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Lindley Type Estimation with Constrains on the Norm

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Abstract

Consider the problem of estimating a $p \times 1$ mean vector $\boldsymbol{\theta}(p \geq 4)$ under the quadratic loss, based on a sample $\mathbf{X_1}, \cdots \mathbf{X_n}$. We find an optimal decision rule within the class of Lindley type decision rules which shrink the usual one toward the mean of observations when the underlying distribution is that of a variance mixture of normals and when the norm $\| \boldsymbol{\theta} - \bar{\theta}\mathbf{1} \|$ is known, where $\bar{\theta} = (1/p)\sum_{i=1}^{p} \theta_i$ and $\mathbf{1}$ is the column vector of ones. When the norm is restricted to a known interval, typically no optimal Lindley type rule exists but we characterize a minimal complete class within the class of Lindley type decision rules. We also characterize the subclass of Lindley type decision rules that dominate the sample mean.

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1. Introduction

The problem considered is that of estimating with quadratic loss function the mean vector of a compound multinormal distribution when the norm $||\boldsymbol{\theta} - \bar{\theta}\mathbf{1}||$ is known. The class of estimation rules considered will consist of Lindley type estimators only. Such a class was introduced by James-Stein (1961) and Lindley (1962) in order to prove that some of its members dominate the sample mean in the multinormal case. Strawderman (1974) also derived a similar result for the more general case considerd in this paper of a compound multinormal distribution.

The problem of estimation of a mean under constraint has an old origin and recently focussed again in the context of curved model in the works of Efron (1975), Hinckley (1977), Amari (1982), Kariya (1989), Perron and Giri (1990), Marchand and Giri (1993) among others. A study of compound multinormal distributions and the estimation of their location vectors was carried out by Berger (1975).

In section 2, we present the general setting of our problem and develop necessary notations. In section 3, we derive the best Lindley type estimator of a mean when the norm $\| \boldsymbol{\theta} - \bar{\theta}\mathbf{1} \|$ is known. Examples of these best estimators are given in section 4. In section 5, we examine the estimation problem based on a Lindley type decision rule when the norm is restricted to a known interval. Particular attention is given to the subclass of Lindley type estimators which dominate the sample mean when the norm is restricted to a known interval.

2. Notation and Preliminaries

Let $\mathbf{x} = (x_1, \cdots, x_p)'$, $p \geq 4$ be an observation from a compound multinormal distribution with unknown location parameter $\boldsymbol{\theta}(p \times 1)$ and mixture parameter $H(\cdot)$, where $H(\cdot)$ represents a known c.d.f. defined on the interval $(0, \infty)$. In other words, we assume that the random variable $\mathbf{X}$ generating our observation x admits the representation,

$$\mathcal{L}(\mathbf{X}|Z=z) = N_p(\boldsymbol{\theta},\ zI_p),\ \forall z>0\ , \tag{2.1}$$

Z being the positive random variable with c.d.f. $H(\cdot)$.

Our problem concerns the estimation of the location parameter $\boldsymbol{\theta}$ with loss function

$$L(\boldsymbol{\theta},\ \boldsymbol{\delta}(\mathbf{x})) = (\boldsymbol{\delta}(\mathbf{x})-\boldsymbol{\theta})'(\boldsymbol{\delta}(\mathbf{x})-\boldsymbol{\theta}),$$

with $\boldsymbol{\theta} \in \Theta_{\lambda_1}^{\lambda_2} = \{\boldsymbol{\theta} \in \mathbb{R}^p |\ \| \boldsymbol{\theta} - \bar{\theta}\mathbf{1} \| \in [\lambda_1,\ \lambda_2],\ 0 \le \lambda_1 \le \lambda_2 \le \infty\}$, where $\bar{\theta} = \frac{1}{p}\sum_1^p \theta_i$, $\mathbf{1} = (1,\cdots,1)'$ and the decision rule $\boldsymbol{\delta}$, $\boldsymbol{\delta}(\cdot) : \mathbb{R}^p \to \mathbb{R}^p$, is of the form

$$\boldsymbol{\delta}(\mathbf{x}) = \bar{x}\mathbf{1} + \left(1 - \frac{c}{(\mathbf{x}-\bar{x}\mathbf{1})'(\mathbf{x}-\bar{x}\mathbf{1})}\right)(\mathbf{x}-\bar{x}\mathbf{1})\ ,\quad c \in \mathbb{R}\ .$$

Restated in terms of the family of probability density functions of $\mathbf{X}$, the distributional assumption given by expression (2.1) and the restriction on the location parameter θ indicate that the p.d.f. of $\mathbf{X}$ is

$$p_{\boldsymbol{\theta}}(\mathbf{x}) = \int_{(0,\infty)} (2\pi z)^{-p/2} exp\left(\frac{-\| \mathbf{x}-\boldsymbol{\theta} \|^2}{2z}\right) dH(z)\ , \tag{2.2}$$

$\mathbf{x} \in \mathbb{R}^p$ and $\boldsymbol{\theta} \in \Theta_{\lambda_1}^{\lambda_2}$. It will be also assumed that $E(Z) < \infty$ which will guarantee the existence of the covariance matrix $\sum = Cov(\mathbf{X}) = E(Z)I_p$ and the mean vector $E(\mathbf{X}) = \boldsymbol{\theta}$. The performance of the estimator $\boldsymbol{\delta}$ will be measured by its risk function

$$R(\boldsymbol{\theta},\boldsymbol{\delta}) = E_{\boldsymbol{\theta}}[L(\boldsymbol{\theta},\boldsymbol{\delta}(\mathbf{X}))] = E_{\boldsymbol{\theta}}[(\boldsymbol{\delta}(\mathbf{X})-\boldsymbol{\theta})'(\boldsymbol{\delta}(\mathbf{X})-\boldsymbol{\theta})],\ \boldsymbol{\theta} \in \Theta_{\lambda_1}^{\lambda_2}\ .$$

3. Optimal Lindley Type Estimation when the norm $\| \theta - \bar{\theta}\mathbf{1} \|$ is known

In this section, the best estimator is derived within

$$\mathcal{D}_{Lind} = \left\{ \boldsymbol{\delta} : \mathbb{R}^p \rightarrow \mathbb{R}^p | \boldsymbol{\delta}^c(\mathbf{X}) = \bar{X}\mathbf{1} + \left(1 - \frac{c}{(\mathbf{X} - \bar{X}\mathbf{1})'(\mathbf{X} - \bar{X}\mathbf{1})}\right)(\mathbf{X} - \bar{X}\mathbf{1}),\ c \in \mathbb{R} \right\},$$

where the parameter space is of the form

$$\Theta_{\lambda_1}^{\lambda_2} = \{\boldsymbol{\theta} \in \mathbb{R}^p | \ \| \boldsymbol{\theta} - \bar{\theta}\mathbf{1} \| = \lambda\}, \ \ \lambda \geq 0 \, .$$

The following lemmas will prove useful in the evaluation of the risk function of the decision rule $\boldsymbol{\delta}^c, \ c \in \mathbb{R}$.

Lemma 3.1. Let $\mathbf{X}$ be a random multinormal vector $N_p(\boldsymbol{\theta}, I_p), \ p \geq 4$ and $\boldsymbol{\theta} \in \mathbb{R}^p$. Then

$$\text{(i) } \mathrm{E}_{\boldsymbol{\theta}}\left(\frac{1}{(\mathbf{X} - \bar{\mathrm{X}}\mathbf{1})'(\mathbf{X} - \bar{\mathrm{X}}\mathbf{1})}\right) = \mathrm{E}^{\mathrm{L}}\left(\frac{1}{\mathrm{p} + 2\mathrm{L} - 3}\right)$$

and

$$\text{(ii) } \mathrm{E}_{\boldsymbol{\theta}}\left(\frac{(\mathbf{X} - \boldsymbol{\theta})'(\mathbf{X} - \bar{\mathrm{X}}\mathbf{1})}{(\mathbf{X} - \bar{\mathrm{X}}\mathbf{1})'(\mathbf{X} - \bar{\mathrm{X}}\mathbf{1})}\right) = \mathrm{E}^{\mathrm{L}}\left(\frac{\mathrm{p} - 3}{\mathrm{p} + 2\mathrm{L} - 3}\right),$$

where L is a Poisson random variable with mean $(\boldsymbol{\theta} - \bar{\theta}\mathbf{1})'(\boldsymbol{\theta} - \bar{\theta}\mathbf{1})/2$

Proof. See James and Stein(1961) and use Stein's Identity

Lemma 3.2. Let $\mathbf{X}$ be a compound multinormal vector with location parameter $\boldsymbol{\theta}; p \geq 4$ and $\boldsymbol{\theta} \in \mathbb{R}^p$; and known mixture parameter $H(\cdot)$ with p.d.f. of the form given in (2.2). Then, with $\lambda = \| \boldsymbol{\theta} - \bar{\theta}\mathbf{1} \|$

$$\text{(i) } \mathrm{E}_{\boldsymbol{\theta}}\left(\frac{1}{(\mathbf{X} - \bar{\mathrm{X}}\mathbf{1})'(\mathbf{X} - \bar{\mathrm{X}}\mathbf{1})}\right) = \int_{(0,\infty)} \mathrm{f_p}(\lambda, \ \mathrm{z})\frac{\mathrm{dH(z)}}{\mathrm{z}} \, ,$$

and

$$\text{(ii) } \mathrm{E}_{\boldsymbol{\theta}}\left(\frac{(\mathbf{X}-\boldsymbol{\theta})'(\mathbf{X}-\bar{\mathrm{X}}\mathbf{1})}{(\mathbf{X}-\bar{\mathrm{X}}\mathbf{1})'(\mathbf{X}-\bar{\mathrm{X}}\mathbf{1})}\right) = (\mathrm{p}-3)\int_{(0,\infty)} \mathrm{f_p}(\lambda,\ \mathrm{z})\mathrm{dH}(\mathrm{z})\ ,$$

where the function $f_p(\cdot,\cdot) : [0,\infty) \to (0,\infty)$, is defined by the relation

$$f_p(\lambda,\ z) = \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^j}{j!(p+2j-3)}\ .$$

Proof. (i) Using both the representation given in (2.1) and part(i) of Lemma 3.1, we obtain

$$\begin{aligned}
E_{\boldsymbol{\theta}}\left(\frac{1}{(\mathbf{X}-\bar{X}\mathbf{1})'(\mathbf{X}-\bar{X}\mathbf{1})}\right) &= E^Z\left\{Z^{-1}E_{\boldsymbol{\theta}}^{X|Z}\left[\frac{Z}{(\mathbf{X}-\bar{X}\mathbf{1})'(\mathbf{X}-\bar{X}\mathbf{1})}\right]\right\} \\
&= \int_{(0.\infty)} z^{-1}\sum_{j=1}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^j}{j!(p+2j-3)}dH(z) \\
&= \int_{(0,\infty)} f_p(\lambda,\ z)\frac{dH(z)}{z}\ .
\end{aligned}$$

(ii) Again, combining the representation given in (2.1) and part(ii) of Lemma 3.1, we obtain

$$\begin{aligned}
E_{\boldsymbol{\theta}}\left(\frac{(\mathbf{X}-\boldsymbol{\theta})'(\mathbf{X}-\bar{X}\mathbf{1})}{(\mathbf{X}-\bar{X}\mathbf{1})'(\mathbf{X}-\bar{X}\mathbf{1})}\right) &= E^Z\left\{E_{\boldsymbol{\theta}}^{X|Z}\left[\frac{\left(\frac{(\mathbf{X}-\boldsymbol{\theta})}{\sqrt{Z}}\right)'\left(\frac{\mathbf{X}-\bar{X}\mathbf{1}}{\sqrt{Z}}\right)}{\left(\frac{\mathbf{X}-\bar{X}\mathbf{1}}{\sqrt{Z}}\right)'\left(\frac{\mathbf{X}-\bar{X}\mathbf{1}}{\sqrt{Z}}\right)}\right]\right\} \\
&= \int_{(0,\infty)}\sum_{j=0}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^j}{j!}\frac{p-3}{p+2j-3}dH(z) \\
&= (p-3)\int_{(0,\infty)} f_p(\lambda,\ z)dH(z)\ .
\end{aligned}$$

The main result of this section now follows.

Theorem 3.3. Let x be a single observation from a p-dimensional location parameter with p.d.f. of the form given by (2.2). Under the assumptions $\boldsymbol{\theta} \in \Theta_\lambda,\ p \geq 4$ and $E[Z] < \infty$, the unique best estimator within the class

$\mathcal{D}_{Lind}$ is given by $\boldsymbol{\delta}^{c^*(\lambda)}$ where

$$c^*(\lambda) = (p-3)\frac{\int_{(0,\infty)} f_p(\lambda, z)dH(z)}{\int_{(0,\infty)} f_p(\lambda, z)\frac{dH(z)}{z}} . \tag{3.1}$$

Proof. Under the assumptions above, we can easily derive the result $E_{\boldsymbol{\theta}}(\mathbf{X}'\mathbf{X}) = \boldsymbol{\theta}'\boldsymbol{\theta} + pE(Z)$. Combining this with Lemma 3.2, we have

$$\begin{aligned}
R(\boldsymbol{\theta},\, \boldsymbol{\delta}^c) &= E_{\boldsymbol{\theta}}[(\boldsymbol{\delta}^c(\mathbf{X}) - \boldsymbol{\theta})'(\boldsymbol{\delta}^c(\mathbf{X}) - \boldsymbol{\theta})] \\
&= E_{\boldsymbol{\theta}}\left[\left\{\bar{X}\mathbf{1} + \left(1 - \frac{c}{(\boldsymbol{X} - \bar{X}\mathbf{1})'(\boldsymbol{X} - \bar{X}\mathbf{1})}\right)(\boldsymbol{X} - \bar{X}\mathbf{1}) - \boldsymbol{\theta}\right\}' \right. \\
&\quad \left. \left\{\bar{X}\mathbf{1} + \left(1 - \frac{c}{(\boldsymbol{X} - \bar{X}\mathbf{1})'(\boldsymbol{X} - \bar{X}\mathbf{1})}\right)(\boldsymbol{X} - \bar{X}\mathbf{1}) - \boldsymbol{\theta}\right\}\right] \\
&= E_{\boldsymbol{\theta}}\left[\left\{(\mathbf{X} - \boldsymbol{\theta}) - \frac{c(\mathbf{X} - \bar{X}\mathbf{1})}{(\mathbf{X} - \bar{X}\mathbf{1})'(\mathbf{X} - \bar{X}\mathbf{1})}\right\}' \right. \\
&\quad \left. \left\{(\mathbf{X} - \boldsymbol{\theta}) - \frac{c(\mathbf{X} - \bar{X}\mathbf{1})}{(\mathbf{X} - \bar{X}\mathbf{1})'(\mathbf{X} - \bar{X}\mathbf{1})}\right\}\right] \\
&= pE(Z) + \left[c^2 E_{\boldsymbol{\theta}}\left\{\frac{1}{(\mathbf{X} - \bar{X}\mathbf{1})'(\mathbf{X} - \bar{X}\mathbf{1})}\right\} - 2cE_{\boldsymbol{\theta}}\left\{\frac{(\mathbf{X} - \boldsymbol{\theta})'(\mathbf{X} - \bar{X}\mathbf{1})}{(\mathbf{X} - \bar{X}\mathbf{1})'(\mathbf{X} - \bar{X}\mathbf{1})}\right\}\right] \\
&= pE(Z) + \left[c^2 \int_{(0,\infty)} f_p(\lambda,\, z)\frac{dH(z)}{z} - 2c(p-3)\int_{(0,\infty)} f_p(\lambda,\, z)dH(z)\right] \\
&= pE(Z) + \left\{\int_{(0,\infty)} \left[\frac{c^2}{z} - 2c(p-3)\right] f_p(\lambda,\, z)dH(z)\right\} .
\end{aligned} \tag{3.2}$$

From this last equality, we obtain easily that

$$\inf_{c\in R} R(\boldsymbol{\theta},\, \boldsymbol{\delta}^c) = R(\boldsymbol{\theta},\, \boldsymbol{\delta}^{c^*(\lambda)})$$

with $c^*(\lambda)$ given by expression (3.1).

Using expression (3.2), the minimum risk attained by the best Lindley type estimator is equal to

$$R(\boldsymbol{\theta},\, \boldsymbol{\delta}^{c^*(\lambda)}) = pE(Z) - (p-3)^2\frac{[\int_{(0,\infty)} f_p(\lambda,\, z)dH(z)]^2}{\int_{(0,\infty)} f_p(\lambda,\, z)\frac{dH(z)}{z}} , \quad \boldsymbol{\theta} \in \Theta_\lambda .$$

When $\| \boldsymbol{\theta} - \bar{\theta}\mathbf{1} \| = \lambda$, the use of other estimators of the Lindley class other than $\delta^{c^*(\lambda)}$ will incur risk which is a strictly increasing function of distance $|c - c^*(\lambda)|$. To see this, we can define $h(\lambda)$ such that $c = h(\lambda)c^*(\lambda)$ and, using expression (3.2), express $R(\boldsymbol{\theta},\ \delta^c)$ as

$$pE(Z) + (p-3)^2[h^2(\lambda) - 2h(\lambda)]\frac{[\int_{(o,\infty)} f_p(\lambda,\ z)dH(z)]^2}{\int_{(o,\infty)} f_p(\lambda,\ z)\frac{dH(z)}{z}}\ . \tag{3.3}$$

From this we can write

$$R(\boldsymbol{\theta},\ \delta^c) - R(\boldsymbol{\theta},\ \delta^{c^*(\lambda)}) = |c - c^*(\lambda)|^2 \int_{(0,\infty)} f_p(\lambda,\ z)\frac{dH(z)}{z}\ . \tag{3.4}$$

The natural estimator $\delta^0(\mathbf{X}) = \mathbf{X}$ is a member of the Lindley class and has a constant risk function equal to $pE(Z)$. We can also characterize the estimators of the Lindley type that dominate the natural estimator δ^0.

Corollary 3.4. Under the conditions of Theorem 3.3, the decision rule δ^c will dominate the natural estimator δ^0 if and only if $0 < c < 2c^*(\lambda)$.

Proof. Using expression (3.3), one easily sees that, for $\boldsymbol{\theta} \in \Theta_\lambda$,

$$\begin{aligned} R(\boldsymbol{\theta},\ \delta^c) &< R(\boldsymbol{\theta},\ \delta^0) = pE(Z) \\ &\Leftrightarrow h^2(\lambda) - 2h(\lambda) < 0 \\ &\Leftrightarrow 0 < h(\lambda) < 2 \\ &\Leftrightarrow 0 < c < 2c^*(\lambda)\ . \end{aligned}$$

4. Examples

The class of compound multinormal distributions is quite large and, in this section, we present some examples of the evaluation of the best Lindley type estimator for different choices of the underlying distribution of $\mathbf{X}$ or, equivalently, of the mixture parameter $H(\cdot)$.

Example 4.1. For $\mathbf{X} \sim N_p(\boldsymbol{\theta},\ \sigma^2 I_p),\ p \geq 4$, (i.e. $H(Z) = 1_{(\sigma^2,\infty)}(Z)$ with $1_A(\cdot)$ being the indicator function of the set A); we deduce from Theorem 3.3

that

$$c^*(\lambda) = (p-3)\frac{f_p(\lambda,\, \sigma^2)}{f_p(\lambda,\, \sigma^2)/\sigma^2} = (p-3)\sigma^2\,,$$

and that the best estimator within the Lindley class $\mathcal{D}_{Lind}$ is equal to

$$\delta^{(p-3)\sigma^2}(\mathbf{X}) = \bar{X}\mathbf{1} + \left(1 - \frac{(p-3)\sigma^2}{(\mathbf{X}-\bar{X}\mathbf{1})'(\mathbf{X}-\bar{X}\mathbf{1})}\right)(\mathbf{X}-\bar{X}\mathbf{1})\,,$$

irregardless of the value of the norm $\lambda = \| \boldsymbol{\theta} - \bar{\theta}\mathbf{1} \|$.

For non-normal cases, the following explicit formula for the quantity $f_p^*(\gamma) = E^L[(p+2L-3)^{-1}],\ L \sim \text{Poisson}(\gamma)$, given by Egerton and Laycock (1982) prove useful for the evaluation of the function $c^*(\lambda),\ \lambda \geq 0$.

Lemma 4.1. Let L be a Poisson random variable with mean $\gamma,\ \gamma > 0$, and $f_p^*(\gamma) = E^L[(p+2L-3)^{-1}]\ ;\ p \geq 4$; then

$$\text{(i)} \qquad \mathrm{f}_\mathrm{p}^*(\gamma) = \mathrm{e}^{-\gamma}\int_{[0,1]} \mathrm{t}^{\mathrm{p}-4}\mathrm{e}^{\gamma \mathrm{t}^2}\mathrm{dt}\,,$$

and

$$\text{(ii)} \quad \mathrm{f}_{\mathrm{p}+2}^*(\gamma) = (2\gamma)^{-1}[1-(\mathrm{p}-3)\mathrm{f}_\mathrm{p}^*(\gamma)]\,. \tag{4.1}$$

For even values of the dimension p, the recurrence formula given by expression (4.1) permits the expression of the function $f_p^*(\cdot)$ as a function of $f_4^*(\cdot)$. From part(i) of the preceding lemma,

$$\begin{aligned} f_4^*(\gamma) &= e^{-\gamma}\int_{[0,1]} e^{\gamma t^2}dt \\ &= \gamma^{-\frac{1}{2}}D(\gamma^{-\frac{1}{2}})\,, \end{aligned}$$

where $D(x) = e^{-x^2}\int_{(0,x)} e^{t^2}dt,\ x > 0$, is known as Dawson's integral which is tabulated in Abramowitz and Stegun (1965). For odd values of the dimension p, the recurrence formula given by expression (4.1) permits the expression of the function $f_p^*(\cdot)$ as a function of $f_5^*(\cdot)$. From part(i) of Lemma 4.1,

$$\begin{aligned} f_5^*(\gamma) &= e^{-\gamma}\int_{[0,1]} te^{rt^2}dt \\ &= (2\gamma)^{-1}(1-e^{-\gamma})\,. \end{aligned} \tag{4.2}$$

We now proceed with the evaluation of the best Lindley type estimator in the contaminated multinormal case.

Example 4.2. Setting $H(z) = \sum_{j=1}^{n} \epsilon_j 1_{[\sigma_j^2,\infty)}(z)$ in expression (2.2), where $0 < \epsilon_j < 1,\ \sigma_j^2 > 0$ for $j \in \{1,\cdots,n\}$ and $\sum_{j=1}^{n}\epsilon_j = 1$, we obtain the family of contaminated multinormal distributions with mean parameter $\boldsymbol{\theta}$ and known dispersion parameters $(\sigma_1^2,\ \epsilon_1),\cdots,(\sigma_n^2,\ \epsilon_n)$. The function $c^*(\lambda),\ \lambda \geq 0$, defined by (3.1) becomes

$$c^*(\lambda) = (p-3)\frac{\sum_{j=1}^{n}\epsilon_j f_p(\lambda,\ \sigma_j^2)}{\sum_{j=1}^{n}\frac{\epsilon_j}{\sigma_j^2} f_p(\lambda,\ \sigma_j^2)}\,,$$

and the decision rule $\boldsymbol{\delta}^{c^*(\lambda)}$ represents, by Theorem 3.3, the best Lindley type estimator when $\boldsymbol{\theta} \in \Theta_\lambda$. The quantities $f_p(\lambda,\ \sigma_j^2)$ can be evaluated by using the results of Lemma 4.1. In particular, for $p = 7$, using expressions (4.1) and (4.2), we obtain

$$\begin{aligned} f_7(\lambda,\ z) &= f_7^*\left(\frac{\lambda^2}{2z}\right) \\ &= \lambda^{-4}z(\lambda^2 - 2z + 2ze^{-\lambda^2/2z})\,, \quad \lambda > 0\,, \quad z > 0\,, \end{aligned}$$

and,

$$c^*(\lambda) = 4\,\frac{\sum_{j=1}^{n} \epsilon_j \sigma_j^2 (\lambda^2 - 2\sigma_j^2 + 2\sigma_j^2 e^{-\lambda^2/2\sigma_j^2})}{\sum_{j=1}^{n} \epsilon_j (\lambda^2 - 2\sigma_j^2 + 2\sigma_j^2 e^{-\lambda^2/2\sigma_j^2})}\;.$$

Example 4.3. Setting $\mathcal{L}(z^{-1}) = \text{Gamma}(a, b),\ a > 1$ and $b > 0$, in the representation given by expression (2.1), we obtain the family of multivariate student distributions with mean parameter $\boldsymbol{\theta}$ (the condition $a > 1$ guaranteeing the existence of a covariance matrix) and known dispersion parameter (a, b). Here, we extend the usual class of multivariate student location families with n degrees of freedom, where $n = 2a = 2b$ and $n \in \{1, 2, \cdots\}$, to include other values of the dispersion parameter (a, b). For the particular case where $p = 5$, we obtain by expressions (3.1) and (4.2),

$$f_5(\lambda,\, z) = f_5^*\left(\frac{\lambda^2}{2z}\right) = \lambda^{-2} z (1 - e^{-\lambda^2/2z})\,, \quad \lambda > 0\,, \quad z > 0\,,$$

and

$$\begin{aligned} c^*(\lambda) &= 2\,\frac{\int_{(0,\infty)} z(1 - e^{-\lambda^2/2z}) dH(z)}{\int_{(0,\infty)} (1 - e^{-\lambda^2/2z}) dH(z)} \\ &= 2\,\frac{\int_{(0,\infty)} (v^{-1} - v^{-1} e^{-\lambda^2 v/2}) v^{a-1} e^{-bv} dv}{\int_{(0,\infty)} (1 - e^{-\lambda^2 v/2}) v^{a-1} e^{-bv} dv} \\ &= \frac{2b}{a-1}\,\frac{\left[1 - \left(\frac{2b}{2b+\lambda^2}\right)^{a-1}\right]}{\left[1 - \left(\frac{2b}{2b+\lambda^2}\right)^{a}\right]}\,. \end{aligned}$$

5. Estimation when the Norm is Restricted to an Interval

In this section, we study the more general case where the mean $\boldsymbol{\theta}$ is restricted to a known interval $[\lambda_1, \lambda_2]$ with $0 \leq \lambda_1 \leq \lambda_2 \leq \infty$. Except for

the multinormal case, no optimal Lindley type decision rule will exist whenever $\lambda_1 \leq \lambda_2$(but see the discussion following corollary 5.6 for asymptotic considerations). We can also characterize the subclass of Lindley type decision rules that dominate the natural estimator $\boldsymbol{\delta}^0$ when $\boldsymbol{\theta} \in \Theta_{\lambda_1}^{\lambda_2}$. In the following, we will denote

$$\underline{c}^*[\lambda_1, \lambda_2] = \inf_{\lambda \in [\lambda_1, \lambda_2]} c^*(\lambda) \text{ and } \bar{c}^*[\lambda_1, \lambda_2] = \sup_{\lambda \in [\lambda_1, \lambda_2]} c^*(\lambda).$$

Theorem 5.1. Let be a single observation from a p-dimensional location parameter with p.d.f. of the form given by expression (2.1). Under the assumptions $\boldsymbol{\theta} \in \Theta_{\lambda_1}^{\lambda_2}$; $0 \leq \lambda_1 \leq \lambda_2 \leq \infty$; $p \geq 4$ and $E(Z) < \infty$,
(a) the subclass $\{ \boldsymbol{\delta}^c \in \mathcal{D}_{Lind} | \ \underline{c}^*[\lambda_1, \lambda_2] \leq c \leq \bar{c}^*[\lambda_1, \lambda_2] \}$ is a minimal complete class within the class $\mathcal{D}_{Lind}$,
and
(b) the decision rule $\boldsymbol{\delta}^c$ will dominate the natural estimator $\boldsymbol{\delta}^0$ if $0 < c < 2\underline{c}^*[\lambda_1, \lambda_2]$.

Proof. (a) Let c_0 be a real number such that $c_0 \notin [\underline{c}^*[\lambda_1, \lambda_2], \ \bar{c}^*[\lambda_1, \lambda_2]]$. Then, using expression (3.4), if $c_0 < \underline{c}^*[\lambda_1, \lambda_2]$, we may write the difference in risks

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}^{c_0}) - R(\boldsymbol{\theta}, \boldsymbol{\delta}^{\underline{c}^*[\lambda_1,\lambda_2]})$$

as

$$\begin{aligned} &= [R(\boldsymbol{\theta}, \boldsymbol{\delta}^{c_0}) - R(\boldsymbol{\theta}, \boldsymbol{\delta}^{c^*(||\boldsymbol{\theta}-\bar{\theta}\mathbf{1}||)})] - [R(\boldsymbol{\theta}, \boldsymbol{\delta}^{\underline{c}^*[\lambda_1,\lambda_2]}) - R(\boldsymbol{\theta}, \boldsymbol{\delta}^{c^*(||\boldsymbol{\theta}-\bar{\theta}\mathbf{1}||)})] \\ &= \int_{(0,\infty)} f_p(\lambda, z) \frac{dH(z)}{z} \{|c_0 - c^*(||\boldsymbol{\theta} - \bar{\theta}\mathbf{1}||)|^2 - |\underline{c}^*[\lambda_1, \lambda_2] - c^*(||\boldsymbol{\theta} - \bar{\theta}\mathbf{1}||)|^2\} \ ; \end{aligned}$$

this last expression being positive for all $\boldsymbol{\theta} \in \Theta_{\lambda_1}^{\lambda_2}$ given that $c_0 < \underline{c}^*[\lambda_1, \lambda_2]$. In the same manner, the decision rule $\boldsymbol{\delta}^c$ with $c = \bar{c}^*[\lambda_1, \lambda_2]$ will dominate the decision rule $\boldsymbol{\delta}^{c_0}$ if $c_0 > \bar{c}^*[\lambda_1, \lambda_2]$. Also, if $c_0 \in [\underline{c}^*[\lambda_1, \lambda_2], \bar{c}^*[\lambda_1, \lambda_2]]$, the intermediate value theorem ($c^*(\lambda)$ is easily shown to be continuous) assures us that

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}^c) - R(\boldsymbol{\theta}, \boldsymbol{\delta}^{c_0}) > 0, \ \ \forall c \neq c_0 ,$$

when $c^*(||\boldsymbol{\theta} - \bar{\theta}\mathbf{1}||) = c_0$. These last results guarantee that all the rules $\boldsymbol{\delta}^c$ with $c \notin [\underline{c}^*[\lambda_1, \lambda_2], \ \bar{c}^*[\lambda_1, \lambda_2]]$ are inadmissible within the class $\mathcal{D}_{Lind}$ and the rules $\boldsymbol{\delta}^c$ with c belonging to the interval $[\underline{c}^*[\lambda_1, \lambda_2], \ \bar{c}^*[\lambda_1, \lambda_2]]$ cannot be improved upon by another rule of the class $\mathcal{D}_{Lind}$. Thus, the result of part (a) follows.
(b) Using corollary 3.4, the decision rule $\boldsymbol{\delta}^c$ will dominate the decision rule $\boldsymbol{\delta}^0$ if

$$\begin{aligned} R(\boldsymbol{\theta}, \boldsymbol{\delta}^c) &< R(\boldsymbol{\theta}, \boldsymbol{\delta}^0), \forall \boldsymbol{\theta} \in \Theta_{\lambda_1}^{\lambda_2} \\ &\Leftrightarrow 0 < c < 2c^*(\lambda), \ \ \forall \lambda \in [\lambda_1, \lambda_2] \\ &\Leftrightarrow 0 < c < 2\underline{c}^*[\lambda_1, \lambda_2]. \end{aligned}$$

It may also be remarked that the rule $\boldsymbol{\delta}^c$ with $c = 2\underline{c}^*[\lambda_1, \lambda_2]$ will also dominate $\boldsymbol{\delta}^0$ under the conditions of the theorem when $\lambda_1 < \lambda_2$ and that all the decisions rules $\boldsymbol{\delta}^c$ with $c > 2\underline{c}^*[\lambda_1, \lambda_2]$ do not dominate $\boldsymbol{\delta}^0$ under the conditions of the theorem. The results above would be more explicit if the function $c^*(\lambda)$ was an increasing function. This would imply $\underline{c}^*[\lambda_1, \lambda_2] = c^*(\lambda_1)$ and $\bar{c}^*[\lambda_1, \lambda_2] = c^*(\lambda_2)$.

The case with no restrictions on the norm $||\boldsymbol{\theta} - \bar{\theta}\mathbf{1}||$ (i.e. $\lambda_1 = 0$ and $\lambda_2 = \infty$) can be expanded using by Strawderman's result(1974) and it can be showed that the decision rules $\boldsymbol{\delta}^c$ with $0 \leq c \leq 2(p-3)E^{-1}(z^{-1})$ are minimax rules by showing that their risk functions are uniformly less than or equal to the risk function ($= pE(Z)$) of the minimax decision rule $\boldsymbol{\delta}^0$. This result is derived below as a particular case of Theorem 5.1. To do so, we need to determine the quantity $\underline{c}^*[0,\infty] = \inf_{\lambda\geq 0} c^*(\lambda)$. The following two lemmas will prove useful in determining $\underline{c}^*[0,\infty]$ and, also, $\bar{c}^*[\lambda_1,\lambda_2]$.

Lemma 5.2. Let $\mathbf{X}$ be an arbitrary random variable and let f and g be two real nondecreasing functions on the support of $\mathbf{X}$. Then, if the quantities $E[f(\mathbf{X})]$ and $E[g(\mathbf{X})]$ exist, $Cov(f(\mathbf{X}),\ g(\mathbf{X})) \geq 0$ with the inequality being strict if f and g are strictly increasing and $\mathbf{X}$ is nondegenerate.

Proof. A neat proof of Lemma 5.2 is given by Chow and Wang(1990).

Lemma 5.3. Let $f_p^*(\cdot), p \in \{4,5,6,\ \cdots\}$ be a function defined on $[0,\ \infty)$ and equal to

$$f_p^*(\gamma) = E^L[(p+2L-3)^{-1}]\ ,\ \gamma \geq 0\ ,$$

where L is a Poisson random variable with mean γ. Then,
(i) $f_p^*(\cdot)$ is a strictly decreasing function,
(ii) $\lim_{\gamma\to 0^+} f_p^*(\gamma) = (p-3)^{-1}\ ,\ \lim_{\gamma\to\infty} f_p^*(\gamma) = 0$,
(iii) if $p \geq 5$, $\gamma f_p^*(\gamma)$ is a strictly increasing function for $\gamma \geq 0$.

Proof. (i) Using part (i) of Lemma 4.1, we have for $\gamma_2 > \gamma_2 > 0$.

$$f_p^*(\gamma_2) - f_p^*(\gamma_1) = \int_{[0,1]} t^{p-4}(e^{\gamma_2(t^2-1)} - e^{\gamma_1(t^2-1)})dt < 0\ .$$

(ii) By the dominated convergence theorem,

$$\begin{aligned}\lim_{\gamma\to 0^+} f_p^*(\gamma) &= \lim_{\gamma\to 0^+}\int_{[0,1]} t^{p-4}e^{\gamma(t^2-1)}dt \\ &= \int_{[0,1]} t^{p-4}(\lim_{\gamma\to 0^+} e^{\gamma(t^2-1)})dt \\ &= \int_{[0,1]} t^{p-4}dt = (p-3)^{-1}\ ,\end{aligned}$$

and

$$\lim_{\gamma\to\infty} f_p^*(\gamma) = \lim_{\gamma\to\infty} \int_{[0,1]} t^{p-4} e^{\gamma(t^2-1)} dt$$
$$= \int_{[0,1]} t^{p-4} (\lim_{\gamma\to\infty} e^{\gamma(t^2-1)}) dt = 0 .$$

(iii) Using expression (4.2), we have

$$\gamma f_5^*(\gamma) = \frac{1}{2}(1 - e^{-\gamma}) .$$

which is easily seen to be strictly increasing. For $p \geq 6$ we obtain by the recurrence formula given by expression (4.1),

$$\gamma f_p^*(\gamma) = \frac{1}{2}(1 - (p-5) f_{p-3}^*(\gamma)),\ \gamma > 0 ,$$

which must be strictly increasing given that function $f_{p-3}^*(\cdot)$ is strictly decreasing by part (i).

In the following, we will set $E^{-1}[Z^{-1}]$ equal to zero if the expectation $E[Z^{-1}] = \infty$.

Theorem 5.4. The function $c^*(\cdot)$ defined by expression(3.1) satisfies the following properties:

(a) $\inf_{\lambda \geq 0} c^*(\lambda) = (p-3)E[Z^{-1}]$,

(b) $c^*(\lambda) = k \Rightarrow Z$ is constant with probability one

and,

(c) for $p \geq 5$, $\sup_{\lambda \geq 0} c^*(\lambda) = (p-3)E(Z)$.

Proof. (a) Expression (3.1) can be rewritten as,

$$c^*(\lambda) = (p-3)\frac{E^Z[f_p(\lambda, Z)]}{E^Z[Z^{-1} f_p(\lambda, Z)]} ,\ \lambda \geq 0 .$$

By applying Lemma 5.2 to the functions $f_p(\lambda, z)$ and , z^{-1} the function $f_p(\lambda, \cdot)$ being an increasing function by virtue of part(i) of Lemma 5.3, we have for

$\lambda \geq 0$,

$$\begin{aligned} & Cov(f_p(\lambda, Z)), -Z^{-1}) \geq 0 \\ \Rightarrow \quad & E^Z[Z^{-1} f_p(\lambda, Z)] \geq E[Z^{-1}] E^Z[f_p(\lambda, Z)] \\ \Rightarrow \quad & c^*(\lambda) \ \geq \ (p-3) E^{-1}[Z^{-1}] \\ \Rightarrow \quad & \inf_{\lambda \geq 0} c^*(\lambda) \ \geq \ (p-3) E^{-1}[Z^{-1}] \,. \end{aligned}$$

The reverse inequality is obtained by observing that $c^*(0) = (p-3)E^{-1}[Z^{-1}]$.
(b) The constancy of $c^*(\lambda)$ implies

$$c^*(\lambda) = k = c^*(0) = (p-3)E^{-1}(Z^{-1}), \ \ \forall \lambda > 0 \,,$$

and

$$\int_{(0,\infty)} (p - 2 - \frac{k}{z}) f_p(\lambda, z) dH(z) = 0 \,.$$

Since both $f_p(\lambda, z)$ and $-kz^{-1}$ are strictly increasing function of z, we have by Lemma 5.2, for nondegenerate Z,

$$\begin{aligned} & Cov(f_p(\lambda, Z)), \ p - 3 - kZ^{-1}) > 0 \\ \Rightarrow \quad & E[(p-3-kZ^{-1}) f_p(\lambda, Z)] > E[(p-3-kZ^{-1})] E[f_p(\lambda, Z)] = 0 \,, \end{aligned}$$

which results in a contradiction implying Z is constant with probability one.
(c) By applying Lemma 5.2 to the functions $-z^{-1} f_p(\lambda, z)$ and z the function $-z^{-1} f_p(\lambda, z)$ being an increasing function by virtue of part(iii) of Lemma 5.3, we have for $p \geq 5$ and $\lambda \geq 0$.

$$\begin{aligned} & Cov(-Z^{-1} f_p(\lambda, Z)), Z) \geq 0 \\ \Rightarrow \quad & E^Z[f_p(\lambda, Z)] \leq E[Z^{-1} f_p(\lambda, Z)] E[Z] \\ \Rightarrow \quad & c^*(\lambda) \ \leq \ (p-3) E[Z] \\ \Rightarrow \quad & \sup_{\lambda \geq 0} c^*(\lambda) \ \leq \ (p-3) E[Z] \,. \end{aligned}$$

The reverse inequality is obtained by verifying that $\lim_{\lambda\to\infty} c^*(\lambda) = (p-3)E[Z]$ whenever $p \geq 5$. To do so, it will be useful to express the function $c^*(\cdot)$ in the following way,

$$c^*(\lambda) = (p-3)\frac{\int_{(0,\infty)} \sum_{j=0}^{\infty} \frac{e^{\frac{-\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^j}{j!(p+2j-3)} z dH(z)}{\int_{(0,\infty)} \sum_{j=0}^{\infty} \frac{e^{\frac{-\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^j}{j!(p+2j-3)} dH(z)}, \quad \lambda > 0\,.$$

$$= (p-3)\frac{\int_{(0,\infty)} \sum_{j=1}^{\infty} \frac{e^{\frac{-\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^j}{j!} \frac{2j}{(p+2j-5)} z dH(z)}{\int_{(0,\infty)} \sum_{j=1}^{\infty} \frac{e^{\frac{-\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^j}{j!} \frac{2j}{(p+2j-5)} dH(z)}, \quad \lambda > 0\,.$$

Moreover, we can write

$$\lim_{\lambda\to\infty} c^*(\lambda) = (p-3)\frac{\lim_{\lambda\to\infty}\left\{\int_{(0,\infty)} \sum_{j=1}^{\infty} \frac{e^{\frac{-\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^j}{j!} \frac{2j}{(p+2j-5)} z dH(z)\right\}}{\lim_{\lambda\to\infty}\left\{\int_{(0,\infty)} \sum_{j=1}^{\infty} \frac{e^{\frac{-\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^j}{j!} \frac{2j}{(p+2j-5)} z dH(z)\right\}}$$

if both limits exist and the denominator is not equal to zero. By the dominated convergence theorem, we can then write $\lim_{\lambda\to\infty} c^*(\lambda)$ as

$$(p-3)\frac{\int_{(0,\infty)} \lim_{\lambda\to\infty} E^{L_z}\left[\frac{2L_z}{(p+2L_z-5)} 1_{((1,2,\cdots)}(L_z)\right] z dH(z)}{\int_{(0,\infty)} \lim_{\lambda\to\infty} E^{L_z}\left[\frac{2L_z}{(p+2L_z-5)} 1_{(1,2,\cdots)}(L_z)\right] dH(z)},$$

where, for $z > 0$, L_z is a Poisson random variable with mean $\lambda^2/2z$. Finally, by noticing that,

$$\forall z > 0\ , \ \lim_{\lambda \to \infty} E^{L_z}[\frac{2L_z}{(p+2L_z-5)} 1_{(1,2,\cdots)}(L_z)] = 1\ ,$$

because the integrand tends $2L_z(p+2L_z-5)^{-1}$ tends to one when $L_z \to \infty$, we obtain

$$\lim_{\lambda \to \infty} c^*(\lambda) = (p-3)\frac{\int_{(0,\infty)} z dH(z)}{\int_{(0,\infty)} dH(z)} = (p-3)E(Z)\ .$$

Having evaluated the quantities $\underline{c}^*[0,\infty]$ and $\bar{c}^*[0,\infty]$, and Theorem 5.1 yields the following result.

Corollary 5.5. Let x be a single observation from a p-dimensional location parameter family with p.d.f. of the form given by expression(2.1). With $p \geq 4$, and under the assumptions $\boldsymbol{\theta} \in \mathbb{R}^p$ and $E[Z] < \infty$,
(a) the subclass $\left\{ \boldsymbol{\delta}^c \in \mathcal{D}_{Lind} | (p-3)E^{-1}[Z^{-1}] \leq c \leq (p-3)E[Z] \right\}$ is a minimal complete class within the class $\mathcal{D}_{Lind}$ for $p \geq 5$,
and
(b) the decision rule $\boldsymbol{\delta}^c$ will dominate the decision rule $\boldsymbol{\delta}^0$ if $0 < c < 2(p-3)E^{-1}[Z^{-1}]$.

Proof. These results above are a direct application of Theorems 5.1 and 5.4. We pursue with some remarks.

Remark 5.1. Under the conditions of Corollary 5.5, the decision rule $\boldsymbol{\delta}^c$ is a minimax rule if and only if $0 \leq c \leq 2(p-3)E^{-1}[Z^{-1}]$; can also be obtained using part (a) of Theorem 5.4. and an analogous version of Corollary 3.4. which, under the same conditions, would specify that

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}^c) \leq p \ \Leftrightarrow 0 \leq c \leq 2c^*(||\boldsymbol{\theta} - \bar{\theta}\mathbf{1}||).$$

It is interesting to note that the natural estimator $\boldsymbol{\delta}^0$ represents the only minimax rule within the class $\mathcal{D}_{Lind}$ when the quantity $E[Z^{-1}]$ does not exist.

Remark 5.2. The results above of Theorem 5.1 and Corollary 5.5 can be extended to the case where the experimental information consists of a sample

$\mathbf{X}_1, \cdots, \mathbf{X}_n$ with p.d.f of the form in (2.1) and the class of decision rules considered consists of the decision rules of the form

$$\delta^c(\mathbf{X}_1, \cdots, \mathbf{X}_n) = \bar{\bar{X}}\mathbf{1} + \left(1 - \frac{c}{(\bar{\mathbf{X}} - \bar{\bar{X}}\mathbf{1})'(\bar{\mathbf{X}} - \bar{\bar{X}}\mathbf{1})}\right)(\bar{\mathbf{X}} - \bar{\bar{X}}\mathbf{1}), \ c \in R,$$

where $\bar{\mathbf{X}}$ is the sample mean and $\bar{\bar{X}} = (np)^{-1}\sum_{i=1}^{p}\sum_{j=1}^{n} X_{ij} = p^{-1}\sum_{i=1}^{p}\bar{X}_i$. This can be seen by noting that the probability law of the sample mean $\bar{\mathbf{X}} = n^{-1}\sum_{i=1}^{p}\mathbf{X}_i$; $\mathbf{X}_1, \cdots, \mathbf{X}_n$ being n independently and identically distributed random vectors admitting the representations

$$\mathcal{L}(\mathbf{X_j}|Z_j = z_j) = N_p(\boldsymbol{\theta}, z_j I_p), \ i = i, \ \cdots, n,$$

for all values $z_1, \cdots, z_n$ of n independent copies $Z_1, \cdots, Z_n$ of a positive random variable Z ; admits the representation

$$\mathcal{L}(\bar{\mathbf{X}}|Z_1 = z_1, \ \cdots, Z_n = z_n) = N_p(\boldsymbol{\theta}, n^{-2}\sum_{j=1}^{n} z_j I_p),$$

or,

$$\mathcal{L}(\bar{\mathbf{X}}|W = w) = N_p(\boldsymbol{\theta}, wI_p), \ \forall w > 0,$$

where W is a random variable such that

$$\mathcal{L}(W) = \mathcal{L}(n^{-2}\sum_{j=1}^{n} z_j). \tag{5.1}$$

Thus, the optimal estimator of the Lindley type is ; with the conditions $\boldsymbol{\theta} \in \Theta_\lambda$, $E[Z] < \infty, p \geq 4$; given by theorem 3.3, and is equal to

$$\delta^{c_n^*(\lambda)}(\bar{\mathbf{X}}) = \bar{\bar{X}}\mathbf{1} + \left(1 - \frac{c_n^*(\lambda)}{(\bar{\mathbf{X}} - \bar{\bar{X}}\mathbf{1})'(\bar{\mathbf{X}} - \bar{\bar{X}}\mathbf{1})}\right)(\bar{\mathbf{X}} - \bar{\bar{X}}\mathbf{1})$$

where

$$c_n^*(\lambda) = (p-3)\frac{\int_{(0,\infty)} f_p(\lambda, w) dH_n^*(w)}{\int_{(0,\infty)} f_p(\lambda, w)\frac{dH_n^*(w)}{w}}$$

$H_n^*(\cdot)$ representing the c.d.f. of the random variable W defined by expression(5.1). Furthermore, the result specifying a minimal complete class within the class

$$\mathcal{D}_{Lind} = \left\{ \delta^c : R^p \to R^p | \delta^c(\bar{\mathbf{X}}) = \bar{\bar{X}}\mathbf{1} + \left(1 - \frac{c}{(\bar{\mathbf{X}} - \bar{\bar{X}}\mathbf{1})'(\bar{\mathbf{X}} - \bar{\bar{X}}\mathbf{1})}\right)(\bar{\mathbf{X}} - \bar{\bar{X}}\mathbf{1}) \right\}$$

as well as the result giving a subclass of Lindley type rules that dominate the sample mean $\delta^0(\bar{\mathbf{X}}) = \bar{\mathbf{X}}$ can be applied to the case where the experimental information consists of a sample. In particular, by rewriting corollary 5.5, we obtain the following result. Part(b) of this corollary has been proved by Bravo and MacGibbon(1988) under a more general setting.

Corollary 5.6. Let $\mathbf{X_1}, \cdots, \mathbf{X_n}$ be a sample generated by a common random vector $\mathbf{X}$ which admits the representation given by expression(2.1). Under the conditions $\boldsymbol{\theta} \in \mathbb{R}^p$, $p \geq 4$ and $E[Z] < \infty$

(a) for $p \geq 5$, the subclass

$\left\{ \delta^c \in \mathcal{D}_{Lind} | n^{-2}(p-3)E^{-1}[(\sum_{i=1}^{n} z_i)^{-1}] \leq c \leq n^{-1}(p-3)E[Z] \right\}$ is a minimal complete class within the class $\mathcal{D}_{Lind}$, and

(b) the decision rule δ^c will dominate the sample mean if

$$0 < c < 2n^{-2}(p-3)E^{-1}[(\sum_{i=1}^{n} Z_i)^{-1}] . \tag{5.2}$$

Proof. These results are a direct application of Corollary 5.5 and the discussion above(expression 5.1)

However, the results concerning the minimax criteria given by Stawderman cannot be applied to the decision rules $\delta^c(\bar{\mathrm{x}})$ since the statistic $\bar{\mathrm{x}}$ does not represent in general a sufficient statistic (the multinormal case being a well known exception). Finally it is interesting to note that,

$$E^{-1}[(\sum_{i=1}^{n} Z_i)^{-1}] \leq E[\sum_{i=1}^{n} Z_i] = nE[Z] ,$$

(the above inequality can be seen as a consequence of Lemma5.2), implying that the interval

$$(0,\ 2n^{-2}(p-3)E^{-1}[(\sum_{i=1}^{n} Z_i)^{-1}] \to \phi \text{ as } n \to \infty ,$$

which, by expression(5.2), indicates that the subclass of Lindley type decision rules dominating the sample mean can be made arbitrarily small by increasing the sample size n.

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