Journal of The Korean Astronomical Society
36: 75 ~ 80, 2003

DERIVATION OF THE GRAVITATIONAL MULTI-LENS EQUATION FROM THE LINEAR APPROXIMATION OF EINSTEIN FIELD EQUATION

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(Received June 24, 2003; Accepted July 22, 2003)

ABSTRACT

When a bright astronomical object (source) is gravitationally lensed by a foreground mass (lens), its image appears to be located at different positions. The lens equation describes the relations between the locations of the lens, source, and images. The lens equation used for the description of the lensing behavior caused by a lens system composed of multiple masses has a form with a linear combination of the individual single lens equations. In this paper, we examine the validity of the linear nature of the multi-lens equation based on the general relativistic point of view.
Key words: gravitation – gravitational lensing

I. INTRODUCTION

The deflection angle of a light ray bent by a point mass lens can be calculated by using the general theory of relativity. Let D_{so} and D_{lo} are the line of sight distances to an astronomical source (marked by 's') and a foreground point mass (lens, marked by 'l'), respectively. Let us consider two planes perpendicular to the line of sight towards the source where one plane is located at the lens distance (lens plane) and the other is located at the source distance (source plane). If the angular position of the lens on the sky is (x_m, y_m), the two components of the projected lens position in physical units on the lens and source planes are, respectively,

$$\begin{aligned} X_M &= x_m D_{lo}, \qquad Y_M = y_m D_{lo}, \\ X_{M_s} &= x_m D_{so}, \qquad Y_{M_s} = y_m D_{so}. \end{aligned} \tag{1}$$

Assume that an observer is looking at a light ray from a lensed source whose apparent angular position on the sky is (x, y). The extension of the line of sight towards this position intersects the lens plane at a point A and the source plane at another point I (corresponding to the position of the image). Then the components of the projected positions of the points A and I on the lens and source planes are, respectively,

$$\begin{aligned} X_A &= x D_{lo}, \qquad Y_A = y D_{lo}, \\ X_I &= x D_{so}, \qquad Y_I = y D_{so}. \end{aligned} \tag{2}$$

From eq. (1) and (2), the projected separation between the lens and the ray on the lens plane is computed by

$$R = [(X_A - X_M)^2 + (Y_A - Y_M)^2]^{1/2}. \tag{3}$$

The general relativity says the light ray is deflected by an angle of

$$\alpha = \frac{4GM}{Rc^2}, \tag{4}$$

with two components of

$$\alpha_x = \alpha \frac{(X_A - X_M)}{R}, \qquad \alpha_y = \alpha \frac{(Y_A - Y_M)}{R}. \tag{5}$$

In physical units, the above relations become

$$X_s = X_I - \alpha_x (D_{so} - D_{lo}), \qquad Y_s = Y_I - \alpha_y (D_{so} - D_{lo}), \tag{6}$$

which is so called the 'lens equation' for a single point mass.

II. MULTI-LENS EQUATION

If a lens system is composed of multiple point masses (multi-lens system), the individual masses influence the bending angle of light. Let us assume that multiple number of point-mass lenses are located at a same distance D_{lo} in the front of a point source at D_{so}. Each lens component has a mass M_i and it is located at (X_i, Y_i) on the lens plane. Then the separation between the ray and each lens component is

$$R_i = [(X - X_i)^2 + (Y - Y_i)^2]^{1/2}. \tag{7}$$

Assuming that each mass contributes to the blending of light with an amount of

$$\alpha_i = \frac{4GM_i}{c^2 R_i}, \tag{8}$$

with the two components of

$$\alpha_{x,i} = \alpha_i \frac{X - X_i}{R_i}, \qquad \alpha_{y,i} = \alpha_i \frac{Y - Y_i}{R_i}, \tag{9}$$

Paczyński (1986) expressed the lens equation for a multi-lens system as a linear combination of individual single lens equations, i.e.

$$X_s = X\frac{D_{so}}{D_{lo}} - \sum_i \alpha_{x,i}(D_{so} - D_{lo}), \qquad Y_s = Y\frac{D_{so}}{D_{lo}} - \sum_i \alpha_{y,i}(D_{so} - D_{lo}), \tag{10}$$

which is called as the 'multi-lens equation'.

The deflecting angle α of the light ray by a point mass, eq. (4), is calculated based on relativistic consideration about the curved space-time caused by the mass. Similarly, the deflecting angle of the light ray caused by a multi-lens system should be also based on the relativistic equation of motion of the light wave in the curved space-time. However, in the multi-lens equation eq. (10), the deflecting angle α is calculated as if the bending angles by the individual masses behave as if they are vectors in flat space-time so that the total deflecting angle can be calculated by a vector sum in the flat space-time. As a result, it is conceptionally confusing whether the multi-lens equation in eq. (10) is based on relativistic consideration. We, therefore, examine the validity of the linear nature of the multi-lens equatin by computing the relativistic equation of motion in a weak gravitational field.

III. DERIVATION OF MULTI-LENS EQUATION BY THE GENERAL RELATIVISTIC CALCULATION

The linear approximation of the field equation for gravitation can be written as (Iranian & Ruffing 1994)

$$\partial_\lambda \partial^\lambda \phi^{\mu\nu} = -\kappa T^{\mu\nu}, \tag{11}$$

with the Gauge condition of

$$\partial_\mu \phi^{\mu\nu} = 0, \tag{12}$$

where κ is a constant and the Greek indexes represent μ, ν = 0, 1, 2, 3, respectively. The adopted summation convention for the repeating indexes and the tensor fields $\phi^{\mu\nu}$ are related to the metric tensor $g_{\mu\nu}$ as follows. In the weak gravitational field, the metric tensor can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{13}$$

where $\eta_{\mu\nu}$ is the Minkowski tensor (the metric tensor of flat space time) and $h_{\mu\nu}$ is the weak gravitational field which can be written in the terms of $\phi^{\mu\nu}$ as

$$h^{\mu\nu} = \phi^{\mu\nu} + \eta^{\mu\nu}\phi, \tag{14}$$

where $\phi = \phi^\mu_\mu$.

In eq. (11) the energy-momentum tensor $T^{\mu\nu}$ is the total energy-momentum tensor of matter plus gravitation, i.e.

$$T^{\mu\nu} = T^{\mu\nu}_{(m)} + t^{\mu\nu}, \tag{15}$$

where $T^{\mu\nu}_{(m)}$ and $t^{\mu\nu}$ are the energy-momentum tensors of matter and gravitation, respectively. The interaction energy of matter and gravitation is always included in $T^{\mu\nu}_{(m)}$ since the interaction energy density is non-zero only at the place where there is matter. The energy-momentum tensor $t^{\mu\nu}$ of gravitational field is quadratic in the field variables. Then, in the weak gravitational field, $t^{\mu\nu}$ can be ignored since the order of $\phi^{\mu\nu}$ is much less than unity.

Consider the time-independent field equation for the weak gravitational field surrounding a discrete mass distribution, where the individual point masses are M_i. In the exterior of each mass, $T^{\mu\nu}_{(m)} = 0$, and thus we can write the field equation and the gauge condition as

$$\partial^{\lambda}_{\lambda}\phi^{\mu\nu} = 0, \tag{16}$$

$$\partial_{\mu}\phi^{\mu\nu} = 0. \tag{17}$$

With the assumption that the field is time independent, eq. (16) reduces into a form of

$$-\nabla^2\phi^{\mu\nu} = 0. \tag{18}$$

Let the position of each mass is located at $\boldsymbol{r}_i = X_i\boldsymbol{i} + Y_i\boldsymbol{j}$ in the lens plane. Then the solution of the equation for each mass is

$$\phi^{\mu\nu} = \sum_i \frac{C_i^{\mu\nu}}{R_i}, \tag{19}$$

where $C_i^{\mu\nu}$ is a constant matrix and $R_i = \sqrt{(X-X_i)^2 + (Y-Y_i)^2 + Z^2}$ is the distance from the point mass to the field point $\boldsymbol{r} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$. One may include a constant term in the solution eq. (19), but since we assume that $\phi^{\mu\nu} \to 0$ as $r \to \infty$, there is no such a term. From eq. (19) and Eq.(16), one finds

$$\sum_i C_i^{\mu\nu}\partial_k\left(\frac{1}{R_i}\right) = 0,$$

or

$$\sum_i \left[C_i^{k\nu}\frac{-(x^k - x_i^k)}{R_i^3}\right] = 0, \tag{20}$$

where $k = 1, 2, 3$ and $x^1 = X$, $x^2 = Y$, $x^3 = Z$, respectively. This expression is valid for all $(x^k - x_i^k)$ as long as $C_i^{k\nu} = 0$. Hence the only nonzero component of $C_i^{\mu\nu}$ is C_i^{00}.

From eq. (14), it follows that

$$h_{00} = \phi_{00} - \frac{1}{2}\eta_{00}\phi = \sum_i \frac{C_{00(i)}}{R_i} - \frac{1}{2}\sum_i \frac{C_{00(i)}}{R_i}. \tag{21}$$

This determines the value of C_i^{00} because h_{00} is proportional to the Newtonian potential in the weak gravitational field and its value is

$$h_{00} = \frac{2}{\kappa}\left(-\sum_i \frac{GM_i}{R_i}\right). \tag{22}$$

Hence $C_{00(i)} = -4GM_i/\kappa$ and

$$\phi^{\mu\nu} = \begin{pmatrix} -\frac{4G}{\kappa}\sum_i \frac{M_i}{R_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{23}$$

The corresponding expression for $h_{\mu\nu}$ is

$$h_{\mu\nu} = \begin{pmatrix} -\frac{2G}{\kappa}\sum_i \frac{M_i}{R_i} & 0 & 0 & 0 \\ 0 & -\frac{2G}{\kappa}\sum_i \frac{M_i}{R_i} & 0 & 0 \\ 0 & 0 & -\frac{2G}{\kappa}\sum_i \frac{M_i}{R_i} & 0 \\ 0 & 0 & 0 & -\frac{2G}{\kappa}\sum_i \frac{M_i}{R_i} \end{pmatrix}. \tag{24}$$

Note that since the Newtonian potential of the mass distribution M_i is $\Phi = -\sum_i GM_i/R_i$, eq. (24) can also be written as

$$h_{\mu\nu} = \begin{pmatrix} \frac{2\Phi}{\kappa} & 0 & 0 & 0 \\ 0 & \frac{2\Phi}{\kappa} & 0 & 0 \\ 0 & 0 & \frac{2\Phi}{\kappa} & 0 \\ 0 & 0 & 0 & \frac{2\Phi}{\kappa} \end{pmatrix}. \tag{25}$$

Substituting eq. (25) into eq. (13) yields the weak field approximation of the metric tensor,

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu} \\ &= \begin{pmatrix} 1+2\Phi & 0 & 0 & 0 \\ 0 & -(1-2\Phi) & 0 & 0 \\ 0 & 0 & -(1-2\Phi) & 0 \\ 0 & 0 & 0 & -(1-2\Phi) \end{pmatrix}, \end{aligned} \tag{26}$$

Now we consider the equation of motion for a particle in the space-time where the metric is given in eq. (26). The equation of motion for a particle in a weak gravitational field is

$$\frac{du_\mu}{d\tau} + (\kappa h_{\mu\alpha,\beta}u^\alpha u^\beta - \frac{\kappa}{2}h_{\alpha\beta,\mu}u^\alpha u^\beta) = 0, \tag{27}$$

where u_μ is the four-velocity vector and $d\tau$ is the proper time interval. Note that according to eq. (27), the motion of a particle in a gravitational field is independent of the mass of the particle. To find out how light propagates in a gravitational field, we adopt the view that light is composed of particles (photons). This permits us to use the relativistic equation of motion for a particle in the gravitational field [see eq. (27)]. However, since the mass of the photon is zero, and thus one must rewrite the equation of motion into a different form.

Multiplication of $md\tau$ into eq. (27) yields

$$d(mu_\mu) + (\kappa h_{\mu\alpha,\beta}mu^\alpha u^\beta d\tau - \frac{\kappa}{2}h_{\alpha\beta,\mu}mu^\alpha u^\beta d\tau) = 0. \tag{28}$$

By introducing $p_\mu = mu_\mu$ and $dx^\beta = u^\beta d\tau$, the above equation becomes

$$dp_\mu + \left(\kappa h_{\mu\alpha,\beta}p^\alpha - \frac{\kappa}{2}h_{\alpha\beta,\mu}p^\alpha\right)dx^\beta = 0. \tag{29}$$

In this equation, neither m nor $d\tau$ appears explicitly. Hence, the equation can be applied to a massless photon. This tells us how the photon momentum is changed by the gravitational field, enabling us to calculate the deflection of a light ray passing through the gravitational field. For such a calculation, we can treat the factor p^α in the second term of eq. (29) as a constant because any non-constant portion in p^α is of order κ, and thus would make a negligible contribution. The net change in p_μ produced during the photon's passage through the gravitational field is, therefore, given as a simple integral

$$\Delta p_\mu = -\kappa p^\alpha \int_{-\infty}^{\infty}\left(h_{\mu\alpha,\beta} - \frac{1}{2}h_{\alpha\beta,\mu}\right)dx^\beta. \tag{30}$$

Here it is assumed that the integration is performed along a straight trajectory under the assumption that the light is deflected by a small angle. The first term on the right side of eq. (30) can be omitted since $\int h_{\mu\alpha,\beta}dx^\beta = h_{\mu\alpha}(\infty) - h_{\mu\alpha}(-\infty)$, which is zero because the field $h_{\mu\alpha}$ is zero at an infinite distance from the gravitating mass. Then the expression for the momentum change can be written into a more compact form of

$$\Delta p_\mu = -\frac{\kappa}{2}p^\alpha \int_{-\infty}^{\infty} h_{\alpha\beta,\mu}dx^\beta. \tag{31}$$

Let us consider a ray of light passing a point $\boldsymbol{r}_0 = X_0\boldsymbol{i} + Y_0\boldsymbol{j}$ on the lens plane. Taking the z-axis along the direction of the light incidence, we see that the displacement along the light trajectory and the momentum vector are, respectively,

$$\begin{aligned} dx^\beta &\simeq (dt, 0, 0, dz) = (dz, 0, 0, dz), & (32)\\ p^\alpha &\simeq (p^3, 0, 0, p^3). & (33) \end{aligned}$$

Here we adopt natural units, where $c = 1$. Inserting eq. (32) and (33) into eq. (31), we obtain

$$\triangle p_0 = 0, \tag{34}$$

since $h_{\alpha\beta}$ is time independent and

$$\triangle p_k = \frac{\kappa}{2} p^3 \int_{-\infty}^{\infty} (h_{00,k} + h_{03,k} + h_{30,k} + h_{33,k}) dz. \tag{35}$$

For a weak gravitational field, the terms h_{03} and h_{30} vanish and $h_{00} = h_{33} = 2\Phi/\kappa$, where Φ is the Newtonian gravitational potential, cf. eq. (25). For a multi-lens system, the Newtonian potential has a form

$$\Phi = -\sum_i \frac{GM_i}{\sqrt{(X - X_i)^2 + (Y - Y_i)^2 + Z^2}}. \tag{36}$$

Hence eq. (35) can be rewritten as

$$\begin{aligned} \triangle p_k &= \frac{\kappa}{2} p^3 \int_{-\infty}^{\infty} (h_{00,k} + h_{33,k}) dz \\ &= -2p^3 \int_{-\infty}^{\infty} \frac{\partial \Phi}{\partial x^k}\Big|_{\mathbf{r}_0} dZ, \qquad \text{for } k = 1,\ 2. \end{aligned} \tag{37}$$

One find that $\triangle p_3 = 0$ and

$$\begin{aligned} \frac{\partial \Phi}{\partial x}\Big|_{\mathbf{r}_0} &= -\sum_i \frac{GM_i(X_0 - X_i)}{\Big((X_0 - X_i)^2 + (Y_0 - Y_i)^2 + Z^2\Big)^{3/2}}, \\ \frac{\partial \Phi}{\partial y}\Big|_{\mathbf{r}_0} &= -\sum_i \frac{GM_i(Y_0 - Y_i)}{\Big((X_0 - X_i)^2 + (Y_0 - Y_i)^2 + Z^2\Big)^{3/2}}. \end{aligned} \tag{38}$$

From eq. (38), the components of the photon momentum can be written as

$$\begin{aligned} \triangle p_x &= -p_1 = 2p^3 \int_{-\infty}^{\infty} \Big[\sum_i \frac{GM_i(X_0 - X_i)}{\Big((X_0 - X_i)^2 + (Y_0 - Y_i)^2 + Z^2\Big)^{3/2}}\Big] dZ \\ &= -4p_z \Big[\sum_i \frac{GM_i(X_0 - X_i)}{(X_0 - X_i)^2 + (Y_0 - Y_i)^2}\Big], & (39)\\ \triangle p_y &= -p_2 = 2p^3 \int_{-\infty}^{\infty} \Big[\sum_i \frac{GM_i(Y_0 - Y_i)}{\Big((X_0 - X_i)^2 + (Y_0 - Y_i)^2 + Z^2\Big)^{3/2}}\Big] dZ \\ &= -4p_z \Big[\sum_i \frac{GM_i(Y_0 - Y_i)}{(X_0 - X_i)^2 + (Y_0 - Y_i)^2}\Big]. & (40) \end{aligned}$$

From these expressions, one can compute the deflection angle caused by a multi-lens plane,

$$\theta_x = \frac{\triangle p_x}{p_z} = -\Big[\frac{4G}{c^2} \sum_i \frac{M_i(X_0 - X_i)}{(X_0 - X_i)^2 + (Y_0 - Y_i)^2}\Big], \tag{41}$$

and

$$\theta_y = \frac{\Delta p_y}{p_z} = -\left[\frac{4G}{c^2}\sum_i \frac{M_i(Y_0 - Y_i)}{(X_0 - X_i)^2 + (Y_0 - Y_i)^2}\right]. \tag{42}$$

Substituting eq. (41) and (42) into the eq. (6), one finds the expression for a multi-lens system,

$$X_s = X_0\frac{D_{so}}{D_{lo}} - \theta_x(D_{so} - D_{lo}), \tag{43}$$

$$Y_s = Y_0\frac{D_{so}}{D_{lo}} - \theta_y(D_{so} - D_{lo}). \tag{44}$$

In terms of physical separations, the lens equation becomes

$$\begin{aligned} X_s &= X_0\frac{D_{so}}{D_{lo}} - \theta_x(D_{so} - D_{lo}) \\ &= X_0\frac{D_{so}}{D_{lo}} - (D_{so} - D_{lo})\frac{4G}{c^2}\sum_i \frac{M_i(X_0 - X_i)}{(X_0 - X_i)^2 + (Y_0 - Y_i)^2}, \end{aligned} \tag{45}$$

$$\begin{aligned} Y_s &= Y_0\frac{D_{so}}{D_{lo}} - \theta_y(D_{so} - D_{lo}) \\ &= Y_0\frac{D_{so}}{D_{lo}} - (D_{so} - D_{lo})\frac{4G}{c^2}\sum_i \frac{M_i(Y_0 - Y_i)}{(X_0 - X_i)^2 + (Y_0 - Y_i)^2}. \end{aligned} \tag{46}$$

One finds that these expressions are identical to the form in eq. (10).

IV. CONCLUSION

We derive an expression for the deflection angle of the light ray caused by multiple masses by using the relativistic calculation and confirm that the lens equation for a multi-lens system can be written as a linear combination of the single lens equations of the individual lens components.

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