positive integer N_2. Then $\frac{n}{z(n)} < p + \varepsilon$ for all $n \geq N$ $(N = max(N_1, N_2))$, where N_1 is in Remark 3.2.
By Remark 3.3, $\underline{\lim}_{x\downarrow t} \frac{f(x)-f(t)}{x-t} = \infty$. □

THEOREM 3.5. *Let t be a non-end point of $C_{a,b}$ and let $t \in F(p)$; then*

$$\textit{if} \quad \overline{\lim_{n\to\infty}} \frac{z(n+1)}{z(n)} > \frac{\ln \frac{1}{C_p}}{\ln 2}, \quad \textit{then} \quad t \in \mathcal{N}^+.$$

Proof. There is $\delta > 0$ such that $\overline{\lim}_{n\to\infty} \frac{z(n+1)}{z(n)} > \frac{\ln \frac{1}{C_p}}{\ln 2} + \delta$. There is
$\varepsilon > 0$ such that $\frac{\ln \frac{1}{a^{p+\varepsilon} b^{1-p+\varepsilon}}}{\ln 2} + \delta \leq \frac{z(n+1)}{z(n)}$ for infinitely many n. That is,
$\frac{z(n+1)}{z(n)} + \frac{(p+\varepsilon)\ln a + (1-p+\varepsilon)\ln b}{\ln 2} > \delta$ for infinitely many n. Fix $t \in F(p)$. For such $\varepsilon > 0$, there is a positive integer N such that $p - \varepsilon < \frac{n}{z(n)} < p + \varepsilon$ for all $n \geq N$. For such $n (\geq N)$, define a sequence of points $u(n)$ in $C_{a,b}$, decreasing to t, by specifying

$$\{u(n)\} = (t_1, t_2, \cdots, t_{z(n)-1}, 2, 0, 0, 0, 0, 0, \cdots),$$
$$\{t(n)\} = (t_1, t_2, \cdots, t_{z(n)-1}, 0, 2, \cdots, 2, 0, ***).$$

Then $\frac{f(u(n)) - f(t)}{u(n) - t} \leq \frac{(\frac{1}{2})^{z(n+1)-1}}{(1-a-b) b (a^{p+\varepsilon} b^{1-p+\varepsilon})^{z(n)}}$.

Now, $\ln \frac{(\frac{1}{2})^{z(n+1)}}{(a^{p+\varepsilon} b^{1-p+\varepsilon})^{z(n)}} = z(n+1) \ln \frac{1}{2} - z(n) \ln(a^{p+\varepsilon} b^{1-p+\varepsilon})$

$= -z(n) \ln 2 \left(\frac{z(n+1)}{z(n)} + \frac{(p+\varepsilon)\ln a + (1-p+\varepsilon)\ln b}{\ln 2}\right) \to -\infty$ as $n \to \infty$.

That is, $\underline{\lim}_{x \downarrow t} \frac{f(x) - f(t)}{x - t} = 0$, which consequently implies that t is in $\mathcal{N}^+$. □

COROLLARY 3.6. *Let $t \in F(p_0)$. For $p = p_0$ such that $p_0 = a^s$ with $a^s + b^s = 1$,*

$$\textit{if} \quad \overline{\lim_{n\to\infty}} \frac{z(n+1)}{z(n)} > \frac{\ln \frac{1}{C_{p_0}}}{\ln 2}, \quad \textit{then} \quad t \in \mathcal{N}^+, \ \textit{where}\ c_{p_0} = a^{p_0} b^{1-p_0}.$$

REMARK 3.7. *Considering Theorem 3.4 and Corollary 3.6, we positively conjecture that Hausdorff dimension of the set $\mathcal{N}^+$ of points at which a self-similar Cantor function is not differentiable is $\left(\frac{ln 2}{ln \frac{1}{C_p}}\right)^2$ where $C_p = a^p b^{(1-p)}$ with $p = a^s$ and $a^s + b^s = 1$.*

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Department of Mathematics
Pusan University of Foreign Studies
Pusan 608-738, Korea
E-mail: isbaek@taejo.pufs.ac.kr