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Conditions on Operators Satisfying Weyl's Theorem

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Abstract

In this note it is shown that if T satisfies (G_1)-condition with finite spectrum then Weyl's theorem holds for T. If T is totally $*$-paranormal then $T-\lambda$ has finite ascent for all $\lambda \in \mathbb{C}$, T is isoloid, and Weyl's theorem holds for T

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1. Introduction

Let H denote an infinite dimensional separable Hilbert space. Let $B(H)$ and $K(H)$ denote, respectively, the algebra of bounded operators and the ideal of compact operators on H. If $T \in B(H)$ write $N(T)$ and $R(T)$ for the null space and range of T; $\alpha(T) = \dim N(T)$; $\beta(T) = \dim N(T^*)$; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T; $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity.

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An operator $T \in B(H)$ is called *Fredholm* if it has closed range with finite dimensional null space and its range of finite co-dimension. The *index* of a Fredholm operator $T \in B(H)$ is given by

$$i(T) := \alpha(T) - \beta(T).$$

An operator $T \in B(H)$ is called *Weyl* if it is Fredholm of index zero. The Weyl spectrum $\omega(T)$ of $T \in B(H)$ is defined by ([**4**],[**5**],[**6**])

$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}.$$

According to Schechter's characterization([**6**])

$$\omega(T) = \cap_{K \in K(H)} \sigma(T + K).$$

If we write iso $K = K \setminus$ acc K then

$$\pi_{00}(T) := \{\lambda \in \text{iso}\, \sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$$

where acc$K$ means the accumulation points of K.

We say that *Weyl's theorem holds for* $T \in B(H)$ if the following equality holds :

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T).$$

H. Weyl ([**19**]) has shown that the above equality holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal operators and to Toeplitz operators ([**6**]), and to several classes of operators including seminormal operators ([**4**],[**5**]).

The null space of $T \in B(H)$ is just the leading terms of monotonic sequence of subspaces([**11**]):

$$\{0\} \subseteq N(T) \subseteq N(T^2) \subseteq \cdots \subseteq N(T^n) \subseteq N(T^{n+1}) \subseteq \cdots$$

T is said to have *finite ascent* if $N(T^k) = \cup_{n=1}^{\infty} N(T^n)$ for some k.

An operator $T \in B(H)$ is called *normaloid* if $\gamma(T) = ||T||$ and *isoloid* if iso $\sigma(T) \subset \pi_0(T)$. An operator $T \in B(H)$ is said to satisfy (G_1)-*condition* if $(T - \lambda I)^{-1}$ is normaloid for all $\lambda \notin \sigma(T)$.

An operator $T \in B(H)$ is said to be *paranormal* if

$$||Tx||^2 \leq ||T^2x|| \quad \text{for every unit vector } x \in H.$$

An operator $T \in B(H)$ is said to be $*$- *paranormal* if

$$||T^*x||^2 \leq ||T^2x|| \quad \text{for every unit vector } x \in H.$$

Generally, $*$-paranormal operator is not isoloid, so we need an another operator which will be stronger than it: An operator $T \in B(H)$ is said to be *totally $*$-paranormal* if $T - \lambda$ is $*$-paranormal for every $\lambda \in \mathbb{C}$. We can note that totally $*$-paranormality implies $*$-paranormality, but the converse is not true(consider unilateral shift operator on l^2). It will be proved that this new operator is isoloid([Theorem 2.5]).

The $*$-paranormality of operators has been studied in (**[2]**),(**[3]**) and (**[17]**). If $T \in B(H)$ and we define P_F through Riesz functional calculus

$$P_F = \frac{1}{2\pi} \int_{\Gamma} (\lambda - T)^{-1} d\lambda$$

where F is the isolated part of $\sigma(T)$ and Γ is a Cauchy contour containing F, then P_F is a projection with $TP_F = P_F T$ and we can decompose T into $T = T_1 \oplus T_2$ such that $\sigma(T_1) = F$ and $\sigma(T_2) = \sigma(T) \setminus F$, which is called the *spectral projection* (**[11]**).

For an arbitrary operator $T \in B(H)$, we define the *local spectral subspaces* and *glocal spectral subspaces* of T respectively as follows:

$$H_T(F) := \{x \in X : \sigma_T(x) \subseteq F\} \quad \text{for each set } F \subset \mathbb{C};$$

$$\mathcal{H}_T(G) = \{x \in X : \text{there exists an analytic function } f : \mathbb{C} \setminus G \longrightarrow H \\ \text{that satisfies } (T-\lambda)f(\lambda) = x \quad \text{for all } \lambda \in \mathbb{C} \setminus G\}$$

for each closed set $G \subseteq \mathbb{C}$. It is well known([**14**]) that if $\lambda \in \sigma(T)$ is isolated, $H = \mathcal{H}_T(\{\lambda\}) + H_T(\mathbb{C} \setminus \{\lambda\})$.

S. Prasanna ([**18**]) showed that Weyl's theorem holds for every paranormal operator. Evidently, hyponormal operators are both paranormal and $*$-paranormal. But $*$-paranormality is independent of paranormality ([**3**, Examples 2.2 and 2.3]). But both of them are normaloid. In this paper we show that if T is totally $*$-paranormal then $T-\lambda$ has finite ascent for all $\lambda \in \mathbb{C}$, T is isoloid, and Weyl's theorem holds for T.

2. Main results

Lemma 2.1([2]) Let $T \in B(H)$. If T is $*$-paranormal, then T is normaloid.
Lemma 2.2([2]) If T is $*$-paranormal, then $N(T-\lambda) \subseteq N(T^* - \bar{\lambda})$ for each $\lambda \in \mathbb{C}$. Thus $T - \lambda$ is reduced by its eigenspaces for each $\lambda \in \mathbb{C}$.
Theorem 2.3 If T is $*$-paranormal, then $T-\lambda$ has finite ascent for each $\lambda \in \mathbb{C}$.

proof. Suppose T is $*$-paranormal. By Lemma 2.2, for each $\lambda \in \mathbb{C}$, $T-\lambda$ is reduced by its eigenspaces. Thus we can represent $T-\lambda$ as the following 2×2 operator matrix with respect to the decomposition $N(T-\lambda) \oplus N(T-\lambda)^{\perp}$:

$$T - \lambda = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}.$$

Let $x \in N((T-\lambda)^2)$. Write $x = y + z$, where $y \in N(T-\lambda)$ and $z \in N(T-\lambda)^{\perp}$. Then $0 = (T-\lambda)^2 x = (T-\lambda)^2 z$, so that $(T-\lambda)z \in N(T-\lambda) \cap N(T-\lambda)^{\perp} = \{0\}$, which implies that $z \in N(T-\lambda)$, and hence $x \in N(T-\lambda)$. Therefore $N(T-\lambda) = N(T-\lambda)^2$. □

Lemma 2.4 Let T be totally $*$-paranormal. Then

$$\mathcal{H}_T(\{\lambda\}) = N(T-\lambda) \quad \text{for every } \lambda \in \mathbb{C}.$$

proof. Observe that $\mathcal{H}_T(\{\lambda\}) = \{x \in H : \lim_{n\to\infty} ||(T-\lambda)^n x||^{\frac{1}{n}} = 0\}$ for each $\lambda \in \mathbb{C}$. Since T is totally $*$-paranormal, it follows from Lemma 2.1 that $T - \mu$ is normaloid for each $\mu \in \mathbb{C}$. Therefore $||(T-\lambda)x|| \leq ||(T-\lambda)^n x||^{\frac{1}{n}}$ for all $x \in H$ and $n \in N$, and hence $\mathcal{H}_T(\lambda) \subseteq N(T-\lambda)$ for every $\lambda \in \mathbb{C}$. The converse is clear. □

Theorem 2.5 Let T be totally $*$-paranormal. Then T is isoloid.

proof. Suppose that λ is an isolated point of $\sigma(T)$. Then it follows from [**14**, Theorem 1.4.7 and Proposition 3.3.2] that $H = \mathcal{H}_T(\{\lambda\}) + H_T(\mathbb{C} \setminus \{\lambda\})$. Assume to the contrary that $T - \lambda$ is injective. Then it follows Lemma 2.4 that $H = H_T(\mathbb{C} \setminus \{\lambda\})$. But $(T-\lambda)H_T(\mathbb{C} \setminus \{\lambda\}) = H_T(\mathbb{C} \setminus \{\lambda\})$; hence $T - \lambda$ is surjective. Therefore $T - \lambda$ is invertible. This is a contradiction; thus T is isoloid. □

Theorem 2.6 If T satisfies (G_1)-condition and $\sigma(T)$ is a finite set then Weyl's theorem holds for T.

proof. Let $\sigma(T) = \{\lambda_1, \lambda_1, \cdots, \lambda_n\}$ and for each λ_j let $C_{j,\epsilon}$ be a circle with the center at λ_j with radius ϵ; moreover ϵ is so small that for all j, the intersection of the circle with $\sigma(T)$ contains no other points. Since the points are isolated, we can use the spectral projection

$$P_j := \frac{1}{2\pi i} \int_{C_{j,\epsilon}} (T-z)^{-1} dz$$

which commutes with T. Since $||P_j|| \leq 1$, P_j is a contraction; thus a hermition projection. Let $x \in P_j H$ and thus

$$||(T-\lambda_j)x|| = ||\frac{1}{2\pi i} \int_{C_j,\epsilon} (z-\lambda_j)(T-z)^{-1} dz|| < \epsilon$$

Letting $\epsilon \to 0$, we obtain $Tx = \lambda_j x$. By the statements about spectral projection $T = \sum \lambda_j P_j$; thus T is normal; thus hyponormal; thus Weyl's theorem holds for T ([**6**]). □

Theorem 2.7 Let T be totally $*$-paranormal. Then Weyl's theorem holds for T.

proof. Suppose T is totally $*$-paranormal. We first prove that $\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T)$. Since totally $*$-paranormality is translation-invariant, it suffices to show that

$$0 \in \pi_{00}(T) \Longrightarrow T \text{ is Weyl but not invertible.}$$

Suppose $0 \in \pi_{00}(T)$. Now using the spectral projection as the above

$$P := \frac{1}{2\pi i} \int_{\partial B_0} (\lambda I - T)^{-1} d\lambda$$

where B_0 is an open disk of center 0 which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } \sigma(T_1) = \{0\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{0\}.$$

Since $P(H) = \{x \in H : \lim_{n \to \infty} ||T^n x||^{\frac{1}{n}} = 0\} = \mathcal{H}_T(\{0\}) = N(T)$ is a finite-dimensional subspace of H, $\omega(T) = \omega(T_2)$. But T_2 is invertible; hence T is Weyl. Therefore $0 \in \sigma(T) \setminus \omega(T)$. Conversely, suppose that $0 \in \sigma(T) \setminus \omega(T)$. Then it follows from Lemma 2.4 that $\mathcal{H}_T(\{0\}) = N(T)$. Since $\mathcal{H}_T(\{0\})$ is a closed invariant subspace for T, T can be represented as the following 2×2 operator matrix with respect to the decomposition $\mathcal{H}_T(\{0\}) \bigoplus \mathcal{H}_T(\{0\})^{\perp}$:

$$T = \begin{pmatrix} 0 & T_1 \\ 0 & T_2 \end{pmatrix}.$$

Since $\mathcal{H}_T(\{0\})$ is a finite-dimensional subspace of H, T is Weyl if and only if T_2 is Weyl. But $\mathcal{H}_T(\{0\}) = N(T)$; hence T_2 is injective, and so T_2 is invertible. It follows from the punctured neighborhood theorem that $0 \in \pi_{00}(T)$. This completes the proof. □

References

[1] P. Aiena and O. Monsalve, *Operators which do not have the single valued extension property*, J. Math. Anal. Appl **250** (2000), 435–448.

[2] S.C. Arora and J.K. Thukral, *On a class of operators*, Glasnik Math. **21(41)** (1986), 381–386.

[3] S.C. Arora and J.K. Thukral, *M^*-paranormal operators*, Glasnik Math. **22(42)** (1987) 123–129.

[4] S.K. Berberian, *An extension of Weyl's theorem to a class of not necessarily normal operators*, Michigan Math. J. **16** (1969), 273–279.

[5] S.K. Berberian, *The Weyl Spectrum of an Operator*, Indiana Univ. Math. J. **20** (1970), 529–544.

[6] L.A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288.

[7] S.V. Djordjević and D.S. Djordjević, *Weyl's theorems: continuity of the spectrum and quasihyponormal operators*, Acta Sci. Math. (Szeged) **64** (1998), 259–269.

[8] S.V. Djordjević and Y.M. Han, *Browder's theorems and spectral continuity*, Glasgow Math. J. **42** (2000), 479–486.

[9] J.K. Finch, *The single valued extension property on a Banach space Pacific*, J. Math. **58** (1975), 61–69.

[10] R.E. Harte, *Fredholm, Weyl and Browder theory*, Proc. Royal Irish Acad. **85A (2)** (1985), 151–176.

[11] R.E. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Dekker, New York, (1988).

[12] R.E. Harte and W.Y. Lee, *Another note on Weyl's theorem*, Trans. Amer. Math. Soc. **349** (1997), 2115–2124.

[13] K.B. Laursen *Operators with finite ascent*, Pacific J. Math. **152** (1992), 323–336.

[14] K.B. Laursen and M.M. Neumann, *An Introduction to Local Spectral Theory*, London Mathematical Society Monographs New Series 20, Clarendon Press, Oxford (2000).

[15] W.Y. Lee and S.H. Lee, *A spectral mapping theorem for the Weyl spectrum*, Glasgow Math. J. **38(1)** (1996), 61–64.

[16] S.H. Lee and W.Y. Lee, *On Weyl's theorem (II)*, Math. Japo. **43** (1996), 549–553.

[17] S.M. Patel, *Contributions to the study of spectraloid operators*, Ph. D. Thesis, Delhi Univ. (1974).

[18] S. Prasanna, *Weyl's theorem and thin spectra*, Indian Acad. Sci. (Math. Sci.) **91** (1982), 59–63.

[19] H. Weyl, *Über beschränkte quadratische Formen, deren Differenz vollsteig ist*, Rend. Circ. Mat. Palermo **27** (1909), 373–392.