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Unique Continuation Property for C^∞ Functions

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Abstract

We prove a unique continuation theorem for C^∞ functions in pseudoconvex domains in $\mathbb{C}^n$. More specifically, we show that if Ω is a pseudoconvex domain in $\mathbb{C}^n$, if f is in $C^\infty(\Omega)$ such that for all multi-indexes α, β with $|\beta| \geq 1$ and for any positive integer k, there exists a positive constant $C_{\alpha,\beta,k}$ such that

$$\left|\frac{\partial^{|\alpha|+|\beta|} f}{\partial z^\alpha \partial \bar{z}^\beta}\right| \leq C_{\alpha,\beta,k}|f|^k \quad \text{in } \Omega,$$

and if there exists $z_0 \in \Omega$ such that f vanishes to infinite order at z_0, then f is identically zero. We also have a sharp result for the case of strongly pseudoconvex domains.

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1. Introduction

A unique continuation property for a holomorphic function defined in a domain of $\mathbb{C}^n$ means that the function cannot vanish to infinite order at any

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point in the domain unless it is identically equal to zero. We now want to ask to what extent C^∞ functions share this property with some assumptions. This problem is related to a type of Carleman theorem on quasi-analytic classes discovered by T. Carleman[Car26](See also [Man52]). Bell and Lempert[BL90] proved using the Cauchy integral formula that if f is a C^∞ function on the unit disc in the complex plane such that $\left|\frac{\partial f}{\partial \bar{z}}(z)\right| \le C|f(z)|$ for some constant C, and if f vanishes to infinite order at the origin, then f must be identically equal to zero. In section 2, I will prove that C^∞ functions defined in a pseudoconvex domain in $\mathbb{C}^n$ with certain conditions also satisfy the unique continuation property. And I will give a sharper result in the case of strongly pseudoconvex domains. In section 3, I will give a motivation from which this research has come out.

2. Main Results

Theorem 2..1 *Suppose that Ω is a pseudoconvex domain in $\mathbb{C}^n$. Suppose that f is a C^∞ smooth function in Ω such that for all multi-indexes α, β with $|\beta| \ge 1$ and for any positive integer k, there exists a positive constant $C_{\alpha,\beta,k}$ such that*

$$\left|\frac{\partial^{|\alpha|+|\beta|} f}{\partial z^\alpha \partial \bar{z}^\beta}\right| \le C_{\alpha,\beta,k}|f|^k \quad \text{in } \Omega. \tag{2.1}$$

If there exists $z_0 \in \Omega$ such that f vanishes to infinite order at z_0, then f is identically zero.

Proof. Let V denote the zero set of the function f, i.e., $V = \{z \in \Omega \,|\, f(z) = 0\}$. We define a $(0,1)$-form v on the set $\Omega \setminus V$ by

$$v = \frac{1}{f}\bar{\partial} f$$

and extend it to V by setting $v = 0$ on V. We first claim that v is a C^∞ $(0,1)$-

form in Ω. Let $z \notin V$. Then for $j = 1, 2, \cdots, n$,

$$\begin{aligned}\left|\frac{1}{f(z)}\frac{\partial f}{\partial \overline{z_j}}(z)\right| &\leq C_{(0,\cdots,0),(0,\cdots,1,\cdots,0),2}\frac{|f(z)|^2}{|f(z)|} \\ &= C_{(0,\cdots,0),(0,\cdots,1,\cdots,0),2}|f(z)|\end{aligned}$$

Hence v is continuous in the whole set Ω. Similarly, since given $w = (w_1, \cdots, w_n) \in V$, for any $z = (z_1, \cdots, z_n) \in \Omega \setminus V$ which is sufficiently close to w,

$$\begin{aligned}\left|\frac{\left(\frac{1}{f}\frac{\partial f}{\partial \overline{z_j}}\right)(w_1, \cdots, z_l, \cdots, w_n)}{z_l - w_l}\right| &\leq C_{(0,\cdots,0),(0,\cdots,1,\cdots,0),3}\frac{|f(z)|^2}{|z_l - w_l|} \\ &\leq C\left|\frac{\partial f}{\partial z_l}(w)\right| |f(z)|,\end{aligned}$$

we know that all partial derivatives of order 1 of the coefficients of v exist in V and they are all zero. Continuing this process, we conclude that all coefficients of the form v is C^∞ smoothly differentiable in Ω.

Note that $\overline{\partial} v = 0$. Since Ω is a pseudoconvex domain, one can find $u \in C^\infty(\Omega)$ such that $\overline{\partial} u = v$.

Now we define $g = fe^{-u}$. Then g is a C^∞ function in Ω and on the set $\Omega \setminus V$ we have

$$\begin{aligned}\overline{\partial} g &= (\overline{\partial} f)e^{-u} - fe^{-u}\overline{\partial} u \\ &= \left(\overline{\partial} f - f\,\frac{\overline{\partial} f}{f}\right)e^{-u} \\ &= 0.\end{aligned}$$

It thus follows that g is holomorphic in $\Omega \setminus V$ and hence by the Classical Radó's Theorem g is holomorphic in Ω. Since f vanishes to infinite order at $z_0 \in \Omega$, g also vanishes to infinite order at z_0, and hence g is identically equal to zero, so must f on Ω. □

Since holomorphic functions obviously satisfy the condition (2.1), we have the following uniqueness theorem.

Corollary 2..2 *Suppose that Ω is a domain in $\mathbb{C}^n$. Suppose that f is a holomorphic function in Ω. If there exists $z_0 \in \Omega$ such that f vanishes to infinite order at z_0, then f is identically zero.*

Proof. Choose a pseudoconvex neighborhood $D \subset \Omega$ of z_0 and apply Theorem 2..1 to D. Then f is identically zero in D and hence by analyticity so is f in Ω. □

We can give a weaker condition on (2.1) in Theorem 2..1 in the case of a smooth bounded pseudoconvex domain.

Theorem 2..3 *Suppose that Ω is a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. Suppose that $f \in C^\infty(\overline{\Omega})$ such that for all multi-indexes α, β with $0 \leq |\alpha| \leq n+2, 1 \leq |\beta| \leq n+3$, there exists a positive constant $C_{\alpha,\beta}$ such that*

$$\left| \frac{\partial^{|\alpha|+|\beta|} f}{\partial z^\alpha \partial \bar{z}^\beta} \right| \leq C_{\alpha,\beta} |f|^{n+3} \quad \text{in } \Omega. \tag{2.2}$$

If there exists $z_0 \in \Omega$ such that f vanishes to infinite order at z_0, then f is identically zero.

Proof. As in the proof of Theorem 2..1, it is easy to see that the $(0,1)$-form v defined by $v = \frac{1}{f}\bar{\partial} f$ is in $W^{n+2}_{(0,1)}(\Omega)$, the space of $(0,1)$-form whose coefficients are in the Sobolev space of order $n+2$ in Ω. It is known from the global regularity of the $\bar{\partial}$-Neumann problem on smoothly bounded pseudoconvex domains (see [CS01],[Koh73]) that the solution u is in $W^{n+2}(\Omega)$, which is by the Sobolev Embedding Theorem a C^1 function. Hence the proof completes if we follow the proof of Theorem 2..1. □

We have even the following theorem which is sharper than Theorem2..3 in the case of strongly pseudoconvex domains in virtue of the subelliptic $1/2$-estimates and boundary regularity for the $\bar{\partial}$-Neumann operator(See [Koh63], [CS01]).

Theorem 2..4 *Suppose that Ω is a smooth bounded strongly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. Suppose that $f \in C^\infty(\overline{\Omega})$ such that for all multi-indexes α, β with $0 \leq |\alpha| \leq n+1, 1 \leq |\beta| \leq n+2$, there exists a positive constant $C_{\alpha,\beta}$ such that*

$$\left| \frac{\partial^{|\alpha|+|\beta|} f}{\partial z^\alpha \partial \bar{z}^\beta} \right| \leq C_{\alpha,\beta} |f|^{n+3} \quad \text{in } \Omega. \tag{2.3}$$

If there exists $z_0 \in \Omega$ such that f vanishes to infinite order at z_0, then f is identically zero.

3. Remark

In this section, we give a motivation for the results in section 2. Let Ω be a bounded domain in $\mathbb{C}^n$. Let $H^2(\Omega)$ be the subspace of $L^2(\Omega)$ which consists of holomorphic functions in Ω. The Bergman kernel function $K(z, w)$ of Ω is the kernel for $H^2(\Omega)$ which has the reproducing property: for all $f \in H^2(\Omega)$,

$$f(z) = \int_\Omega \overline{K(w,z)} f(w) \, dV, \tag{3.1}$$

where dV is the volume measure. In one complex variable, the Bergman kernel can be written in terms of the Green's function. Hence we can show that, using the boundary regularities of the Green's function, if Ω is a smoothly bounded m-connected domain in $\mathbb{C}$ and if $w_0 \in \Omega$ is sufficiently near to $b\Omega$, $K(\cdot, w_0)$ has exactly $(m-1)$ zeroes in Ω. (See [SY76], [Bel92], [Chu93]). In order to raise the question in several variables, we need the notion of a unique continuation property for $\overline{\partial}$-operator introduced by Bell [Bel93]. Suppose that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and that the boundary $b\Omega$ of Ω is C^∞ near a point $z_0 \in \Omega$. The operator ϑ is the formal adjoint of the $\overline{\partial}$-operator defined by

$$\vartheta u = -\sum_{j=1}^n \frac{\partial u_j}{\partial z_j} \quad \text{for a } (0,1)\text{−form } u = \sum_{j=1}^n u_j d\overline{z_j} \, .$$

Let $\epsilon > 0$ be small enough so that the intersection $B_\epsilon(z_0) \cap \Omega$ of the Ball $B_\epsilon(z_0)$ and Ω is connected. We say that the ϑ-unique continuation property holds at z_0 if, for any $(0,1)$-form $u \in C^\infty_{(0,1)}(\overline{\Omega} \cap B_\epsilon(z_0))$ whose coefficients vanish on $b\Omega \cap B_\epsilon(z_0)$, ϑu vanishes identically on $B_\epsilon(z_0) \cap \Omega$ whenever ϑu is holomorphic on $B_\epsilon(z_0) \cap \Omega$ and ϑu vanishes to infinite order at z_0.

Bell asked in [Bel93] whether given $z_0 \in b\Omega,\ w_0 \in \overline{\Omega}$ and a multi-index β, or not there exists a multi-index α such that

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \overline{w}^\beta} K(z_0, w_0) \neq 0?$$

and proved the following.

Proposition 3..1 *Suppose that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and the boundary of Ω is C^∞ near a boundary point z_0 that is of finite type in the sense of D'Angelo. Suppose that the ϑ-unique continuation property holds at z_0. If $w_0 \in \overline{\Omega}$ is such that either $w_0 \in \Omega$ or $w_0 \in b\Omega, w_0 \neq z_0$, the boundary of Ω is C^∞ near w_0, and w_0 is a point of finite type in the sense of D'Angelo, then given a multi-index β, there exists a multi-index α such that*

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \overline{w}^\beta} K(z_0, w_0) \neq 0.$$

Hence we are interested in asking for which domain the ϑ-unique continuation property holds at boundary points. Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Let P be the Bergman projection of Ω which is the orthogonal projection of $L^2(\Omega)$ onto $H^2(\Omega)$ and which has a reproducing property: for $h \in L^2(\Omega)$

$$Ph(z) = \int_\Omega K(z,w)h(w)dV_w.$$

Let $w_0 \in \Omega$ and let $d = \mathrm{dist}(w_0, b\Omega)$. Choose $\chi \in C_0^\infty(B(0;1))$ which is radially symmetric about the origin with the normalized condition $\int \chi = 1$. Let χ_{w_0} be the compactly supported function near w_0 defined by $\chi_{w_0} = d^{-2n}\chi\left(\frac{\cdot - w_0}{d}\right)$. Then it follows from the polar coordinate representation that

$$K(\cdot, w_0) = P(\chi_{w_0}).$$

Furthermore,

$$\frac{\partial^{|\beta|}}{\partial \overline{w}^{\beta}} K(z, w_0) = P\left((-1)^{|\beta|} \frac{\partial^{|\beta|}}{\partial \overline{\zeta}^{\beta}} \chi_{w_0}(\zeta)\right)(z).$$

Hence in order to study about the finite order vanishing property of the Bermgan kernel, it is natural to consider the vanishing property of the Bergman projection P too.

Now let $\phi \in C^\infty(\Omega)$ be compactly supported. Let's assume that $P\phi \in C^\infty(\overline{\Omega})$. Then if $u \in C^\infty_{0,1}(\overline{\Omega})$ vanishes on the boundary of Ω, ϑu is orthogonal to the space $H^2(\Omega)$ by the Stoke theorem. Notice that $\vartheta\alpha = P\phi - \phi$ is orthogonal to $H^2(\Omega)$. According to Rosay's theorem(see [Ros82], [Bel93]), there is $u \in C^\infty_{0,1}(\overline{\Omega})$ such that the coefficients of u vanish on $b\Omega$ satisfying $\vartheta u = P\phi - \phi$. Since $\vartheta u = P\phi - \phi$ is orthogonal to $H^2(\Omega)$, it follows that $P(\vartheta u) = 0$ in Ω. But the Kohn's formula $P = I - \vartheta N \overline{\partial}$ for the $\overline{\partial}$-Neumann operator N ([Koh63], [Koh64], [Koh84]) gives the identity $\vartheta u = \vartheta N \overline{\partial}(\vartheta u)$ on Ω. Thus we have an important identity

$$\vartheta u = P\phi - \phi = \vartheta N \overline{\partial}(\vartheta u) \quad \text{on } \Omega. \tag{3.2}$$

Hence the ϑ-unique continuation property holds at a strictly pseudoconvex boundary point z_0 if Ω has real analytic boundaries near z_0 because the analytic hypoellipticity of the $\overline{\partial}$-Neumann problem holds at z_0. See [Tar80].

Bell pointed out in [Bel93] that one might "not" need the global analytic hypoellipticity of $\overline{\partial}$-Neumann problem to get an affirmative answer for the ϑ-unique continuation property. The author realized that in order to study the ϑ-unique continuation property, by identity (3.2), we need a finite order vanishing property of $N(u)$ for a C^∞ $(0,1)$-form u such that u vanishes in Ω near z_0.

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