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On Generalized Quasi-preclosed Sets and Quasi Preseparation Axioms*

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Abstract

In this paper, we define generalized quasi-preclosed sets and *gqp*-closed functions and obtain some new characterizations of quasi P-normal spaces and quasi P-regular spaces due to Tapi et al. [9,11]. It is shown that the pairwise continuous pre *gqp*-closed (resp. pairwise preopen pre *gqp*-closed) surjective image of quasi P-normal (resp. quasi P-regular) space is quasi P-normal (resp. quasi P-regular).

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1. Introduction

The notion of quasi-open sets in bitopological space was introduced by Dutta [1]. Tapi et al. [8] have defined and investigated the notion of quasi-preopen sets as a generalization of quasi-open sets. In [9-11], Tapi and coworker used quasi-preopen sets to define quasi P-normal and quasi P-regular spaces, and obtained characterizations of those spaces, and introduced the notions of quasi precontinuous and quasi pre-irresolute functions in bitopological spaces and obtained some of their properties. In this paper, we define and investigate the notions of generalized quasi-preclosed sets and generalized quasi-preopen sets, which are implied by quasi-preclosed sets and quasi-preopen sets respectively, and use these notions to obtain new characterizations of quasi P-normal and quasi P-regular spaces. We also define pre gqp-closed functions in bitopological spaces and use these functions to obtain certain preservation theorems of quasi P-normal and quasi P-regular spaces.

The triple (X, τ_1, τ_2) where X is a set and τ_1, τ_2 are topologies on X, will always denote a bitopological space (for short space), while (X, τ) denotes a single topological space. For a subset A of (X, τ), $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ represent the closure of A and the interior of A with respect to τ. A subset A is called preopen [4] (resp. α-open [6]) if $A \subset \mathrm{int}(\mathrm{cl}(A))$ (resp. $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$). The complement of preopen (resp. α-open) set is called preclosed (resp. α-closed). For a subset A of (X, τ_1, τ_2), τ_i-$\mathrm{cl}(A)$ and τ_j-$\mathrm{int}(A)$ represent the τ_i-closure of A and τ_j-interior of A with respect to the topologies τ_i and τ_j, respectively, where the indices i and j take valued in $\{1, 2\}$ and $i \neq j$. A subset A of a space (X, τ_1, τ_2) is called bi-open (resp. bi-closed, bi-α-open) if it is both τ_1- and τ_2-open (resp. closed, α-open). A subset A of a space (X, τ_1, τ_2) is said to be quasi-open [1] (resp. quasi-preopen [8]) if for each $x \in A$ there exists either a τ_1-open (resp. τ_1-preopen) set U such that $x \in U \subset A$ or a τ_2-open (resp. τ_2-preopen) set V such that $x \in V \subset A$. Every quasi-open set is quasi-preopen but the converse may not be true. The complement of quasi-open (resp. quasi-preopen) set is called quasi-closed (resp. quasi-preclosed). The intersection of all quasi-closed (resp. quasi-preclosed) sets containing A is called the quasi-closure [1] (resp. quasi-preclosure [7]) of A and is denoted by qcl(A) (resp. qpcl(A)). Dually, the quasi-interior [1] (resp. quasi-preinterior) of A, denoted by qint(A) (resp. qpint(A)), is defined to be the union of all quasi-open (resp. quasi-preopen) sets contained in A.

Lemma 1.1 [1,8] Let A be a subset of a space (X, τ_1, τ_2) and $x \in X$. Then the following properties hold:

(a) If $A \subset B$, then qcl$(A) \subset$ qcl(B) and qpcl$(A) \subset$ qpcl(B).

(b) qcl(qcl(A)) = qcl(A) and qpcl(qpcl(A)) = qpcl(A).

(c) A is quasi-closed (resp. quasi-preclosed) if and only if $A =$ qcl(A) (resp. $A =$ qpcl(A)).

(d) qcl(A) (resp. qpcl(A)) is quasi-closed (resp. quasi-preclosed).

(e) $x \in$ qcl(A) (resp. $x \in$ qpcl(A)) if and only if $A \cap U \neq \phi$ for every quasi-open (resp. quasi-preopen) set U containing x.

Lemma 1.2 If A is a bi-α-open set and B is a quasi-preopen set of a space (X, τ_1, τ_2), then $A \cap B$ is quasi-preopen in X.

Proof. Let A be bi-α-open and B be quasi-preopen in X. By [8, Theorem 2.1], there exist a τ_1-preopen set U and a τ_2-preopen set V such that $B = U \cup V$. By [2, Lemma 4.2], $A \cap U$ is τ_1-preopen and $A \cap V$ is τ_2-preopen. Hence by [8, Theorem 2.2], $A \cap B = (A \cap U) \cup (A \cap V)$ is quasi-preopen. □

Corollary 1.3 [8] If A is a bi-open set and B is a quasi-preopen set of (X, τ_1, τ_2), then $A \cap B$ is quasi-preopen in X.

Lemma 1.4 Let $(Z, (\tau_1)_Z), (\tau_2)_Z)$ be a bi-α-open subspace of a space (X, τ_1, τ_2). If A is quasi-preopen in X, then $A \cap Z$ is quasi-preopen in Z.

Proof. It follows from Lemma 1.2 and [8, Theorem 2.4]. □

Lemma 1.5 If $(Z, (\tau_1)_Z, (\tau_2)_Z)$ is a bi-α-open subspace of a space (X, τ_1, τ_2), then for any subset A of Z, qpcl$_Z(A) =$ qpcl$(A) \cap Z$, where qpcl$_Z(A)$ denotes the quasi-preclosure of A in the subspace $(Z, (\tau_1)_Z, (\tau_2)_Z)$.

Proof. Let $x \in$ qpcl$_Z(A)$ and U be any quasi-preopen set of X containing x. Since Z is bi-α-open, by [2, Lemma 4.2] and [8, Theorem 2.2], $U \cap Z$ is quasi-preopen in Z. So, by Lemma 1.1 (e), $(U \cap Z) \cap A \neq \phi$ and consequently $U \cap A \neq \phi$. Hence $x \in$ qpcl$(A) \cap Z$. On the other hand, let $x \in$ qpcl$(A) \cap Z$

and V be any quasi-preopen set of Z containing x. Since Z is bi-α-open, by [8, Lemma 1.4 and Theorem 2.1], U is quasi-preopen in X and $U \cap A \neq \phi$. Hence $x \in \text{qpcl}_Z(A)$. □

Corollary 1.6 [8] If $(Z, (\tau_1)_Z, (\tau_2)_Z)$ is a bi-open subspace of a space (X, τ_1, τ_2), then for any subset A of Z, $\text{qpcl}_Z(A) = \text{qpcl}(A) \cap Z$.

2. Generalized quasi-preclosed sets

Definition 2.1 A subset A of a space (X, τ_1, τ_2) is said to be generalized quasi-preclosed (briefly *gqp*-closed) if $\text{qpcl}(A) \subset U$ whenever $A \subset U$ and U is τ_i-open in X. The comlement of a *gqp*-closed set is said to be generalized quasi-preopen (briefly *gqp*-open).

Every quasi-preclosed set is *gqp*-closed but the converse may not true.

Example 2.2 Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, X\}$. Then $\{a, c\}$ is *gqp*-closed but it is not quasi-preclosed in X.

Proposition 2.3 A subset A of a space (X, τ_1, τ_2) is *gqp*-closed if and only if $\text{qpcl}(A) \subset \tau_i\text{-ker}(A)$, where $\tau_i\text{-ker}(A)$ denotes the kernel of A with respect to τ_i.

Proof. Since A is *gqp*-closed, $\text{qpcl}(A) \subset G$ for any τ_i-open set G with $A \subset G$ and hence $\text{qpcl}(A) \subset \tau_i\text{-ker}(A)$. Conversely, let G be a τ_i-open set of X and $A \subset G$. By hypothesis, $\text{qpcl}(A) \subset \tau_i\text{-ker}(A) \subset G$ and hence A is *gqp*-closed. □

Proposition 2.4 If A is a bi-open and *gqp*-closed set of (X, τ_1, τ_2), then A is quasi-preclosed.

Proof. Since A is bi-open and *gqp*-closed, $\text{qpcl}(A) \subset A$ and then $\text{qpcl}(A) = A$. By Lemma 1.1 (c), A is quasi-preclosed. □

Proposition 2.5 Let $F \subset Z \subset X$, where Z is bi-open and *gqp*-closed in (X, τ_1, τ_2). If F is *gqp*-closed relative to Z, then F is *gqp*-closed relative to X.

Proof. Let U be a τ_i-open set of X such that $F \subset U$. Since $F \subset U \cap Z$, $U \cap Z$ is $(\tau_i)_Z$-open in Z and F is *gqp*-closed in Z, we have $\text{qpcl}_Z(F) \subset U \cap Z$. By Lemma 1.5, $\text{qpcl}(F) \cap Z \subset U \cap Z$ and then by Proposition 2.4, $\text{qpcl}(F) \subset$

$\text{qpcl}(Z) = Z$ and hence $\text{qpcl}(F) = \text{qpcl}(F) \cap Z \subset U \cap Z \subset U$. This implies that F is *gqp*-closed in X. □

Proposition 2.6 Let $F \subset Z \subset X$, where Z is bi-open in (X, τ_1, τ_2). If F is *gqp*-closed relative to X, then F is *gqp*-closed relative to Z.

Proof. Let U be a $(\tau_i)_Z$-open set of A such that $F \subset U$. Since $U = V \cap Z$ for some τ_i-open set V of X and Z is bi-open in X, U is τ_i-open in X. Using assumption, we have $\text{qpcl}(F) \subset U$ and so $\text{qpcl}_Z(F) = \text{qpcl}(F) \cap Z \subset U \cap Z = U$. Hence F is *gqp*-closed in Z. □

Proposition 2.7 For a subset A of a space (X, τ_1, τ_2) in which every *gqp*-closed set is τ_i-closed, the following are equivalent:

(a) A is *gqp*-closed.

(b) For each $x \in \text{qpcl}(A)$, $\tau_i\text{-cl}(\{x\}) \cap A \neq \phi$.

(c) $\text{qpcl}(A) \setminus A$ contains no nonempty τ_i-closed set.

Proof. (a)⇒(b): Let $x \in \text{qpcl}(A)$. If $\tau_i\text{-cl}(\{x\}) \cap A = \phi$, then $A \subset X \setminus \tau_i\text{-cl}(\{x\})$ and so $\text{qpcl}(A) \subset X \setminus \tau_i\text{-cl}(\{x\})$, contradicting $x \in \text{qpcl}(A)$.

(b)⇒(c): Let F be a τ_i-closed set such that $F \subset \text{qpcl}(A) \setminus A$. If there exists a $x \in F$, then by (b), $\phi \neq \tau_i\text{-cl}(\{x\}) \cap A \subset F \cap A \subset (\text{qpcl}(A) \setminus A) \cap A$, a contradiction. Hence $F = \phi$.

(c)⇒(a): Let $A \subset G$ and G be τ_i-open in X. If $\text{qpcl}(A) \not\subset G$, then $\text{qpcl}(A) \cap (X \setminus G)$ is nonempty quasi-preclosed. By hypothesis, $\text{qpcl}(A) \cap (X \setminus G)$ is nonempty τ_i-closed subset of $\text{qpcl}(A) \setminus A$, a contradiction. Hence $\text{qpcl}(A) \subset G$. □

Proposition 2.9 A subset A of a space (X, τ_1, τ_2) is *gqp*-open in X if and only if $F \subset \text{qpint}(A)$ whenever $F \subset A$ and F is τ_i-closed in X.

Proof. Let F be τ_i-closed in X and $F \subset A$. Since $X \setminus A$ is *gqp*-closed, $\text{qpcl}(X \setminus A) \subset X \setminus F$. Then $X \setminus \text{qpint}(A) \subset X \setminus F$, i.e. $F \subset \text{qpint}(A)$. Conversely, let $X \setminus A \subset U$ and U be any τ_i-open in X. By hypothesis, $X \setminus U \subset \text{qpint}(A)$, i.e. $\text{qpcl}(X \setminus A) \subset U$. This implies that $X \setminus A$ is *gqp*-closed and so A is *gqp*-open. □

Proposition 2.10 For a subset A of a space (X, τ_1, τ_2), the following are true:

(a) If A is *gqp*-open in X, then $U = X$ whenever $\text{qpint}(A) \cup (X \setminus A) \subset U$

and U is τ_i-open in X.

(b) If A is gqp-closed in X, then $\mathrm{qpcl}(A) \setminus A$ is gqp-open.

Proof. Straightforward. □

3. gqp-closed functions

Definition 3.1 A function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(a) pairwise continuous [3] (resp. pairwise closed [5], pairwise open [5], pairwise preopen) if the induced function $f : (X, \tau_i) \to (Y, \sigma_i)$ is continuous (resp. closed, open, preopen);

(b) quasi pre-irresolute [10] if for each quasi-preopen set V of Y, $f^{-1}(V)$ is quasi-preopen in X.

Definition 3.2 A function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(a) gqp-closed if for each quasi-closed set F of X, $f(F)$ is gqp-closed in Y;

(b) pre gqp-closed if for each quasi-preclosed set F of X, $f(F)$ is gqp-closed in Y.

It is obvious that both pairwise closedness and pre gqp-closedness imply gqp-closedness. However, the converses are false as the following example shows.

Exapmple 3.3 (a) Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$, $\tau_2 = \{\phi, X\}$, $\sigma_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma_2 = \{\phi, X, \{b\}, \{a, b\}\}$. Let $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ be the identity. Then f is pairwise closed and hence gqp-closed, but it is not pre gqp-closed because there exists a quasi-preclosed set $\{b\}$ of (X, τ_1, τ_2) such that $\{b\}$ is not gqp-closed in (X, σ_1, σ_2).

(b) Let $X = \{a, b, c\}$, $\tau_1 = \tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma_1 = \{\phi, X, \{a\}, \{a, b\}\}$ and $\sigma_2 = \{\phi, X\}$. Let $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ be the identity. Then f is pre gqp-closed and hence gqp-closed, but it is not pairwise closed because there exists a τ_2-closed set $\{a, c\}$ of X such that $\{a, c\}$ is not σ_2-closed in X.

Theorem 3.4 A surjective function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is gqp-closed (resp. pre gqp-closed) if and only if for each subset B of Y and each quasi-open (resp. quasi-preopen) set U of X containing $f^{-1}(B)$, there exists a gqp-

open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

Proof. Let B be any subset of Y and U be a quasi-open (resp. quasi-preopen) set of X containing $f^{-1}(B)$. Put $V = Y \setminus f(X \setminus U)$. Then V is *gqp*-open in Y, $B \subset V$ and $f^{-1}(V) \subset U$. Conversely, let F be any quasi-closed (resp. quasi-preclosed) set of X. Put $B = Y \setminus f(F)$, then we have $f^{-1}(B) \subset X \setminus F$ and $X \setminus F$ is quasi-open (resp. quasi-preopen) in X. There exists a *gqp*-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset X \setminus F$. Then we obtain $f(F) = Y \setminus V$ and so $f(F)$ is *gqp*-closed in Y. Hence f is *gqp*-closed (resp. pre *gqp*-closed). □

Necessity of Theorem 3.4 is proved without assuming that f is surjective. Therefore, we can obtain the following corollary.

Corollary 3.5 A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is *gqp*-closed (resp. pre *gqp*-closed), then for any σ_i-closed set F of Y and each quasi-open (resp. quasi-preopen) set U of X containing $f^{-1}(F)$, there exists a quasi-preopen set V of Y such that $F \subset V$ and $f^{-1}(V) \subset U$.

Proof. By Theorem 3.4, there exists a *gqp*-open set W of Y such that $F \subset W$ and $f^{-1}(W) \subset U$. Since F is σ_i-closed, by Proposition 2.9 we have $F \subset \mathrm{qpint}(W)$. Put $V = \mathrm{qpint}(W)$, then V is quasi-preopen in Y, $F \subset V$ and $f^{-1}(V) \subset U$. □

Proposition 3.6 If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise continuous and pre *gqp*-closed and A is *gqp*-closed in X, then $f(A)$ is *gqp*-closed in Y.

Proof. Let V be any σ_i-open set of Y containing $f(A)$. Then $A \subset f^{-1}(V)$ and $f^{-1}(V)$ is τ_i-open in X. Since A is *gqp*-closed in X, $\mathrm{qpcl}(A) \subset f^{-1}(V)$ and hence $f(A) \subset f(\mathrm{qpcl}(A)) \subset V$. Since f is pre *gqp*-closed and $\mathrm{qpcl}(A)$ is quasi-preclosed in X, $f(\mathrm{qpcl}(A))$ is *gqp*-closed in Y and hence $\mathrm{qpcl}(f(A)) \subset \mathrm{qpcl}(f(\mathrm{qpcl}(A))) \subset V$. This shows that $f(A)$ is *gqp*-closed in Y. □

Proposition 3.7 If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise open and quasi pre-irresolute bijection and B is *gqp*-closed in Y, then $f^{-1}(B)$ is *gqp*-closed in X.

Proof. Let U be any τ_i-open set of X containing $f^{-1}(B)$. Then $B \subset f(U)$ and $f(U)$ is σ_i-open in Y. Since B is *gqp*-closed in Y, $\mathrm{qpcl}(B) \subset f(U)$

and hence $f^{-1}(B) \subset f^{-1}(\text{qpcl}(B)) \subset U$. Since f is quasi pre-irresolute, $f^{-1}(\text{qpcl}(B))$ is quasi-preclosed in X and hence by Lemma 1.1 we have qpcl $(f^{-1}(B)) \subset f^{-1}(\text{qpcl}(B)) \subset U$. This shows that $f^{-1}(B)$ is gqp-closed in X. □

4. Quasi P-normal bitopological spaces

Definition 4.1 A space (X, τ_1, τ_2) is said to be quasi P-normal [9] if for each τ_1-closed set A and τ_2-closed set B of X such that $A \cap B = \phi$, there exist quasi-preopen sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$.

Every pairwise normal space is quasi P-normal but not conversely as shown by Tapi et al. [9]. By using gqp-open sets, we obtain new characterization of quasi P-normal spaces.

Theorem 4.2 For a space (X, τ_1, τ_2), the following properties are equivalent:

(a) X is quasi P-normal.

(b) For any τ_1-closed set A and τ_2-closed set B of X such that $A \cap B = \phi$, there exist gqp-open sets U and V such that $A \subset U$ and $B \subset V$.

(c) For any τ_i-closed set A and τ_j-open set V containing A, there exists a gqp-open set U such that $A \subset U \subset \text{qpcl}(U) \subset V$.

Proof. (a)⇒(b): It is obvious since every quasi-preopen set is gqp-open.

(b)⇒(c): Let A be any τ_i-closed set and V be a τ_j-open set containing A. Since A and $X \setminus V$ are disjoint, there exist gqp-open sets U and W of X such that $A \subset U$, $X \setminus V \subset W$ and $U \cap W = \phi$. By Proposition 2.9, we have $X \setminus V \subset \text{qpint}(W)$. Since $U \cap \text{qpint}(W) = \phi$, we have $\text{qpcl}(U) \cap \text{qpint}(W) = \phi$ and then $\text{qpcl}(U) \subset X \setminus \text{qpint}(W) \subset V$. Hence we obtain $A \subset U \subset \text{qpcl}(U) \subset V$.

(c)⇒(a): Let A be a τ_1-closed set and B be a τ_2-closed set such that $A \cap B = \phi$. Since $X \setminus B$ is τ_2-open set containing A, there exist a gqp-open set G such that $A \subset G \subset \text{qpcl}(G) \subset X \setminus B$. By Proposition 2.9, we have $A \subset \text{qpint}(G)$. Put $U = \text{qpint}(G)$ and $V = X \setminus \text{qpcl}(G)$. Then U and V are disjoint quasi-preopen sets such that $A \subset U$ and $B \subset V$. Hence X is quasi P-normal. □

Theorem 4.3 Every bi-α-open, bi-closed subspace $(Z, (\tau_1)_Z, (\tau_2)_Z)$ of a quasi P-normal space (X, τ_1, τ_2) is quasi P-normal.

Proof. Let A and B be any disjoint sets of Z such that A is $(\tau_1)_Z$-closed and B is $(\tau_2)_Z$-closed. Since Z is bi-closed, A is τ_1-closed and B is τ_2-closed. By the quasi P-normality of X, there exist quasi-preopen sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$. Since Z is bi-α-open, by Lemma 1.4 $U \cap Z$ and $V \cap Z$ are disjoint quasi-preopen sets in Z such that $A \subset U \cap Z$ and $B \subset V \cap Z$. Hence Z is quasi P-normal. □

Corollary 4.4 [9] Every bi-open, bi-closed subspace of quasi p-normal space is quasi P-normal.

Theorem 4.5 If $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise continuous gqp-closed surjection and X is pairwise normal, then Y ia quasi P-normal.

Proof. Let A and B be disjoint sets of Y such that A is σ_1-closed and B is σ_2-closed. Then $f^{-1}(A)$ is τ_1-closed and $f^{-1}(B)$ is τ_2-closed and $f^{-1}(A) \cap f^{-1}(B) = \phi$ since f is pairwise continuous. Since X is pairwise normal, there exist a τ_2-open set U and a τ_1-open set V such that $f^{-1}(A) \subset U$, $f^{-1}(B) \subset V$ and $U \cap V = \phi$. By Theorem 3.4, there exist gqp-open sets G and H of Y such that $A \subset G$, $B \subset H$, $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Then, we have $f^{-1}(G) \cap f^{-1}(H) = \phi$ and thus $G \cap H = \phi$. It follows from Theorem 4.2 that Y is quasi P-normal. □

Theorem 4.6 If $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise continuous pre gqp-closed surjection and X is quasi P-normal, then Y ia quasi P-normal.

Proof. Let A and B be disjoint sets of Y such that A is σ_1-closed and B is σ_2-closed. Then $f^{-1}(A)$ is τ_1-closed and $f^{-1}(B)$ is τ_2-closed and $f^{-1}(A) \cap f^{-1}(B) = \phi$ since f is pairwise continuous. By the quasi P-normality of X, there exist disjoint quasi-preopen sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Since f is pre gqp-closed, by Corollary 3.5, there exist quasi-preopen sets G and H of Y such that $A \subset G$, $B \subset H$, $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Since U and V are disjoint, we have $G \cap H = \phi$. This shows that Y is quasi P-normal. □

5. Quasi P-regular bitopological spaces

Definition 5.1 A space (X, τ_1, τ_2) is said to be quasi P-regular [11] if for each τ_i-closed set F of X and each point $x \in X \setminus F$, there exist disjoint quasi-preopen sets U and V such that $F \subset U$ and $x \in V$.

Theorem 5.2 For a space (X, τ_1, τ_2) the following properties are equivalent:

(a) X is quasi P-regular.

(b) For each τ_i-closed set F and each point $x \in X \setminus F$, there exist a quasi-preopen set U and a gqp-open set V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$.

(c) For each subset A of X and each τ_i-closed set F such that $A \cap F = \phi$, there exist a quasi-preopen set U and a gqp-open set V such that $A \cap U = \phi$, $F \subset V$ and $U \cap V = \phi$.

(d) For each τ_i-closed set F of X, $F = \{\text{qpcl}(V) : F \subset V \text{ and } V \text{ is } gqp\text{-open}\}$.

Proof. (a)⇒(b): It is obvious since every quasi-preopen set is gqp-open.

(b)⇒(c): Let A be a subset of X and F be a τ_i-closed set of X such that $A \cap F = \phi$. For a point $x \in A$, $x \in X \setminus F$ and hence there exist a quasi-preopen set U and a gqp-open set V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$.

(c)⇒(a): Let F be any τ_i-closed set of X and $x \in X \setminus F$. Then $\{x\} \cap F = \phi$ and there exist a quasi-preopen set U and a gqp-open set W such that $x \in U$, $F \subset W$ and $U \cap W = \phi$. Put $V = \text{qpint}(W)$, then by Proposition 2.9, V is quasi-preopen set such that $F \subset V$ and $U \cap V = \phi$. Hence X is quasi P-regular.

(a)⇒(d): For any τ_i-closed set F of X, we have

$$\begin{aligned} F &\subset \cap\{\text{qpcl}(V) : F \subset V \text{ and } V \text{ is } gqp\text{-open}\} \\ &\subset \cap\{\text{qpcl}(V) : F \subset V \text{ and } V \text{ is quasi-preopen}\} = F. \end{aligned}$$

Hence $F = \cap\{\text{qpcl}(V) : F \subset V \text{ and } V \text{ is } gqp\text{-open}\}$.

(d)⇒(a): Let F be any τ_i-closed set of X and $x \in X \setminus F$. By (d), there exists a gqp-open set W of X such that $F \subset W$ and $x \in X \setminus \text{qpcl}(W)$. Since F is τ_i-closed, by Proposition 2.9 we have $F \subset \text{qpint}(W)$. Put $V = \text{qpint}(W)$, then V is quasi-preopen in X and $F \subset V$. Since $x \in X \setminus \text{qpcl}(W)$, we have

$x \in X \setminus \mathrm{qpcl}(V)$. Put $U = X \setminus \mathrm{qpcl}(V)$, then U is quasi-preopen in X, $x \in U$ and $U \cap V = \phi$. This shows that X is quasi P-regular. □

Theorem 5.3 Every bi-α-open subspace $(Z, (\tau_1)_Z, (\tau_2)_Z)$ of a quasi P-regular space (X, τ_1, τ_2) is quasi P-regular.

Proof. Let F be any $(\tau_i)_Z$-closed set of Z and $x \in Z \setminus F$. Then there exists a τ_i-closed set H of X such that $F = H \cap Z$ and $x \notin H$. Since X is quasi P-regular, there exist disjoint quasi-preopen sets U_x and U_H such that $x \in U_x$ and $H \subset U_H$. Now, put $V_x = U_x \cap Z$ and $V_F = U_H \cap Z$, then by Lemma 1.4 V_x and V_F are quasi-preopen in Z, $x \in V_x$, $F \subset V_F$ and $V_x \cap V_F = \phi$. This shows that Z is quasi P-regular. □

Corollary 5.4 Every bi-open subspace of a quasi P-regular space is quasi P-regular.

Lemma 5.5 If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise continuous pairwise preopen and U is quasi-preopen in X, then $f(U)$ is quasi-preopen in Y.

Proof. It follows from [7, Lemma 2] and [8, Theorem 2.2]. □

Theorem 5.6 If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise continuous pairwise preopen pre gqp-closed surjection and X is quasi P-regular, then Y is quasi P-regular.

Proof. Let F be any σ_i-closed set of Y and $y \in Y \setminus F$. Then $f^{-1}(y) \cap f^{-1}(F) = \phi$ and $f^{-1}(F)$ is τ_i-closed in X. Since X is quasi P-regular, for a point $x \in f^{-1}(y)$ there exist quasi-preopen sets U and V of X such that $x \in U$, $f^{-1}(F) \subset V$ and $U \cap V = \phi$. Since F is σ_i-closed in Y, by Corollary 3.5 there exists a quasi-preopen set W of Y such that $F \subset W$ and $f^{-1}(W) \subset V$. By Lemma 5.5, $f(U)$ is quasi-preopen in Y and $y \in f(x) \in f(U)$. Since $U \cap V = \phi$, $f^{-1}(W) \cap U = \phi$ and hence $W \cap f(U) = \phi$. This shows that Y is quasi P-regular. □

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