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SUNSPOT MODELING AND SCALING LAWS

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ABSTRACT

In an early paper Skumanich suggested the existence of a scaling law relating the mean sunspot magnetic field with the square-root of the photospheric pressure. This was derived from an analysis of a variety of theoretical spot models including those by Yun (1968). These were based on the Schlüter-Temesvary (S-T) similarity assumption. To answer criticisms that such modeling may have unphysical (non-axial maxima) solutions, the S-T model was revisited, Moon et al. (1998), with an improved vector potential function. We consider here the consequences of this work for the scaling relation. We show that by dimensionalizing the lateral force balance equation for the S-T model one finds that a single parameter enters as a *characteristic value* of the solution. This parameter yields Skumanich's scaling directly. Using an observed universal flux-radius relation for dark solar magnetic features (spots and pores) for comparison, we find good to fair agreement with Yun's *characteristic value*, however the Moon et al. values deviate significantly.

Key words : Sun: sunspot models —Sun: magnetic fields—Sun: scaling law

I. INTRODUCTION

In a consideration of the physical implications of sunspot models the author found that their mean magnetic field, scaled by square-root of the photospheric pressure, was essentially a constant whether modeled with distributed currents or current sheaths (Skumanich 1992a). This constancy was further supported by an analysis of Stokes vector observations of spots (Skumanich 1999), but with a somewhat different coefficient of proportionality.

Here we reexamine this issue in the context of the more recent distributed current models of Moon et al. (1998). These authors apply the methodology of Yun (1968, 1970, 1971, 1972) but use an apparently more physically representative vector potential. The parameters of this self-similar potential are obtained from a fit to the radial distribution of observed vertical fields, scaled by axial field and penumbral radius, but corrected for the Wilson depression to a common physical depth. It is commonly accepted that the indicated scaling of the observed fields yields a near self-similar shape function. The use of a self-similar potential allows one to reduce the pointwise lateral force balance to one holding solely on the axis of the spot.

In §2 we review and discuss the Yun and Moon et al. (hereafter Y&M) basic equations and boundary conditions. In §3 we derive a non-dimensional form of their lateral magnetic force balance equation and show that the mean spot field appears as a *characteristic value* of the solution. In §4 we compare such values, normalized by the square-root of the quiet sun photospheric pressure, with the observed scaling values obtained by Skumanich (1992a, 1999). Finally in §5 we discuss the results and suggest changes in the boundary conditions that may improve the S-T models. We conclude in §6 with a comment on model validation.

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II. BASIC EQUATIONS

(a) Magnetic Potential and Field

Consider a cylindrically symmetric spot with a vector potential $\boldsymbol{U}(z,r) = (1/r)u(z,r)\boldsymbol{e}_\phi$ where z is *downward*, r radial and $\boldsymbol{e}_\phi$ the unit azimuthal vector. The magnetic field is given by $\boldsymbol{B} = \nabla \times \boldsymbol{U}$. To obtain a 1-D problem assume the Schlüter- Temesvary similarity transformation $z \to z, \quad r \to x = \xi(z)\, r$ where x is the similarity variable and ξ is the radial scaling factor. For given spot flux Φ, Moon et al. take

$$u(z,r) \equiv (\Phi/2\pi) \int_0^r \exp[-(\xi(z)r)^n\,]rdr \Big/ \int_0^\infty \exp[-(\xi(z)r)^n\,]rdr. \quad (1)$$

Note that $u(z,r) \to \tilde{u}(x)$ so that $x = constant$ defines a field line. Since $B_z\,(z,r) = (1/r)(\partial u(z,r)/\partial r)_z$ one finds that

$$\begin{aligned} B_z(z,r) &= (\Phi/\pi N_n)\, \xi(z)^2 \, \exp[-(\xi(z)r)^n\,] \quad (2) \\ &\equiv B_z(z,0)\, \exp[-(\xi(z)r)^n\,], \end{aligned}$$

where $N_n = 2\int_0^\infty \exp[-x^n\,]xdx$. In the case of B_r we use, rather than $\nabla \times \boldsymbol{U}$,

$$\begin{aligned} B_r(z,r) &= B_z(z,r)\, \tan\phi \quad (3) \\ &= -(\Phi/\pi N_n)\, \xi'(z)\, \xi(z) r\, \exp[-(\xi(z)r)^n\,], \end{aligned}$$

where the field line tangent is

$$\begin{aligned} \tan\phi &= (\partial r/\partial z)_x = -x\xi'/\xi^2 = -r\xi'/\xi, \quad (4) \\ &= \tan(\pi-\psi) = -\tan\psi, \end{aligned}$$

and ψ is the field line inclination to the *outward* direction. We have expressed the tangent in both (z,x) and (z,r) space as in the latter $\tan\psi$ can be written as $(r/2)\, d\ \ln\ B_z(z,0)/dz$.

(b) Reduced Lateral Force Balance

With an explicit radial dependence for the field components one can derive the axial force balance equation, refer to Yun (1968), which has the form

$$a_n\,\xi\xi'' - b_n\,\xi^4 + 8\pi\Delta P = 0 \quad (5)$$

where

$$\begin{aligned} a_n &= 2(\Phi/\pi N_n)^2 \int_0^\infty \exp[-x^{2n}\,]x dx \\ b_n &= (\Phi/\pi N_n)^2 \\ \Delta P(z\,|\xi) &= P(z,\infty) - P(z,0\,|\xi). \end{aligned}$$

Here $P(z,\infty) = P_{qs}(z)$. ΔP is a functional of the solution ξ via the auxiliary equations for axial temperature (heat flow) and pressure (hydrostatic balance). Eq. (5) differs from that of Y&M in that they use the variable $y(z) = B_z(z,0)^{1/2} = (\Phi/\pi N_n)^{1/2}\xi(z)$ rather than the more physically meaningful scale factor $\xi(z)$, cf. Flå et al. (1982). The case of $n = 2$ yields the Schlüter-Temesvary equation solved by Yun.

(c) Parameter Assignment and Boundary Conditions

To specify the vector potential, i.e. the parameter n, Moon et al. fit the radial shape of the observed vertical field scaled by the axial field and penumbral radius and mapped from optical depth to physical depth at *constant radius*. If $B_z^o(\tilde{r}) = B_z^o(z(\tau(\tilde{r}) = 2/3), \tilde{r})$ is the observed field, where $\tilde{r} = r/R_p$, then one can write

$$\begin{aligned} B_z^o(z(\tau(\tilde{r}) = 2/3), \tilde{r}) &\approx B_z^o(z_0 - \Delta Z(\tilde{r}), \tilde{r}) \\ &\approx B_z^o(z_0, \tilde{r}) - (\partial B_z/\partial z)_0\, \Delta Z(\tilde{r}), \end{aligned}$$

with $z_0 = z(\tau(0) = 2/3, 0), \Delta Z(\tilde{r}) = D(0) - D(\tilde{r})$, and D, the Wilson depression. Both the gradient and D are not treated self-consistently but are assigned reasonable observable values. One fits $B_z^o(z_0,\tilde{r})/B_z^o(z_0,0)$ with $\exp[-(\xi(z_0)R_p\,\tilde{r})^n\,]$ to obtain n and the value $x_p = \xi(z_0)R_p$. Note that the latter defines the field line passing through the penumbral radius projected down to z_0. This field line defines R_p if the surface value of $\xi(z_0)$ is known.

For a reasonable linear gradient and an assumed radial variation for D, Moon et al. find $n = 1.7$ and $x_p = 1.39$ (case 5). In this case the flux contained within R_p $\Phi_p = \Phi(n, x_p) = .67\ \Phi$, an overly low value compared to observed values and raises a question about the meaning of R_p. For the case $n = 2$, $D = 0$, Yun determined x_p by equating $\Phi(n = 2, x_p)$ with an empirical flux scaling value for spots (see §5) and found $x_p = 1.63$. Here $\Phi_p = .93\ \Phi$. This correctly defines the penumbral flux line. It is important to note that a fit to the *shape* of $B_z^o(z_0,\tilde{r})$ with $n = 2$, $D = 0$ yields $x_p = 1.8$, Skumanich (1992b).

For a two point boundary value problem as we have here one must specify ξ at the surface and at depth. In this case the derivative ξ' becomes a *characteristic value* of the solution, i.e. a generalized eigenvalue. It is this parameter that must be adjusted to its *characteristic value* if the solution by a Runge-Kutta or "trajectory" solution method is to satisfy the lower boundary condition. However Y&M not only assign ξ at the surface and at depth but also assign ξ_0' at the spot surface z_0 and use R_p as the *characteristic value*. With Φ and x_p given they assign the field inclination ψ_p at the penumbral radius, i.e. for the field line given by x_p. This specifies ξ_0'. Note that the scaled surface gradient, $G = R_p\,(d\ \ln\ B_z(z,0)/dz)_0 = 2\tan\psi_p$, is being specified. Thus at the surface

$$\begin{aligned} z_0 : R_p \rightarrow \xi_0 &= x_p/R_p \\ \xi_0' &= x_p\ \tan\ \psi_p\ /R_p^2, \end{aligned}$$

while at depth they impose a constant asymptotic field

$$z \rightarrow \infty : \quad \xi \rightarrow constant.$$

The asymptotic condition was imposed at a selected lower boundary which was adjusted downward until the photospheric solution changed by $\leq 1\%$. This error also applies to R_p.

The lower boundary conditions for the auxiliary equations follow from applying the asymptotic field condition to the derivative of Eq. (5). The result is that, for sufficient depth,

$$\rho(z, r=0) \ \rightarrow\ \rho(z,\infty) = \rho_{qs}(z),$$

where $\rho_{qs}(z)$ is the quiet sun density. With $\rho(z, r = 0)$ known the asymptotic value for the pressure gradient for the axial hydrostatic equation is known. The asymptotic value for the pressure is obtained from applying Eq. (5) with $\xi'' = 0$. The surface pressure, $P(z_0, 0)$, to be fit is given by a $P(T_{\rm eff})$ relation from atmospheric models. This relation may be represented by the requirement that $P(z_0) \cong \frac{2}{3}\, g/\kappa(P(z_0), T_{\rm eff})$ [1] for the given $T_{\rm eff}$. Here κ is the Rosseland opacity. The asymptotic boundary condition for the heat equation follows from applying the equation of state to the pressure and density while the heat flux is fixed by $T_{\rm eff}$.

[1] Note that this equation is based on Vitense's (1951b) and Bohm-Vitense's (1958) models and Vitense's (1951a) opacities. Yun's opacities are a factor of 2 smaller. Such differences in the surface boundry condition are probably not significant.

The mixing length becomes the *characteristic value* for the auxiliary system of equations.

Note that the asymptotic constant flux tube size is problematic as the difference in temperatures between interior and exterior drops rapidly since the density rises rapidly inwards. This can only be due to lateral radiative heat exchange which is very inefficient in the convection zone. In addition the plasma beta becomes large and thus the exterior fluid dominates the nature and evolution of the flux tube. It is unlikely that the constant cross-section with its attendant small presure difference will be maintained.

III. NORMALIZED LATERAL FORCE BALANCE

To obtain the non-dimensional form of the 1-D force balance equation we scale both spatial coordinates by the penumbral radius R_p and the pressure by the quiet sun photospheric pressure $P_{qs}(z=0)$. This yields the variables $\zeta = z/R_p, \eta = R_p\,\xi$, and $\Delta\tilde{P} = \Delta P/P_{qs}$ The lateral force balance equation becomes

$$A_n\,\eta(\zeta)\,\eta''(\zeta) - \eta^4(\zeta) + \beta\,\Delta\tilde{P}(\zeta|\,\eta) = 0 \qquad (6)$$

where

$$\beta = 8\pi\,P_{qs}\,/\,(\Phi/\pi R_p^2 N_n)^2\,,$$
$$A_n = 2\int_0^\infty \exp[-2\,x^n\,]x dx = O(1/2).$$

The parameter β (or alternatively, the mean field $\Phi/(\pi R_p^2 N_n)$) now replaces R_p as the *characteristic value* for a solution. It is directly the scaling parameter considered by Skumanich. Here the boundary conditions become known constants with

$$\zeta_0 \;:\; \eta_0 = x_p \textit{ and } \eta_0' = x_p\,\tan\psi_p \qquad (7)$$
$$\zeta\to\infty \;:\; \eta \to \textit{constant} \qquad (8)$$

We note that rather than using a Runge-Kutta method to solve all of the equations and iterating between the magnetic force equation and the auxiliary equations as Yun does one could use finite element representation of the variables with the boundary conditions incorporated in the representation and a Newton-Raphson linearization technique. This would be similar to the Henyey method used in stellar interiors.

IV. CHARACTERISTIC VALUES

We now consider the solutions obtained individually by Y&M restricting, in the case of Moon et al., our attention to case (5) as being the most physically realistic situation. Only two flux values are presented by Moon et al. for this case. We note that two of Yun's studies are directly comparable as differences in the assigned parameters between them are not significant. The associated $\sqrt{\beta}$ values are listed in Table 1.

Table 1. Characteristic value of $\sqrt{\beta}$

\\Φ $T_{\rm eff}$	4×10^{21} Y	5.6×10^{21} M	2.2×10^{22} Y	2.4×10^{22} M
4000	–	–	0.96	0.82
4400	1.20	1.02	–	–
	n	x_p	ψ_p	
M	1.70	1.39	67°.0	
Y	2.00	1.63	67°.2	

The value of $P_{qs} = 0.830\times10^5$ dyn cm^{-2} is taken from Yun (1968). One finds that $\sqrt{\beta}\approx 1$ in agreement with the scaling proposed by Skumanich (1992a). However using the more recent observationally derived square-root relation between penumbral radius and flux, Skumanich (1999), where a characteristic value of $B = 870$ G appears, one finds that $\sqrt{\beta_{\rm obs}} = 1.65$. Thus both calculations fall below the observed value especially at high flux values with Moon et al. deviating the most. It appears that the characteristic penumbral radii are too small for the given flux.

A more direct comparison is shown in Figure 1 where Skumanich's radius-flux relation is illustrated. For a detailed explanation of the data refer to Skumanich (1999). Here we have included the Yun solutions (with $\psi_p =$ 67°.2) for $T_{\rm eff} = 5000$ K plotted as '+' while his 4400 K solution is plotted as a □. Moon et al. result for 4400 K is shown by the ◇. The Yun R_p values appear to fit the data at 5000 K & only tolerably at 4400 *K* whereas the Moon et al. value is deviant. At $T = 4000$ K both authors R_p deviate even more significantly, cf. Table 1.

V. DISCUSSION

A consideration of Yun's solutions for different fluxes Φ at a given $T_{\rm eff}$ shows that R_p varies essentially as the square-root of Φ but with a coefficient of proportionality dependent on $T_{\rm eff}$. Thus $\sqrt{\beta}$, is essentially independent of Φ, for fixed ψ_p, but falls below the observations as the temperature falls below 5000 K. This may indicate that the model of the heat flow in a magnetic field is not well modeled at lower temperatures.

One might also ask whether the correct surface boundary condition is being used. Is it self-consistent to fix ψ_p, i.e. η_0', arbitrarily? Should one, as in the case of the surface value of the spot pressure, use the solution of lateral force balance in the atmosphere above ζ_0 to assign a consistent value to η_0'?

Consider the potential field solution for a given distribution of flux at ζ_0. It is easy to show that for the $n=2$ Yun case with $\Delta P(\zeta) = 0$, the potential solution

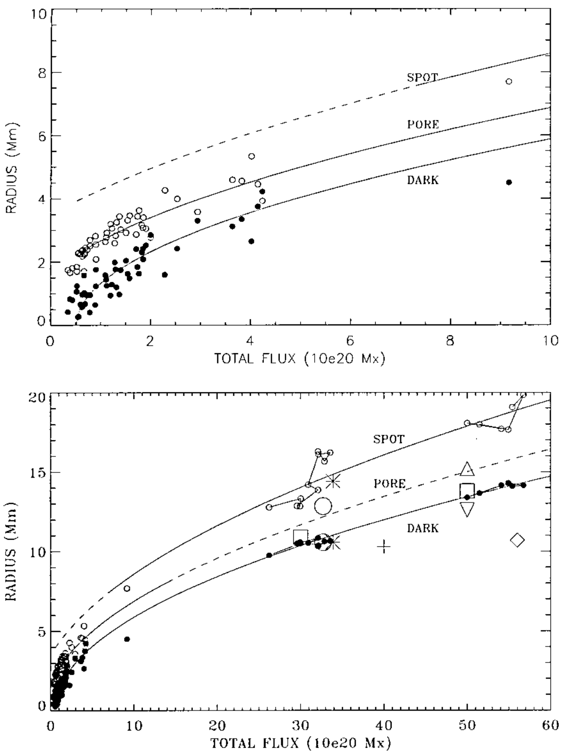

Fig. 1.— Comparison of S-T model penumbral radii with the Skumanich radius-flux (R, Φ) relation. The curve marked "DARK" is the dark radius square-root relation for pores and spots (penumbra) ; those marked "SPOT" and "PORE" are the relations for the magnetic radius which contains all the observed flux. Yun's radius values for 5000 K are given by the $\square$, for 4400 K, by the $+$. Moon et al. value for 4400 K is given by $\diamond$. Deinzer's value for 5000 K is given by the $\triangle$, for 4400 K, by the ∇.

in the upper half-space, $\zeta \leq \zeta_0$, is

$$\eta(\zeta) = x_p/(1 + x_p\,(\zeta_0 - \zeta)) \tag{9}$$

This yields the condition that $\eta'/\eta^2 = 1$ which requires $\tan\psi(x) = x$. In the Yun case then $\tan\psi_p = x_p = 1.63$, or $\psi_p \doteq 58°$, i.e. the potential flux tube at the spot photosphere is more vertical than Yun's stressed field with its assumed $\psi_p = 67.2°$. A stressed field at this level should have a smaller spread of its field lines than the potential solution, i.e. it should be more vertical and penetrate higher (smaller η'_0).

Deinzer (1965), in the very first study of the similarity S-T model, applies just such a potential condition to $y'(z_0)$ to obtain a solution of the y version of Eq. 5. Using the same asymptotic condition and basic auxiliary equations and boundary conditions[2] as Yun, he finds that his characteristic value $y_0 = y(z_0)$ is independent of Φ, for the range $5 - 10 \times 10^{21}$ Mx, but varies with $T_{\rm eff}$. Since $y_0^2 = (\Phi/\pi)\xi_0^2$ we can introduce Yun's penumbral field line radius here to obtain

$$R_p = (x_p/y_0)\sqrt{\Phi/\pi},$$

i.e. we find, as in the Y&M case, a square-root-flux scaling for R_p. Using Yun's $x_p = 1.63$ and $\Phi = 5 \times 10^{21}$ Mx, we plot the resulting R_p values for Deinzer's solutions in Figure 1. The value at $T_{\rm eff} = 5000$ K is indicated by the $\triangle$ and at 4400 K, by a ∇. It appears that using a potential upper boundary condition increases the penumbral radii compared to the Yun solutions. However this is at the expense of a penumbral inclination which is lower than observed. Note that the differential change with $T_{\rm eff}$ appears to be the same for both calculations.

We now consider a stressed exterior solution. Flå et al. solve Eq. (5) in the upper half-space with $n = 2, \Phi = 2.4 \times 10^{21}$ Mx, $x_p = 1.63$, and $R_p = 10$ Mm. In this case $\xi'(z_0)$ becomes the characteristic value. Their $\Delta P(z\,|D)$ was obtained by assigning a Wilson depression D to an empirical sunspot model relative to an empirical quiet-sun model with a convective subphotosphere appended. The upper boundary condition is that the field becomes potential asymptotically, i.e. ξ varies as $-z^{-1}$. They use $\beta_{\rm eff} = 8\pi\ \Delta P(z_0\,|D)/B_z(z_0,0)^2$ to characterize their models. $B_z(z_0, 0)$ is obtained from the empirical flux relation $\Phi_p = 0.35\pi\ B_z(z_0,0)\ R_p^2 = 0.93\ \Phi$. Here $\beta_{\rm eff}(D = .5\text{ Mm}) = 3.47$. They find the characteristic value $\tan\psi_p = 1.1$ at z_0 which is less than that for the associated potential field, i.e. the field is stressed compared to the associated potential field. In addition a magnetic boundary layer is found where the field deviates the most from the potential solution and which has a thickness of $z_b \approx 0.05\ R_p = D$, i.e. the Wilson depression. Above this layer, i.e. at and above the quiet-sun photosphere, the field is essentially potential with $\tan\psi_p \cong x_p = 1.63$. Since observed inclinations at the penumbral boundary are larger than 58° it would appear that something is missing. It may be that for larger fluxes the upper-space solutions will be different from the one Flå et al. (1982) discussed. It would be interesting to see if their exterior solution applies to the interior solution for large pores. Even the larger $x_p = 1.80$, or $\psi_p = 61°$, obtained by Skumanich for a shape fit to the observed field profiles, may not be quite adequate.

Finally we consider the scaling of spot solutions with $\beta_{\rm eff}$. Although Flå et al. (1982) consider a return flux version of Eq. (5) in the upper half-space, it is instructive to consider the scaling they found for the absolute value of the scaled logarithmic gradient $|G|$. They found that, for $0.5 \leq \beta_{\rm eff} \leq 6.5$,

$$|G| = |G|_0 - b\,\beta_{\rm eff}, \tag{10}$$

where $|G|_0$ is the associated potential gradient and b, a constant (< 1) that may depend only on the flux

[2]The differences in $P(T_{\rm eff})$ are not significant.

parameters. Flå et al. (1982) demonstrate that this scaling represents an underlying "global" force balance. We hold that it should also apply to the exterior solutions for the current S-T cases. This would represent a condition on $\tan \psi_p$.

It is of interest to ask if such a "global" force balance also applies to the interior solutions. Consider $\beta_{\rm eff}$ for the Y&M solutions discussed here. One finds that $\beta_{\rm eff}$ is approximately a constant[3] with an average value of 2.22. Since $\tan \psi_p$ is an imposed constant then $G = 2 \tan \psi_p$ is constant and it must follow from Eq. (10), if a similar form applies, that $\beta_{\rm eff}$ must be constant, as is approximately the case.

We suggest that Eq. (10), calibrated for the Y&M flux parameters, allows one to include the effects of an exterior asymptotic magnetic boundary condition on the interior solution. It represents the equivalent of the $P(T_{\rm eff})$ condition on the hydrostatic equation.

It would appear that the extant S-T models are in reasonable agreement, for the most part, with spot size data. The issue of an improved penumbral inclination may need further study of the effect of the upper magnetic boundary condition. In addition the modeling of heat transfer in a magnetic flux tube may require a better treatment. For example one might consider a mixing-length that depends on some function of the field strength, such as a variation given by $\eta(\zeta_0)/\eta(\zeta)$, with the value at ζ_0 as the characteristic value.

VI. MODEL-OBSERVATIONS VALIDATION

In comparing[4] models of *complex* phenomena with *finite* bandwidth measurements we must do so with a sense of evenhandedness. Models try to capture the essence of a complex problem by a form of reductionism with a subsequent reaggregation in which inevitable fudge factors appear. On the other hand designed, controlled, high resolution experiments are, on the whole, unreachable for solar physics situations. Thus in comparing models and data we must remember the limitations of both.

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[3]A weak $T_{\rm eff}$ variation with a range of 16% is present.

[4]The following is a paraphrase from "Science, Uncertainty and Risk: The Problem of Complex Phenomena." R. L Wagner, Jr., Amer Phys Soc News 2003

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