East Asian Math. J. **19** (2003), No. 2, pp. 165–171

ON F-HARMONIC MAPS AND CONVEX FUNCTIONS

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ABSTRACT. We show that any F-harmonic map from a compact manifold M to N is necessarily constant if N possesses a strictly-convex function, and prove 'Liouville type theorems' for F-harmonic maps. Finally, when the target manifold is the real line, we get a result for F-subharmonic functions.

1. F-harmonic maps and F-subharmonic functions

Recently, M.Ara[1] introduced the concept of F-harmonic maps, and unified the theory of harmonic maps, p-harmonic maps, exponencially harmonic maps and so on. More precisely, let $F : [0, \infty) \to [0, \infty)$ be a C^2 function such that $F' > 0$ on $(0, \infty)$. Let $\phi : (M, g) \to (N, h)$ be a smooth map between riemannian manifolds with metrics g and h respectively. Then ϕ is an F-*harmonic map* if it satisfies the F-*tension field* equation weakly

$$\mathrm{Trace}\nabla(F'(\frac{\|d\phi\|^2}{2})d\phi) = 0,$$

i.e., for every compactly supported vector field X along ϕ

$$\int_M < F'(\frac{\|d\phi\|^2}{2})d\phi, \nabla X >= 0,$$

Received May 9, 2003.
2000 Mathematics Subject Classification: 58E20.
Key words and phrases. F-harmonic maps, convex functions, F-subharmonic functions.

where $\|d\phi\|$ denotes the Hilbert-Schmidt norm of the differential $d\phi$ of ϕ, which is the differential 1-form with values in the induced bundle $\phi^{-1}TN$ over M. It is harmonic, p-harmonic, α-harmonic and exponentially harmonic when $F(t) = t$, $(2t)^{p/2}/p(p \geq 4)$, $(1+2t)^{\alpha}(\alpha > 1, \dim M = 2)$ and e^t, respectively. Finally we define an *F-subharmonic function*. A smooth function $\phi : M \to R$ is an *F-subharmonic function* if ϕ satisfies the inequality

$$\mathrm{Trace}\nabla(F'(\frac{\|d\phi\|^2}{2})d\phi) \geq 0$$

weakly, i.e.,

$$\int_M < F'(\frac{\|d\phi\|^2}{2})d\phi, d\tau > \leq 0$$

for any compactly supported, nonnegative smooth function τ on M. It is subharmonic and p-subharmonic when $F(t) = t$ and $(2t)^{p/2}/p(p \geq 4)$, respectively.

2. Main results

In this article we prove the following theorems.

THEOREM 1. *Suppose that a smooth map $\phi : M \to N$ is F-harmonic. If M is compact and there exists a strictly convex function on N, then ϕ is a constant map.*

THEOREM 2. *Let M and N be riemannian manifolds. Suppose that M is complete and noncompact, and N has a strictly convex function $f : N \to R$ such that the uniform norm $\|df\|$ is bounded. If a smooth map $\phi : M \to N$ is F-harmonic with $\int_M F'(\frac{\|d\phi\|^2}{2})\|d\phi\| < \infty$, then ϕ is a constant map.*

THEOREM 3. *Let M be a complete noncompact manifold. If any F-subharmonic function $\phi : M \to R$ with*

$$\int_M F'(\frac{\|d\phi\|^2}{2})\|d\phi\| < \infty,$$

then ϕ is a constant map.

REMARK. *In the above theorems, for the case of $F(t) = t$, i.e., harmonic maps or $F(t) = (2t)^{p/2}/p$, i.e., p-harmonic maps, see [3] and [6], respectively. In these cases, the energy $\int_M F'(\frac{\|d\phi\|^2}{2})\|d\phi\|$ reduces to $\int_M \|d\phi\|$ (cf.[7]) and $\int_M \|d\phi\|^{p-1}$ (cf. [4,5,6]), respectively.*

3. Proofs

First we show the following lemma.

LEMMA. *Let $\phi : M \to N$ be a smooth map between Riemannian manifolds and $f : N \to R$ be a smooth function. Then the following identity holds for every smooth function η on M.*

$$< F'(\frac{\|d\phi\|^2}{2})d(f \circ \phi), d\eta > = -F'(\frac{\|d\phi\|^2}{2})\text{Trace}(\nabla df)(d\phi, d\phi)\eta$$
$$+ < \nabla(\eta \cdot (\text{grad}f) \circ \phi), F'(\frac{\|d\phi\|^2}{2})d\phi > .$$

Proof. Let $\{e_i\}$ be an orthonomal frame around some point of M which satisfies $\nabla e_i = 0$ at that point. Then

$$< \nabla(\eta \cdot (\text{grad}) \circ \phi), F'(\frac{\|d\phi\|^2}{2})d\phi >$$
$$= \sum_i < \nabla_{e_i}(\eta \cdot (\text{grad}) \circ \phi), F'(\frac{\|d\phi\|^2}{2})d\phi(e_i) >$$
$$= \sum_i d\eta(e_i)F'(\frac{\|d\phi\|^2}{2}) < (\text{grad}f) \circ \phi, d\phi(e_i) >$$
$$+ \sum_i \eta F'(\frac{\|d\phi\|^2}{2}) < \nabla_{d\phi(e_i)}(\text{grad}f) \circ \phi, d\phi(e_i) >$$
$$=< F'(\frac{\|d\phi\|^2}{2})d(f \circ \phi), d\eta > +\eta F'(\frac{\|d\phi\|^2}{2})\text{Trace}(\nabla df)(d\phi, d\phi),$$

where the last term was calculated as follows;

$$\begin{aligned}
&\sum_i < \nabla_{d\phi(e_i)}(\mathrm{grad} f) \circ \phi, d\phi(e_i) > \\
&= \sum_i \nabla_{d\phi(e_i)} < (\mathrm{grad} f) \circ \phi, d\phi(e_i) > \\
&- \sum_i < (\mathrm{grad} f) \circ \phi, \nabla_{d\phi(e_i)} d\phi(e_i) > \\
&= \sum_i \nabla_{d\phi(e_i)} (d\phi(e_i) f) - \sum_i \nabla_{d\phi(e_i)} d\phi(e_i) f \\
&= \sum_i \nabla_{d\phi(e_i)} df(d\phi(e_i)) - \sum_i df(\nabla_{d\phi(e_i)} d\phi(e_i)) \\
&= \sum_i (\nabla_{d\phi(e_i)} df)(d\phi(e_i)) \\
&= \mathrm{Trace}(\nabla df)(d\phi, d\phi). \square
\end{aligned}$$

□

Proof of Theorem 1. Let $f : N \to R$ be a strictly convex function. Taking $\eta = 1$ in Lemma and integrating on M, we obtain

$$\int_M F'(\frac{\|d\phi\|^2}{2}) \mathrm{Trace}(\nabla df)(d\phi, d\phi) = 0,$$

since ϕ is F-harmonic map. Thus we have $F'(\frac{\|d\phi\|^2}{2}) = 0$, which implies that $\frac{\|d\phi\|^2}{2} = 0$, i.e., ϕ is constant.

□

Proof of Theorem 2. Let us fix a point of M and denote B_r the geodesic ball with radius r and centered at this point. Then there exists a smooth function η on M such that

$$0 \le \eta \le 1, \quad \|d\eta\| \le \frac{c}{r},$$

$$\eta = \begin{cases} 1 & \text{on} \quad B_r \\ 0 & \text{on} \quad M \setminus B_{2r}, \end{cases}$$

where c is a positive constant which does not depend on r. Then it follows from Lemma that

$$\begin{aligned}
&\int_M F'(\frac{\|d\phi\|^2}{2})\mathrm{Trace}(\nabla df)(d\phi, d\phi) \\
&= -\int_M F'(\frac{\|d\phi\|^2}{2}) < d(f \circ \phi), d\eta > \\
&\leq \int_M F'(\frac{\|d\phi\|^2}{2})\|df\|\|d\phi\|\|d\eta\| \\
&\leq \frac{c}{r}\int_M F'(\frac{\|d\phi\|^2}{2})\|d\phi\| \quad \to 0 \quad (\text{as} \quad r \to \infty).
\end{aligned}$$

Thus we obtain $F'(\frac{\|d\phi\|^2}{2}) = 0$, which implies that ϕ is constant. □

Proof of Theorem 3. Taking a nondecreasing strictly convex function f with bounded derivative on the real line. Then for any non-negative smooth function η with compact support, we get

$$\begin{aligned}
&\mathrm{div}[\sum_i \{F'(\frac{\|d\phi\|^2}{2})d\phi(e_i) \cdot \eta \cdot (\mathrm{grad} f) \circ \phi\} e_i] \\
&= \sum_i < e_j\{F'(\frac{\|d\phi\|^2}{2})d\phi(e_i) \cdot \eta \cdot (\mathrm{grad} f) \circ \phi\} e_i, e_j > \\
&= \sum_i e_i\{F'(\frac{\|d\phi\|^2}{2})d\phi(e_i) \cdot \eta \cdot (\mathrm{grad} f) \circ \phi\}
\end{aligned}$$

$$= \sum_i e_i \{F'(\frac{\|d\phi\|^2}{2}) d\phi(e_i)\} \cdot \eta \cdot (\mathrm{grad} f) \circ \phi$$

$$+ \sum_i F'(\frac{\|d\phi\|^2}{2}) d\phi(e_i) \nabla_{e_i} \{\eta \cdot (\mathrm{grad} f) \circ \phi\}$$

$$= \sum_i \nabla_{e_i} (F'(\frac{\|d\phi\|^2}{2}) d\phi)(e_i) \{\eta \cdot (\mathrm{grad} f) \circ \phi\}$$

$$+ < F'(\frac{\|d\phi\|^2}{2}) d\phi, \nabla\{\eta \cdot (\mathrm{grad} f) \circ \phi\} > .$$

Integrating this equation over M and using assumptions, we obtain

$$\int_M < \nabla\{\eta \cdot (\mathrm{grad} f) \circ \phi\}, F'(\frac{\|d\phi\|^2}{2}) d\phi >$$

$$= - \int_M \mathrm{trace} \nabla (F'(\frac{\|d\phi\|^2}{2}) d\phi) \cdot \eta \cdot (\mathrm{grad} f \circ \phi) \leq 0.$$

From this inequality and Lemma, we have

$$\int_M F'(\frac{\|d\phi\|^2}{2}) \mathrm{Trace}(\nabla df)(d\phi, d\phi) \eta$$

$$= \int_M < \nabla\{\eta \cdot (\mathrm{grad} f) \circ \phi\}, F'(\frac{\|d\phi\|^2}{2}) d\phi >$$

$$- \int_M < F'(\frac{\|d\phi\|^2}{2}) d(f \circ \phi), d\eta >$$

$$\leq - \int_M < F'(\frac{\|d\phi\|^2}{2}) d(f \circ \phi), d\eta > .$$

Then we can argue as in Theorem 2. □

Acknowledgement
This work was supported by University of Ulsan Research Fund of 2002.

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