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Intuitionistic Fuzzy Subgroups and Subrings

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Abstract

In this paper, we introduce the concepts of intuitionistic fuzzy subgroups and intuitionistic fuzzy subrings and investigate some of their properties.

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0. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [15], several researchers[1,7,10,12,13,14] have applied the notion of fuzzy sets to group theory. As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov[2]. Recently, Çoker and his colleagues [5,6,8], and S.J.Lee and E.P.Lee[11] introduced the concept of intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets. In 1989, R.Biswas[4] introduced the concept of intuitionistic fuzzy subgroups and studied some of it's properties. In 2003, Baldev Banerjee and Dhiren Kr. Basnet[3] investigated intuitionistic fuzzy subrings and ideals using intuitionistic fuzzy sets. Recently, Hur and his colleagues[9] studied intuitionistic fuzzy subgroupoids.

In this paper, we introduce the concepts of intuitionistic fuzzy subgroups and intuitionistic fuzzy subrings and we study some of their properties.

1. Preliminaries

We will list some concepts and results needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I.

Definition 1.1[2]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) on X if $\mu_A + \nu_A \leq 1$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2[2]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\,]A = (\mu_A, 1 - \mu_A)$, $< > A = (1 - \nu_A, \nu_A)$.

Definition 1.3[5]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(a) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.

(b) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4[5]. $0_\sim = (0, 1)$ and $1_\sim = (1, 0)$.

Result 1.A[5, Corollary 2.8]. Let A, B, C, D be IFSs in X. Then

(1) $A \subset B$ and $C \subset D \Rightarrow A \cup C \subset B \cup D$ and $A \cap C \subset B \cap D$.

(2) $A \subset B$ and $A \subset C \Rightarrow A \subset B \cap C$.

(3) $A \subset C$ and $B \subset C \Rightarrow A \cup B \subset C$.

(4) $A \subset B$ and $B \subset C \Rightarrow A \subset C$.

(5) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

(6) $A \subset B \Rightarrow B^c \subset A^c$.

(7) $(A^c)^c = A$.

(8) $1_\sim^c = 0_\sim$, $0_\sim^c = 1_\sim$.

Definition 1.5[5]. Let X and Y be nonempty sets and let $f : X \to Y$ be a mapping. Let $A = (\mu_A, \nu_A)$ be an IFS in X and $B = (\mu_B, \nu_B)$ be an IFS in Y. Then

(a) the *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by:

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)),$$

where $f^{-1}(\mu_B) = \mu_B \circ f$ and $f^{-1}(\nu_B) = \nu_B \circ f$.

(b) the *image* of A under f, denoted by $f(A)$, is the IFS in Y defined by:

$$f(A) = (f(\mu_A), f(\nu_A)),$$

where for each $y \in Y$

$$f(\mu_A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

and

$$f(\nu_A)(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 1 & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

Result 1.B[5, Corollary 2.10]. Let $A, A_i (i \in J)$ be IFSs in X, let $B, B_j (j \in K)$ IFSs in Y and let $f : X \to Y$ a mapping. Then

(1) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

(2) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$.

(3) $A \subset f^{-1}(f(A))$.

If f is injective, then $A = f^{-1}(f(A))$.

(4) $f(f^{-1}(B)) \subset B$.

If f is surjective, then $f(f^{-1}(B)) = B$.

(5) $f^{-1}(\bigcup B_j)) = \bigcup f^{-1}(B_j)$.

(6) $f^{-1}(\bigcap B_j)) = \bigcap f^{-1}(B_j)$.

(7) $f(\bigcup A_i)) = \bigcup f(A_i)$.

(8) $f(\bigcap A_i)) \subset \bigcap f(A_i)$.

If f is injective, then $f(\bigcap A_i)) = \bigcap f(A_i)$.

(9) $f(1_\sim) = 1_\sim$, if f is surjective and $f(0_\sim) = 0_\sim$.

(10) $f^{-1}(1_\sim) = 1_\sim$ and $f^{-1}(0_\sim) = 0_\sim$.

(11) $[f(A)]^c \subset f(A^c)$, if f is surjective.

(12) $f^{-1}(B^c) = [f^{-1}(B)]^c$.

Definition 1.6[5]. Let X be a set and let $\lambda, \mu \in I$ with $0 \leq \lambda + \mu \leq 1$. Then the IFS $C_{(\lambda,\mu)}$ in X is defined by: for each $x \in X$, $C_{(\lambda,\mu)}(x) = (\lambda, \mu)$, i.e., $\mu_{C_{(\lambda,\mu)}}(x) = \lambda$ and $\nu_{C_{(\lambda,\mu)}}(x) = \mu$.

Definition 1.7[9]. Let $(X, \cdot)$ be a groupoid and let $A, B \in IFS(X)$. Then the *intuitionistic fuzzy product* of A and B, $A \circ B$, is defined as follows : for any $x \in X$,

$$\mu_{A\circ B}(x) = \begin{cases} \bigvee_{yz=x}[\mu_A(y) \wedge \mu_B(z)] & \text{for each } (y,z) \in X \times X \text{ with } yz = x, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_{A\circ B}(x) = \begin{cases} \bigwedge_{yz=x}[\nu_A(y) \vee \nu_B(z)] & \text{for each } (y,z) \in X \times X \text{ with } yz = x, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 1.8[9]. Let $(G, \cdot)$ be a groupoid and let $0_\sim \neq A \in IFS(G)$. Then A is called an *intuitionistic fuzzy subgroupoid* in G (in short, $IFGP$ in G) if $A \circ A \subset A$.

Definition 1.8′[9]. Let $(G, \cdot)$ be a groupoid and let $A \in IFS(X)$. Then A is called an *intuitionistic fuzzy subgroupoid* (in short, $IFGP$) of G if for any $x, y \in G$, $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

We will denote the set of all IFGPs of G as IFGP(G).

Definition 1.9[9]. Let $A \in IFS(G)$. Then A is said to have the*sup property* if for any $T \in P(G)$, there exists a $t_0 \in T$ such that $A(t_0) = \bigcup_{t\in T} A(t)$, i.e., $\mu_A(t_0) = \bigvee_{t\in T} \mu_A(t)$ and $\nu_A(t_0) = \bigwedge_{t\in T} \nu_A(t)$, where $P(G)$ denotes the power set of G.

2. Intuitionistic fuzzy subgroups

Definition 2.1. Let G be a group and let $A \in IFGP(G)$. Then A is called an *intuitionistic fuzzy subgroup* (in short, IFG) of G if $A(x^{-1}) \geq A(x)$, i.e., $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$, for each $x \in G$.

We will denote the set of all IFGs of G as IFG(G).

Example 2.1. Consider the additive group $(Z, +)$. Let us define a complex mapping $A = (\mu_A, \nu_A) : Z \to I \times I$ as follows: for set $0 \neq n \in Z$

$$A(0) = 1_\sim,$$

$$\mu_A(n) = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd,} \\ \frac{2}{3} & \text{if } n \text{ is even} \end{cases}$$

and

$$\nu_A(n) = \begin{cases} \frac{1}{3} & \text{if } n \text{ is odd,} \\ \frac{1}{5} & \text{if } n \text{ is even.} \end{cases}$$

Then clearly $A \in IFS(Z)$. Moreover, A satisfies all the conditions of Definition 2.1. Hence A is an IFG of Z.

Remark 2.1. Let G be a group.

(1) If μ_A is a fuzzy subgroup of G, then $A = (\mu_A, {\mu_A}^c) \in IFG(G)$.

(2) If $A \in IFG(G)$, then μ_A and ${\nu_A}^c$ are fuzzy subgroups of G.

(3) If $A \in IFG(G)$, then $[\,]A$, $< >A \in IFG(G)$.

The following two results can be easily proved from Definition 2.1, Proposition 3.8 and Proposition 3.9 in [9]:

Proposition 2.2. Let G be a group. Then $A = (\chi_T, \chi_{T^c}) \in IFG(G)$ if and only if T is a subgroup of G.

Proposition 2.3. Let $\{A_\alpha\}_{\alpha\in\beta} \subset IFG(G)$. Then $\bigcap_{\alpha\in\beta} A_\alpha \in IFG(G)$.

The following two results can be easily seen from Definition 1.7:

Proposition 2.4. Let A be an IFG of a group G. Then $A \circ A = A$.

Proposition 2.5. Let A and B be any two IFGs in a group G. Then the following conditions are equivalent:

(1) $A \circ B \in IFG(G)$.

(2) $A \circ B = B \circ A$.

Proposition 2.6. Let $A \in IFG(G)$. Then $A(x^{-1}) = A(x)$, i.e., $\mu_A(x^{-1}) = \mu_A(x), \nu_A(x^{-1}) = \nu_A(x)$ and $A(x) \leq A(e)$, i.e., $\mu_A(x) \leq \mu_A(e), \nu_A(x) \geq \nu_A(e)$ for each $x \in G$, where e is the identity element of G.

Proof. By the proof of Proposition 5.4 in [12], we have:

$\mu_A(x^{-1}) = \mu_A(x)$ and $\mu_A(x) \leq \mu_A(e)$ for each $x \in G$.

Thus it is enough to show that $\nu_A(x^{-1}) = \nu_A(x)$ and $\nu_A(x) \geq \nu_A(e)$ for each $x \in G$.

Let $x \in G$. Then :

$\nu_A(x) = \nu_A((x^{-1})^{-1}) \leq \nu_A(x^{-1}) \leq \nu_A(x)$.

On the other hand,

$\nu_A(e) = \nu_A(xx^{-1}) \leq \nu_A(x) \vee \nu_A(x^{-1}) = \nu_A(x)$.

Hence $\nu_A(x^{-1}) = \nu_A(x)$ and $\nu_A(e) \leq \nu_A(x)$ for each $x \in G$. This completes the proof.

Proposition 2.7. If $A \in$ IFG(G), then $G_A = \{x \in G : A(x) = A(e)$, i.e., $\mu_A(x) = \mu_A(e)$ and $\nu_A(x) = \nu_A(e)\}$ is a subgroup of G.

Proof. Let $x, y \in G_A$. Then $\mu_A(x) = \mu_A(e), \nu_A(x) = \nu_A(e)$ and $\mu_A(y) = \mu_A(e), \nu_A(y) = \nu_A(e)$.
Thus

$$\begin{aligned} \mu_A(xy^{-1}) &\geq \mu_A(x) \wedge \mu_A(y^{-1}) \\ &= \mu_A(x) \wedge \mu_A(y) \qquad \text{(by Proposition 2.6)} \\ &= \mu_A(e) \wedge \mu_A(e) = \mu_A(e) \end{aligned}$$

and

$$\begin{aligned} \nu_A(xy^{-1}) &\leq \nu_A(x) \vee \nu_A(y^{-1}) \\ &= \nu_A(x) \vee \nu_A(y) \qquad \text{(by Proposition 2.6)} \\ &= \nu_A(e) \vee \nu_A(e) = \nu_A(e). \end{aligned}$$

On the other hand, by Proposition 2.6, $\mu_A(xy^{-1}) \leq \mu_A(e)$ and $\nu_A(xy^{-1}) \geq \nu_A(e)$. So $\mu_A(xy^{-1}) = \mu_A(e)$ and $\nu_A(xy^{-1}) = \nu_A(e)$. Thus $xy^{-1} \in G_A$. Hence G_A is a subgroup of G.

Proposition 2.8. Let $A \in IFG(G)$. If $A(xy^{-1}) = A(e)$, i.e., $\mu_A(xy^{-1}) = \mu_A(e)$ and $\nu_A(xy^{-1}) = \nu_A(e)$ for any $x, y \in G$, then $A(x) = A(y)$, i.e., $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$.

Proof. Let $x, y \in G$. Then

$$\begin{aligned} \mu_A(x) = \mu_A((xy^{-1})y) &\geq \mu_A(xy^{-1}) \wedge \mu_A(y) \\ &= \mu_A(e) \wedge \mu_A(y) = \mu_A(y). \end{aligned}$$

On the other hand, since $\mu_A(x^{-1}) = \mu_A(x)$ by Proposition 2.6, we have $\mu_A(xy^{-1}) = \mu_A((yx^{-1})^{-1}) = \mu_A(yx^{-1})$ and thus

$$\begin{aligned} \mu_A(y) = \mu_A((yx^{-1})x) &\geq \mu_A(yx^{-1}) \wedge \mu_A(x) = \mu_A(xy^{-1}) \wedge \mu_A(x) \\ &= \mu_A(e) \wedge \mu_A(x) = \mu_A(x). \end{aligned}$$

So $\mu_A(x) = \mu_A(y)$. By the similar arguments, we have $\nu_A(x) = \nu_A(y)$. This completes the proof.

Corollary 2.8-1. Let $A \in IFG(G)$. If G_A is a normal subgroup of G, then A is constant on each coset of G_A.

Proof. Let $a \in G$ and let $x \in aG_A$. Then there exists an $x' \in G_A$ such that $x = ax'$. Since G_A is normal and $x' \in G_A, xa^{-1} = ax'a^{-1} \in G_A$. Thus $\mu_A(xa^{-1}) = \mu_A(e)$ and $\nu_A(xa^{-1}) = \nu_A(e)$. By Proposition 2.8, $\mu_A(x) = \mu_A(a)$ and $\nu_A(x) = \nu_A(a)$. So A is constant on aG_A for each $a \in G$. By the similar arguments, we can see that A is constant on G_Aa for each $a \in G$. Hence A is constant on each coset of G_A.

Let H be a subgroup of G. Then the number of right [resp. left] cosets of H in G is called the *index of H in G* and denoted by [G:H]. If G is a finite group, then there can be only a finite number of distinct right [resp. left] cosets of H; hence the index [G:H] is finite. If G is an infinite group, then [G:H] may be either finite or infinite.

Corollary 2.8-2. Let $A \in IFG(G)$ and let G_A be normal. If G_A has a finite index, then A has the sup property.

Proof. Let $T \subset G$. Since G_A has finite index, let the index $[G : G_A] = n$, say $\mathcal{A} = \{a_1G_A, \cdots, a_nG_A\}$, where $a_i \in G(i = 1, \cdots, n)$ and $a_iG_A \cap a_jG_A = \emptyset$ for any $i \neq j$. Let $t \in T$. Since $G = \bigcup \mathcal{A} = \bigcup_{i=1}^{n} a_iG_i$, there exists an $i \in \{1, \cdots, n\}$ such that $t \in a_iG_A$. Since G_A is normal, by Corollary 2.8-1, $\mu_A(t) = \mu_A(a_i)$ and $\nu_A(t) = \nu_A(a_i)$ on a_iG_A, say $\mu_A(t) = \alpha_i$ and $\nu_A(t) = \beta_i$, where $\alpha_i, \beta_i \in I$ and $\alpha_i + \beta_i \leq 1$. Thus there exists a $t_0 \in T$ such that $\mu_A(t_0) = \bigvee_{t\in T} \mu_A(t) = \bigvee_{i=1}^{n} \alpha_i$ and $\nu_A(t_0) = \bigwedge_{t\in T} \nu_A(t) = \bigwedge_{i=1}^{n} \beta_i$. Hence A has the sup property.

The following is the immediate result of Definition 2.1 and Proposition 2.6:

Proposition 2.9[3, Definition 2.3]. $A \in IFG(G)$ if and only if $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y)$ for any $x, y \in G$.

Proposition 2.10. A group G cannot be the union of two proper IFGs.

Proof. Let A and B be proper IFGs of a group G such that $A \cup B = 1_\sim$, $A \neq 1_\sim$ and $B \neq 1_\sim$. Since $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$ and $1_\sim = (1, 0)$, $\mu_A \vee \mu_B = 1$ and $\nu_A \wedge \nu_B = 0$. Then $\mu_A = 1$ or $\mu_B = 1$ and $\nu_A = 0$ or $\nu_B = 0$. Since $A \neq 1_\sim$ and $B \neq 1_\sim$, $\mu_A \neq 1$ or $\nu_A \neq 0$ and $\mu_B \neq 1$ or $\nu_B \neq 0$. In either cases, this is a contradiction. This completes the proof.

Proposition 2.11. If A is an IFGP of a finite group G, then A is an IFG of G.

Proof. Let $x \in G$. Since G is finite, x has the finite order, say n. Then $x^n = e$, where e is the identity of G. Thus $x^{-1} = x^{n-1}$. Since A is an $IFGP$ of G,

$$\mu_A(x^{-1}) = \mu_A(x^{n-1}) = \mu_A(x^{n-2}x) \geq \mu_A(x)$$

and

$$\nu_A(x^{-1}) = \nu_A(x^{n-1}) = \nu_A(x^{n-2}x) \leq \nu_A(x)$$

Hence A is an IFG of G.

Proposition 2.12. Let A be an IFG of a group G and let $x \in G$. Then $A(xy) = A(y)$, i.e., $\mu_A(xy) = \mu_A(x)$ and $\nu_A(xy) = \nu_A(x)$ for each $y \in G$ if and only if $A(x) = A(e)$, i.e., $\mu_A(x) = \mu_A(e)$ and $\nu_A(x) = \nu_A(e)$, where e is the identity of G.

Proof. ($\Rightarrow$): Suppose $A(xy) = A(y)$ for each $y \in G$. Then clearly $A(x) = A(e)$.

($\Leftarrow$): Suppose $A(x) = A(e)$. Then, by Proposition 2.6, $\mu_A(y) \leq \mu_A(x)$ and $\nu_A(y) \leq \nu_A(x)$ for each $y \in G$. Since A is an IFG of G, Then $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$. Thus $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$

for each $y \in G$.

On the other hand, by Proposition 2.6,

$$\mu_A(y) = \mu_A(x^{-1}xy) \geq \mu_A(x) \wedge \mu_A(xy)$$

and

$$\nu_A(y) = \nu_A(x^{-1}xy) \leq \nu_A(x) \vee \nu_A(xy).$$

Since $\mu_A(x) \geq \mu_A(y)$ and $\nu_A(x) \leq \nu_A(y)$ for each $y \in G, \mu_A(x) \wedge \mu_A(xy) = \mu_A(xy)$ and $\nu_A(x) \vee \nu_A(xy) = \nu_A(xy)$. So $\mu_A(y) \geq \mu_A(xy)$ and $\nu_A(y) \leq \nu_A(xy)$ for each $y \in G$. Hence $\mu_A(xy) = \mu_A(y)$ and $\nu_A(xy) = \nu_A(y)$ for each $y \in G$.

Proposition 2.13. Let $f : G \to G'$ be a group homomorphism, let $A \in IFG(G)$ and let $B \in IFG(G')$. Then the following hold :

(1) If A has the sup property, then $f(A) \in$ IFG(G′).

(2) $f^{-1}(B) \in$ IFG(G).

Proof. (1) By Proposition 4.4 in [9], since $f(A) \in IFGP(G)$, it is enough to show that $\mu_{f(A)}(y^{-1}) \geq \mu_{f(A)}(y)$ and $\nu_{f(A)}(y^{-1}) \leq \nu_{f(A)}(y)$ for each $y \in f(G)$.

Let $y \in f(G)$. Then $\phi \neq f^{-1}(y) \subset G$. Since A has the sup property, there exists an $x_0 \in f^{-1}(y)$ such that $\mu_A(x_0) = \bigvee_{t \in f^{-1}(y)} \mu_A(t)$ and $\nu_A(x_0) = \bigwedge_{t \in f^{-1}(y)} \nu_A(t)$.
Thus

$$\begin{aligned} \mu_{f(A)}(y^{-1}) &= f(\mu_A)(y^{-1}) = \bigvee_{t \in f^{-1}(y^{-1})} \mu_A(t) \geq \mu_A(x_0^{-1}) \geq \mu_A(x_0) \\ &= \mu_{f(A)}(y) \end{aligned}$$

and

$$\begin{aligned} \nu_{f(A)}(y^{-1}) &= f(\nu_A)(y^{-1}) = \bigwedge_{t \in f^{-1}(y^{-1})} \nu_A(t) \leq \nu_A(x_0^{-1}) \leq \nu_A(x_0) \\ &= \nu_{f(A)}(y). \end{aligned}$$

Hence $f(A) \in IFG(G)$.

(2) By Proposition 4.1 in [9], since $f^{-1}(B) \in IFGP(G)$, it is enough to show that $f^{-1}(B)(x^{-1}) \geq f^{-1}(B)(x)$ for each $x \in G$.

Let $x \in G$. Then

$$\begin{aligned} \mu_{f^{-1}(B)}(x^{-1}) &= f^{-1}(\mu_B)(x^{-1}) = \mu_B(f(x^{-1})) = \mu_B((f(x))^{-1}) \geq \mu_B(f(x)) \\ &= \mu_{f^{-1}(B)}(x). \end{aligned}$$

and

$$\begin{aligned} \nu_{f^{-1}(B)}(x^{-1}) &= f^{-1}(\nu_B)(x^{-1}) = \nu_B(f(x^{-1})) = \nu_B((f(x))^{-1}) \leq \nu_B(f(x)) \\ &= \nu_{f^{-1}(B)}(x). \end{aligned}$$

Hence $f^{-1}(B) \in IFG(G)$.

Proposition 2.14. Let G_p be the cyclic group of prime order p. Then $A \in IFG(G_p)$ if and only if $A(x) = A(1) \leq A(0)$, i.e., $\mu_A(x) = \mu_A(1) \leq \mu_A(0)$ and $\nu_A(x) = \nu_A(1) \geq \nu_A(0)$ for each $0 \neq x \in G_p$.

Proof. $(\Rightarrow)$: Suppose $A \in IFG(G_p)$ and let $0 \neq x \in G_p$. Then $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$ for any $x, y \in G_p$. Since G_p is the cyclic group of prime order p, $G_p = \{0, 1, 2, \cdots, p-1\}$. Since x is the sum of $1's$ and 1 is the sum of $x's$, $\mu_A(x) \geq \mu_A(1) \geq \mu_A(x)$ and $\nu_A(x) \leq \nu_A(1) \leq \nu_A(x)$. Thus $\mu_A(x) = \mu_A(1)$ and $\nu_A(x) = \nu_A(1)$. Since 0 is the identity element of G_p, $\mu_A(x) \leq \mu_A(0)$ and $\nu_A(x) \geq \nu_A(0)$. Hence the necessary conditions hold.

$(\Leftarrow)$: Suppose the necessary conditions hold, and let $x, y \in G_p$. Then we have four cases : (i) $x \neq 0, y \neq 0$ and $x = y$, (ii) $x \neq 0, y = 0$, (iii) $x = 0, y \neq 0$, (iv) $x \neq 0, y \neq 0$ and $x \neq y$.

Case(i) Suppose $x \neq 0, y \neq 0$ and $x = y$. Then, by the hypothesis, $\mu_A(x) = \mu_A(y) = \mu_A(1) \leq \mu_A(0)$ and $\nu_A(x) = \nu_A(y) = \nu_A(1) \geq \nu_A(0)$. So $\mu_A(x - y) = \mu_A(0) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x - y) = \nu_A(0) \leq \nu_A(x) \vee \nu_A(y)$.

Case(ii) Suppose $x \neq 0$ and $y = 0$. Since $x - y \neq 0$, by the hypothesis,

$\mu_A(x-y) = \mu_A(x) = \mu_A(1) \leq \mu_A(0) = \mu_A(y)$ and $\nu_A(x-y) = \nu_A(x) = \nu_A(1) \geq \nu_A(0) = \nu_A(y)$. So $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$.

Case(iii) is the same as Case(ii).

Case(iv) Suppose $x \neq 0, y \neq 0$and $x \neq y$. Since $x-y \neq 0$, by the hypothesis, $\mu_A(x-y) = \mu_A(x) = \mu_A(y) = \mu_A(1) \leq \mu_A(0)$ and $\nu_A(x-y) = \nu_A(x) = \nu_A(y) \geq \nu_A(0)$. So $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$. In all, $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$. Hence, by Proposition 2.7, $A \in IFG(G_p)$.

Definition 2.15[9]. Let G be a groupoid and let $A \in IFS(G)$. Then A is called an :

(1) *intuitionistic fuzzy left ideal* (in short, $IFLI$) of G if for any $x, y \in G$, $A(xy) \geq A(y)$, i.e., $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$.

(2) *intuitionistic fuzzy right ideal* (in short, $IFRI$) of G if for any $x, y \in G$, $A(xy) \geq A(x)$, i.e., $\mu_A(xy) \geq \mu_A(x)$ and $\nu_A(xy) \leq \nu_A(x)$.

(3) *intuitionistic fuzzy ideal* (in short, IFI) of G if it is both an IFLI and an IFRI.

It is clear that A is an IFI of G if and only if for any $x, y \in G$, $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$. Moreover, an IFI (resp. IFLI, IFRI) is an IFGP of G. Note that for any IFGP A of G we have $\mu_A(x^n) \geq \mu_A(x)$ and $\nu_A(x^n) \leq \nu_A(x)$ for each $x \in G$, where x^n is any composite of $x's$.

We will denote the set of all IFGPs of G as IFGP(G).

Proposition 2.16. The IFIs (resp. IFLIs, IFRIs) in a group G are just the constant mappings.

Proof. Suppose A is an constant mapping and let $x, y \in G$. Then $\mu_A(xy) = \mu_A(x) = \mu_A(y)$ and $\nu_A(xy) = \nu_A(x) = \nu_A(y)$. So A is an IFI of G.

Now suppose A is an IFLI of G. Then $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$

for any $x, y \in G$. In particular, $\mu_A(x) \geq \mu_A(e)$ and $\nu_A(x) \leq \nu_A(e)$ for each $x \in G$.
Moreover, $\mu_A(e) = \mu_A(x^{-1}x) \geq \mu_A(x)$ and $\nu_A(e) = \nu_A(x^{-1}x) \leq \nu_A(x)$ for each $x \in G$. So $\mu_A(x) = \mu_A(e)$ and $\nu_A(x) = \mu_A(e)$ for each $x \in G$. Hence A is a constant mapping.

Definition 2.17[9]. Let A be an IFS in a set X and let $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$. Then the set $X_A^{(\lambda,\mu)} = \{x \in X : A(x) \geq C_{(\lambda,\mu)}(x)\} = \{x \in X : \mu_A(x) \geq \lambda \text{ and } \nu_A(x) \leq \mu\}$ is called a (λ, μ)-*level subset* of A.

Proposition 2.18. Let A be an IFG of a group G. Then for each $(\lambda, \mu) \in I \times I$ with $(\lambda, \mu) \leq A(e)$, i.e., $\lambda \leq \mu_A(e), \mu \geq \nu_A(e)$, $G_A^{(\lambda,\mu)}$ is a subgroup of G, where e is the identity of G.

Proof. Clearly, $G_A^{(\lambda,\mu)} \neq \emptyset$. Let $x, y \in G_A^{(\lambda,\mu)}$. Then $A(x) \geq (\lambda, \mu)$ and $A(y) \geq (\lambda, \mu)$, i.e., $\mu_A(x) \geq \lambda, \nu_A(x) \leq \mu$ and $\mu_A(y) \geq \lambda, \nu_A(y) \leq \mu$. Since $A \in IFG(G)$, $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \geq \lambda$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y) \leq \mu$. Thus $A(xy) \geq (\lambda, \mu)$. So $xy \in G_A^{(\lambda,\mu)}$. On the other hand, $\mu_A(x^{-1}) \geq \mu_A(x) \geq \lambda$ and $\nu_A(x^{-1}) \leq \nu_A(x) \leq \mu$. Thus $A(x^{-1}) \geq (\lambda, \mu)$. So $x^{-1} \in G_A^{(\lambda,\mu)}$. Hence $G_A^{(\lambda,\mu)}$ is a subgroup of G.

Proposition 2.19. Let A be an IFS in a group G such that $G_A^{(\lambda,\mu)}$ is a subgroup of G for each $(\lambda, \mu) \in I \times I$ with $(\lambda, \mu) \leq A(e)$. Then A is an IFG of G.

Proof. For any $x, y \in G$, let $A(x) = (t_1, s_1)$ and let $A(y) = (t_2, s_2)$. Then clearly, $x \in G_A^{(t_1,s_1)}$ and $y \in G_A^{(t_2,s_2)}$. Suppose $t_1 < t_2$ and $s_1 > s_2$. Then $G_A^{(t_2,s_2)} \subset G_A^{(t_1,s_1)}$. Thus $y \in G_A^{(t_1,s_1)}$. Since $G_A^{(t_1,s_1)}$ is a subgroup of G, $xy \in G_A^{(t_1,s_1)}$. Then $A(xy) = (t_1, s_1)$, i.e., $\mu_A(xy) \geq t_1$ and $\nu_A(xy) \leq s_1$. So $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$. For each $x \in G$, let $A(xy) = (\lambda, \mu)$. Then $x \in G_A^{(\lambda,\mu)}$. Since $G_A^{(\lambda,\mu)}$ is a subgroup of G, $x^{-1} \in G_A^{(\lambda,\mu)}$. So $A(x^{-1}) \geq (\lambda, \mu)$, i.e., $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$. Hence A is an IFG of G.

3. Intuitionistic fuzzy normal subgroups

Definition 3.1. Let A be an IFG in a group G. Then A is called an *intuitionistic fuzzy normal subgroup*(in short, $IFNG$) of G if $A(xy) = A(yx)$, i.e., $\mu_A(xy) = \mu_A(yx)$ and $\nu_A(xy) = \nu_A(yx)$ for any $x, y \in G$.

It is clear that if G is abelian, then every IFG of G is an IFNG of G.

Example 3.1. Consider the general linear group of degree n, $GL(n, R)$. Then clearly, $GL(n, R)$ is a non abelian group. Let us define a complex mapping $A : GL(n, R) \to I \times I$ as follows: for set $I_n \neq M \in GL(n, R)$, where I_n is the unit matrix,

$$A(I_n) = 1_{\sim},$$

$$\mu_A(M) = \begin{cases} \frac{2}{3} & \text{if } M \text{ is not a triangular matrix,} \\ \frac{1}{2} & \text{if } M \text{ is a triangular matrix} \end{cases}$$

and

$$\nu_A(M) = \begin{cases} \frac{1}{5} & \text{if } M \text{ is not a triangular matrix,} \\ \frac{1}{3} & \text{if } M \text{ is a triangular matrix.} \end{cases}$$

Then we can easily see that A is an IFNG of $GL(n, R)$.

The following is the immediate result of Definition 1.7 and Definition 3.1:

Proposition 3.2. Let A be an IFS of a group G and let $B \in IFNG(G)$. Then $A \circ B = B \circ A$.

Proposition 3.3. Let A be an IFNG of a group G. If B is an IFG of G, then so is $B \circ A$.

Proof. By Definition 1.8 and Proposition 2.3 in [9], $(A \circ B) \circ (B \circ A) \subset B \circ A$. So $B \circ A$ is an IFGP of G. Moreover, by the proof of Proposition 2.1(ii) in [11], $\mu_{B \circ A}(x^{-1}) \geq \mu_{B \circ A}(x)$ for each $x \in G$. Thus it is sufficient to show that

$\nu_{B\circ A}(x^{-1}) \leq \nu_{B\circ A}(x)$ for each $x \in G$.

Let $x \in G$. Then

$$\begin{aligned} \nu_{B\circ A}(x^{-1}) &= \bigwedge_{yz=x^{-1}} [\nu_B(y) \vee \nu_A(z)] \\ &= \bigwedge_{z^{-1}y^{-1}=x} [\nu_B((y^{-1})^{-1}) \vee \nu_A((z^{-1})^{-1})] \\ &\leq \bigwedge_{z^{-1}y^{-1}=x} [\nu_B(y^{-1}) \vee \nu_A(z^{-1})] \\ &= \nu_{A\circ B}(x) = \nu_{B\circ A}(x). \end{aligned}$$

Hence $B \circ A$ is an IFG of G.

Proposition 3.4. Let G be a group and let $A, B \in IFNG(G)$. Then $A \circ B \in IFNG(G)$.

Proof. By Proposition 2.5, $A \circ B \in IFG(G)$. Let $a, b \in G$. Then there exists $x, y \in G$ such that $ab = xy$. Since $b = a^{-1}xy$, $ba = (a^{-1}xa)(a^{-1}ya)$. Since A and B are IFNGs of G,

$$\begin{aligned} (A \circ B)(ab) &= (\mu_{A\circ B}(ab), \nu_{A\circ B}(ab)) \\ &= (\bigvee_{ab=xy} [\mu_A(x) \wedge \mu_B(y)], \bigwedge_{ab=xy} [\nu_A(x) \vee \nu_B(y)]) \\ &= (\bigvee_{ba=(a^{-1}xa)(a^{-1}ya)} [\mu_A(a^{-1}xa) \wedge \mu_B(a^{-1}ya)], \\ &\qquad \bigwedge_{ba=(a^{-1}xa)(a^{-1}ya)} [\nu_A(a^{-1}xa) \vee \nu_B(a^{-1}ya)]) \\ &= (\mu_{A\circ B}(ba), \nu_{A\circ B}(ba)) \\ &= (A \circ B)(ba). \end{aligned}$$

Hence $A \circ B \in IFNG(G)$.

Proposition 3.5. If A is an IFNG of G, then G_A is a normal subgroup of G.

Proof. By Proposition 2.7, G_A is a subgroup of G. Moreover $G_A \neq \emptyset$.

Let $x \in G_A$ and let $y \in G$. Then

$$\mu_A(yxy^{-1}) = \mu_A((yx)y^{-1}) = \mu_A(y^{-1}(yx)) = \mu_A(x) = \mu_A(e)$$

and

$$\nu_A(yxy^{-1}) = \nu_A((yx)y^{-1}) = \nu_A(y^{-1}(yx)) = \nu_A(x) = \nu_A(e).$$

Thus $yxy^{-1} \in G_A$. Hence G_A is a normal subgroup of G.

It is clear that if A is a (usual) normal subgroup of G, then $A = (\chi_A, \chi_{A^c})$ is an IFNG of G and $G_A = A$.

Definition 3.6. Let A be an IFNG of G. Then the quotient group G/G_A is called the *intuitionistic fuzzy quotient subgroup* (in short, $IFQG$) of X with respect to A.

Now let $\varphi : G \to G/G_A$ be the natural projection.

Proposition 3.7. If A is an IFNG of G and $B \in IFS(G)$, then $\varphi^{-1}(\varphi(B)) = G_A \circ B$.

Proof. Let $x \in G$. Then:

$$\begin{aligned}\mu_{\varphi^{-1}(\varphi(B))}(x) &= \varphi^{-1}(\mu_{\varphi(B)})(x) = \mu_{\varphi(B)}(\varphi(x)) = \varphi(\mu_B)(\varphi(x)) \\ &= \bigvee_{\varphi(y)=\varphi(x)} \mu_B(y) = \bigvee_{xy^{-1}\in G_A} \mu_B(y)\end{aligned}$$

and

$$\begin{aligned}\nu_{\varphi^{-1}(\varphi(B))}(x) &= \varphi^{-1}(\nu_{\varphi(B)})(x) = \nu_{\varphi(B)}(\varphi(x)) = \varphi(\nu_B)(\varphi(x)) \\ &= \bigwedge_{\varphi(y)=\varphi(x)} \nu_B(y) = \bigwedge_{xy^{-1}\in G_A} \nu_B(y).\end{aligned}$$

On the other hand:

$$\mu_{G_A \circ B}(x) = \bigvee_{zy=x} [\mu_{G_A}(z) \wedge \mu_B(y)] = \bigvee_{z=xy^{-1} \in G_A} \mu_B(y)$$

and

$$\nu_{G_A \circ B}(x) = \bigwedge_{zy=x} [\nu_{G_A}(z) \vee \nu_B(y)] = \bigwedge_{z=xy^{-1} \in G_A} \nu_B(y).$$

Thus $\mu_{\varphi^{-1}(\varphi(B))}(x) = \mu_{G_A \circ B}(x)$ and $\nu_{\varphi^{-1}(\varphi(B))}(x) = \nu_{G_A \circ B}(x)$ for each $x \in G$. Hence $\varphi^{-1}(\varphi(B)) = G_A \circ B$.

4. Intuitionistic fuzzy subrings and ideals

Definition 4.1. Let $(R, +, \cdot)$ be a ring and let $0_\sim \neq A \in IFS(R)$. Then A is called an *intuitionistic fuzzy subring* (in short, IFR) in R if it satisfies the following conitions:

(i) A is an IFG with respect to the operation " $+$ " (in the sense of Definition 2.1).

(ii) A is an IFGP with respect to the operation " $\cdot$ " (in the sense of Definition 1.8 or Definition 1.8′).

We will denote the set of all IFRs of R as IFR(R).

It is clear that subrings of R are IFRs of R.

Remark 4.1. (1) If μ_A is a fuzzy subring of a ring R, then $(\mu_A, {\mu_A}^c) \in IFR(R)$.

(2) If $A \in IFR(R)$, then μ_A and ${\nu_A}^c$ are fuzzy subrings of G.

(3) If $A \in IFR(R)$, then $[\,]A, <> A \in IFR(R)$.

The following is the immediate result of Definition 1.8′ and Proposition 2.7:

Proposition 4.2[3, Definition 3.1]. Let R be a ring and let $0_\sim \neq A \in IFS(R)$. Then A is an IFR in R if and only if for any $x, y \in R$,

(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$.

(ii) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

The following is easily seen:

Proposition 4.3. Let R be a ring. Then the following conditions are equivalent:

(1) A is a subring of R.

(2) (χ_A, χ_{A^c}) is an IFR of R.

Definition 4.4. Let R be a ring and let $0_\sim \neq A$ an IFR in R. Then A is called an :

(1) *intuitionistic fuzzy left ideal* (in short, $IFLI$) in R if $A(xy) \geq A(y)$, i.e., $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$ for any $x, y \in R$.

(2) *intuitionistic fuzzy right ideal* (in short, $IFRI$) in X if $A(xy) \geq A(x)$, i.e., $\mu_A(xy) \geq \mu_A(x)$ and $\nu_A(xy) \leq \nu_A(x)$ for any $x, y \in R$.

(3) *intuitionistic fuzzy ideal* (in short,IFI) in X if it both an IFLI and an IFRI in R.

We can see that an example about IFR and IFI of a ring R is given as Example 3.2 and Example 4.1 in [3], respectively.

Remark 4.4. (1) If μ_A is a fuzzy [resp. left, right] ideal of a ring R, then $(\mu_A, {\mu_A}^c)$ is an IFI [resp. an IFLI, an IFRI] of R.

(2) If A is an IFI [resp. an IFLI, an IFRI] of R, then μ_A and ${\nu_A}^c$ are a fuzzy [resp. left, right] ideal of R.

(3) If A is an IFI [resp. an IFLI, an IFRI] of R, then $[\,]A$, $<>A$ are IFIs [resp. IFLIs, IFRIs] of R.

Let $f : R \to R'$ be a ring epimorphism. If A is an IFR (IFI) of R, then so is $f(A)$; and if B is an IFR (IFI) of R', then so is $f^{-1}(B)$ (See Theorem 6.2 and

Theorem 6.3 in [3]).

The following can be directly verified.

Proposition 4.5[3, Definition 4.1]. Let R be a ring and let $0_\sim \neq A \in IFS(R)$. Then A is an IFI [resp. an IFLI, an IFRI] of R if and only if for any $x, y \in R$,

(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$.

(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$ [resp. $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$, $\mu_A(xy) \geq \mu_A(x)$ and $\nu_A(xy) \leq \nu_A(x)$].

The following is easily seen:

Proposition 4.6. Let R be a ring. Then the following conditions are equivalent:

(1) A is an ideal [resp. a left ideal, a right ideal] of R.

(2) (χ_A, χ_{A^c}) is an an IFI [resp. an IFLI, an IFRI] of R.

Proposition 4.7. Let R be a skew field (also division ring) and let $0_\sim \neq A \in IFS(R)$. Then A is an IFI (IFLI, IFRI) of R if and only if $\mu_A(x) = \mu_A(e) \leq \mu_A(0)$ and $\nu_A(x) = \nu_A(e) \geq \nu_A(0)$ for any $0 \neq x \in R$, where 0 is a unity of R for " $+$ " and e is the unity of R for " $\cdot$ ".

Proof. ($\Rightarrow$): Suppose A is an IFLI of R and let $0 \neq x \in R$. Then

$$\mu_A(x) = \mu_A(xe) \geq \mu_A(e), \mu_A(e) = \mu_A(x^{-1}x) \geq \mu_A(x)$$

and

$$\nu_A(x) = \nu_A(xe) \leq \nu_A(e), \nu_A(e) = \nu_A(x^{-1}x) \leq \nu_A(x).$$

Thus $\mu_A(x) = \mu_A(e)$ and $\nu_A(x) = \nu_A(e)$. On the other hand,

$$\mu_A(0) = \mu_A(e-e) \geq \mu_A(e) \wedge \mu_A(e) = \mu_A(e)$$

and

$$\nu_A(0) = \nu_A(e-e) \leq \nu_A(e) \vee \nu_A(e) = \nu_A(e).$$

So $\mu_A(e) \leq \mu_A(0)$ and $\nu_A(e) \geq \nu_A(0)$. Hence the necessary conditions hold.

($\Leftarrow$): Suppose the necessary conditions hold. Let $x \in R$. Then we have four cases:

(i) $x \neq 0, y \neq 0$ and $x \neq y$ (ii) $x \neq 0, y \neq 0$ and $x = y$

(iii) $x \neq 0, y = 0$ (iv) $x = 0, y \neq 0$.

Case (i): Suppose $x \neq 0, y \neq 0$ and $x \neq y$. Then:

$\mu_A(x-y) = \mu_A(e) \geq \mu_A(x) \wedge \mu_A(y), \nu_A(x-y) = \nu_A(e) \leq \nu_A(x) \vee \nu_A(y)$

and

$\mu_A(xy) = \mu_A(e) \geq \mu_A(x) \vee \mu_A(y), \nu_A(xy) = \nu_A(e) \leq \nu_A(x) \wedge \nu_A(y)$.

Case(ii): Suppose $x \neq 0, y \neq 0$ and $x = y$. Then:

$\mu_A(x-y) = \mu_A(0) \geq \mu_A(x) \wedge \mu_A(y), \nu_A(x-y) = \nu_A(0) \leq \nu_A(x) \vee \nu_A(y)$

and

$\mu_A(xy) = \mu_A(e) \geq \mu_A(x) \vee \mu_A(y), \nu_A(xy) = \nu_A(e) \leq \nu_A(x) \wedge \nu_A(y)$.

Case(iii): Suppose $x \neq 0$ and $y = 0$. Then:

$\mu_A(x-y) = \mu_A(x) = \mu_A(e) \geq \mu_A(x) \wedge \mu_A(y)$,

$\nu_A(x-y) = \nu_A(x) = \nu_A(0) = \nu_A(x) \vee \nu_A(y)$

and

$\mu_A(xy) = \mu_A(0) \geq \mu_A(x) \vee \mu_A(y), \nu_A(xy) = \nu_A(0) \leq \nu_A(x) \wedge \nu_A(y)$.

Case(iv): It is similar to case (iii).

In all, A is an IFI of R.

Remark 4.8. Proposition 4.5 shows that an IFLI (IFRI) is an IFI in a skew field.

The following gives a characteristic of a (usual) field by an IFI.

Proposition 4.9. Let R be a commutative ring with a unity e. If for any IFI A of R, $A(x) = A(e) \leq A(0)$, i.e., $\mu_A(x) = \mu_A(e) \leq \mu_A(0)$ and $\nu_A(x) = \nu_A(e) \geq \nu_A(0)$ for each $0 \neq x \in R$, then R is a field.

Proof. Let A be an ideal of R such that $A \neq R$. Then clearly $A = (\chi_A, \chi_{A^c})$ is an IFI in X such that $A \neq 1_{\sim}$. Then there exists $y \in R$ such that $y \notin A$. Thus $\chi_A(y) = 0$ and $\chi_{A^c}(y) = 1$. By the hypothesis, $\chi_A(x) = \chi_A(e) \leq \chi_A(0)$ and

$\chi_{A^c}(x) = \chi_{A^c}(e) \geq \chi_{A^c}(0)$ for each $0 \neq x \in X$. Thus $\chi_A(x) = \chi_A(e) = 0 \leq \chi_A(0)$ and $\chi_{A^c}(x) = \chi_{A^c}(e) = 1 \geq \chi_{A^c}(0)$ for each $0 \neq x \in X$. So $\chi_A(0) = 1$ and $\chi_{A^c}(0) = 0$, i.e., $A = \{0\}$. Hence R is a field.

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