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On the Wave Source Identification of an Wave Maker Problem

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ABSTRACT: *The question of wave source identification in a wave maker problem is the primary objective of the this paper. With the observed wave elevation, the existence of the wave maker velocity is discussed with the help of the mathematical theory of inverse problems. Utilizing the property of the Strum-Liouville system and compactness, the uniqueness and the ill-posedness(in the sense of stability) for the identification are proved.*

1. Introduction

There is extensive literature on the generation of free surface waves due to the wave maker motion. Based on the linearized boundary conditions, perturbation theories have been developed for various types of wave maker velocity (Crapper, 1984; Joo et al., 1990; Miles, 1991; Hocking and Mahdmina, 1991; Chakrabarti, 1994). Without exceptions, they have treated the wave maker problem in a natural way of thinking, which means that they have tried to find the corresponding wave fields with specified wave maker velocity distributions in time or vertical space coordinates. This is the starting point of our question to the wave maker problem. With an observed or measured wave elevation, is it possible to inversely find the wave maker velocity? If so, is it unique? Under what conditions or how can one determine the required velocity distribution?

To resolve these uncommon questions, the so-called wave maker operator is newly introduced by the context of the linear operator theory.

After clarifying the essential concept of the wave maker operator, its ill-poseness is derived from the compactness. Uniqueness of the wave maker velocity has been proved by using a Strum-Liouville system of an eigenvalue problem. When it comes to the topic of uniqueness, we mention that the classical viewpoint in the linearized two-dimensional water wave problems was mainly related with uniqueness (John, 1950; Hulme, 1984; Simon and Ursell, 1984). Compared with these classical papers, the present paper tries to build a functional analytic framework for an inverse wave maker problem. The answer to uniqueness of the problem is also an essential part to be treated with a completely different setting.

An inverse perspective of hydrodynamic problems is also seen in Jang et al. (2000a, 2000b). The former studied a problem similar to ours except that the measurement was done for the velocity on the vertical cut inside the fluid domain and waves were generated by pressure distribution on the free surface. The latter suggested a new mathematical model of an inverse lift problem for hydrofoils with its numerical experiments.

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2. Boundary Value Problem

We consider the wave generation problem in a two-dimensional semi-infinitely long wave tank with constant water depth. Right-going waves are generated in the tank by an oscillating wave maker located at x=0 with a given circular frequency w in a given oscillating mode of small amplitude. The fluid is assumed to be homogeneous, incompressible and inviscid. The wave maker-induced fluid motion is irrotational so that a velocity potential function exists. The wave and the motion of the wave maker are small: the problem considered is assumed linear. The velocity potential function for this simple harmonic motion with a frequency w is expressed by

$$\Phi(x, y, t) = \mathrm{Re}[\varphi(x, y)e^{-i\omega t}] \quad (1)$$

and the elevation H of the free surface by

$$\mathrm{H}(x, t) = \mathrm{Re}[\eta(x)e^{-i\omega t}]\,. \quad (2)$$

Assuming the x-axis to coincide with the free surface in its undisturbed position, with the y-axis positive upward, let an open unbounded domain be

$$\Omega = \{(x,y) \in \mathbf{R}\times\mathbf{R} : -h < y < 0, 0 < x < \infty\}$$

for water depth $h > 0$ with a boundary $\Gamma = \Gamma_F \cup \Gamma_B$:

$$\Gamma_F = \{(x,y) \in \mathbf{R}\times\mathbf{R} : y = 0, 0 < x < \infty\}$$

$$\Gamma_B = \{(x,y) \in \mathbf{R}\times\mathbf{R} : y = -h, 0 < x < \infty\}$$

and $\Gamma_M \subset \Gamma_F$ be an open interval (b,c) of Γ_F with positive real values of b and c.

The complex velocity potential $\varphi(x,y)$ for the wave maker can be described as the boundary value problem:

$$\nabla^2\varphi = 0 \text{ in } \Omega \tag{3}$$

$$\varphi_y - \gamma\varphi = 0 \text{ on } \Gamma_F \tag{4}$$

$$\varphi_y = 0 \text{ on } \Gamma_B \tag{5}$$

$$\varphi_x\big|_{x=0} = f(y) \text{ at } x = 0 \tag{6}$$

$$\varphi \propto e^{ik_0 x} \text{ as } x \to \infty \tag{7}$$

With a given constant $\gamma = \omega^2 / g > 0$ in (4), the wave number $k_0 > 0$ in (7) is determined by the dispersion relation

$$\gamma = k_0 \tanh(k_0 h) \tag{8}$$

where g is the acceleration of gravity. $\varphi(x,y)$ is governed by the Lapace equation (3) and satisfied with the free surface condition (4), the bottom condition (5) and the radiation condition (7). In (6), the wave maker velocity f is imposed at $x = 0$

The free surface wave elevation η on Γ_M is related to the velocity potential φ on the free surface:

$$\eta(x) = \frac{i\omega\varphi}{g} \text{ on } \Gamma_M \tag{9}$$

where ρ is the fluid density.

In order to build the functional analytic framework, necessary definitions and concepts are briefly introduced. Let X be the Hilbert space $L_2(-h,0)$ of all complex-valued functions $x(t)$ defined on $-h \le t \le 0$, for which the Lebesque integral

$$\int_{-h}^{0} |x(t)|^2 \, dt$$

exists. We introduce an inner product $<\cdot,\cdot>_X$ and the corresponding norm $\|\cdot\|_X$ in X : for, $f_1, f_2 \in X$

$$< f_1, f_2 >_X = \int_{-h}^{0} f_1(t)\overline{f_2}(t)dt,$$

$$\|f_1\|_X = < f_1, f_1 >_X^{1/2}. \tag{10}$$

where $\overline{f_2}$ denotes the complex conjugate of a function f_2 . Similarly, Y is defined as the Hilbert space $L_2(b,c)$: for $\eta_1, \eta_2 \in Y$

$$< \eta_1, \eta_2 >_Y = \int_{b}^{c} \eta_1(t)\overline{\eta_2}(t)dt,$$

$$\|\eta_1\|_Y = < \eta_1, \eta_1 >_Y^{1/2}. \tag{11}$$

Since the boundary value problem (3)-(7) is known as well-posed in the sense of Hadamard (Isakov, 1998, p.20), a unique φ exists with a given $f \in X$: there exists an operator $\mathsf{L} : X \to Y$ such that

$$\mathsf{L}f = \eta \tag{12}$$

for the system (3)-(7).

L Because, physically, $\mathsf{L}f$ may be viewed as a wave system generated by a wave maker velocity f , we will call a wave maker operator. And the range of the operator L will be denoted by

$$R(\mathsf{L}) = \{\mathsf{L}f \in Y : f \in L_2(-h,0)\}. \tag{13}$$

If an observed wave elevation belongs to the range $R(\mathsf{L})$ (i.e., $\eta \in R(\mathsf{L})$), then (12) has at least one solution $p \in X$: After showing compactness of L in §4, the question of solvability will be discussed in detail.

Noting that the governing equation (3) and the corresponding boundary conditions (4)-(7) are linear, we have

$$\mathsf{L}(f_1 + f_2) = \mathsf{L}f_1 + \mathsf{L}f_2 ,\ f_1, f_2 \in X$$

$$\mathsf{L}(\alpha p) = \alpha \mathsf{L} f ,\ f \in X \text{ for complex numbers}$$

such that L can be defined as a linear operator from X to Y. The precise relation between the wave maker velocity f and the wave elevation η will be set up in the next section.

3. The Wave Maker Operator Equation

This section is devoted to the realization of the explicit functional form of the wave maker operator we have defined in the previous section. The solution for the boundary value problem (3)-(7) can be obtained by separation of variables. Assuming the following form of a solution

$$\varphi(x,y) = \sum_{j=0}^{\infty} d_j \psi_j(y) \xi_j(x) \tag{14}$$

where d_j (j=0,1,2,⋯) are constants to be determined, we have the solution for the boundary value problem (3)-(7) expressed as

$$\varphi(x,y) = \frac{<f,e_0>_X}{ik_0} e_0(y) e^{ik_0 x} + \sum_{j=1}^{\infty} \frac{<f,e_j>_X}{-k_j} e_j(y) e^{-k_j x} \tag{15}$$

In the above, eigenfunction ψ_j is normalized with its norm to give $e_j = \psi_j / \|\psi_j\|_Y$. Using (15), we can express the wave elevation in term of the wave maker velocity:

$$\eta(x) = A_0 \langle f, e_0 \rangle_X e^{ik_0 x} + \sum_{j=1}^{\infty} A_j \langle f, e_j \rangle_X e^{-k_j x} \tag{16}$$

where

$A_0 = \omega e_0(0)/gk_0$ and

$A_j = -i\omega e_j(0) g k_j$, j=1,2,⋯ (17)

Reminding the definition of the inner product (10), we obtain

$$\eta(x) = \int_{-h}^{0} \{A_0 e_0(y) e^{ik_0 x} + \sum_{j=1}^{\infty} A_j e_j(y) e^{-k_j x}\} f(y) dy \tag{18}$$

or

$$\eta(x) = \int_{-h}^{0} k(x,y) f(y) dy \tag{19}$$

with the kernel

$$k(x,y) = A_0 e_0(y) e^{ik_0 x} + \sum_{j=1}^{\infty} A_j e_j(y) e^{-k_j x} . \tag{20}$$

As shown in (19), the operator equation (12) is a Fredholm integral equation of the first kind. By comparing (18) with (12), the wave maker operator L turns out to be the integral operator:

$$\mathsf{L}f = \int_{-h}^{0} k(x,y) f(y) dy \tag{21}$$

4. Solvability

Since we have constructed the wave maker operator in the previous section, we are going to investigate the stability question in this section. First we will show that the wave maker operator is compact.

By the Schwarz inequality,

$$|\mathsf{L}f| \le \left| \int_{-h}^{0} |k(x,y)| \cdot |f(y)| dy \right| \le \|f\|_X \cdot \left| \int_{-h}^{0} |k(x,y)|^2 dy \right|^{1/2} . \tag{22}$$

This shows $\mathsf{L}f$ is finite almost everywhere, and furthermore

$$\|\mathsf{L}f\|_Y \le \left| \int_b^c \int_{-h}^{0} |k(x,y)|^2 dydx \right|^{1/2} \cdot \|f\|_X . \tag{23}$$

From (23),

$$\sup_{f \ne 0} \frac{\|\mathsf{L}f\|_Y}{\|f\|_X} \le \left| \int_b^c \int_{-h}^{0} |k(x,y)|^2 dydx \right|^{1/2} . \tag{24}$$

Let us define the norm of the operator L as

$$\|L\|_o \overset{def}{=} \sup_{f\neq 0}\frac{\|Lf\|_Y}{\|f\|_X}. \tag{25}$$

By (20), the kernel k is square integrable, that is,

$$k\in L^2(\Gamma_M\times\Gamma_f)\overset{def}{=}\{\kappa(x,y):\int_{-h}^{0}\int_{b}^{c}|\kappa(x,y)|^2\,dxdy<\infty\} \tag{26}$$

From the inequality (24) and (26), $\|L\|_o$ is bounded. Let $\{e_i\}$ and $\{g_i\}$ be an orthonormal basis for $L_2(-h,0)$ and, $L_2(b,c)$ respectively. From Fubini's theorem it follows that $\{e_i\cdot g_j\}$ is an orthonormal basis for $L_2(\Gamma_M\times\Gamma_f)$, and we can expand $k(x,y)$ in the norm convergent Fourier series

$$k(x,y)=\sum_{i,j=1}^{\infty}a_{ij}e_i(y)g_j(x) \tag{27}$$

with $\sum_{i,j=1}^{\infty}|a_{ij}|^2<\infty$ Then for $N=1,2,\ldots,$ we have ,

$$Lf=L_Nf+L_N^{'}f$$

where

$$L_Nf=\sum_{i+i<N}a_{ij}<g_j,f>e_i \tag{28}$$

is an operator of finite rank, while $\|L_N^{'}\|_o^2=\sum_{i+j>N}^{\infty}|a_{ij}|^2\to 0$ as $N\to\infty$. We have shown that a sequence of finite rank operators $\{L_N\}$ such that $\|L_N-L\|_o\to 0$. From this and boundedness of $\|L\|_o$, L is compact(Bassanini and Elcrat 1997).

$R(L)$, the range of the operator L, is concerned with all possible wave elevations the wave maker can generate with all possible $f\in X$. The set of all possible wave elevations $R(L)$ is smaller than Y, i.e., $R(L)$ does not exhaust Y because the wave maker operator L is compact. Therefore, a traditional solution of (12) will exist only in a restricted class of functions $\eta\in R(L)$. If $\eta\notin R(L)$ an $\eta\in Y$, then there is no solution of a wave source f in X satisfying (12): there may exist a wave profile the wave maker cannot generate because $R(L)\subset Y$.

5.Unique Wave Source

In this section, we are going to show the identification of the inverse problem, that is, we will show that the wave maker velocity fis uniquely determined from the observed wave elevation η on Γ_M .

We want to show the null space $N(L)$ of the linear operator L is trivial.
Let us assume

$$\eta=0 \text{ on } \Gamma_M \tag{29}$$

$N(L)$ Noting (9), we obtain (30) from (29).

$$\varphi=0 \text{ on } \Gamma_M \tag{30}$$

Applying (30) to (16), we can have

$$\psi_j(0)=0 \quad j=0,1,2\ldots,. \tag{31}$$

In order to satisfy (30), the condition (31) must be imposed on the Strum-Liouville eigen value problem (32)-(34):

$$\frac{d^2\psi_j}{dy^2}-k_j^2\psi_j=0,\quad -h<y<0 \tag{32}$$

$$\frac{d\psi_j}{dy}-\gamma\psi_j=0,\quad y=0 \tag{33}$$

$$\frac{d\psi_j}{dy}=0,\quad y=-h \tag{34}$$

which are obtained by substituting (14) into the boundary value problem (3)-(5). Kreisel (1949) has proven that the set $\{\psi_j\}, j=0,1,2\ldots$ consists of nontrivial orthogonal eigenfunctions which is complete:

$$\psi_0(y) = \frac{\cosh\{k_0(y+h)\}}{\cosh\{k_0 h\}} \quad (35)$$

$$\psi_j(y) = \cos\{k_j(y+h)\}, \quad j=1,2,\cdots \quad (36)$$

and each $k_j h \in ((j-1/2)\pi, j\pi)$,i.e.,

$$(j-1/2)\pi < k_j h < j\pi \ , j = 0,1,2\ldots. \quad (37)$$

The equations of (8) and $\gamma = -k_j \tan k_j h$ provide the dispersion relations for the wave numbers k_0 and $k_j > 0 \ (j = 1,2,\ldots)$, respectively. Because of (37), the only solution that satisfies all the requirements (30)-(34) and (31) is trivial:

$$\psi_j = 0, j = 0,1,2\ldots \quad (38)$$

If we substitute (38) into (15), we obtain $\varphi = 0$ in Ω: finally (6) gives us

$$f = 0 \text{ on } \Gamma_p \ . \quad (39)$$

We have shown that $N(\mathrm{L})$ is trivial. Because L is linear, the trivial null space means that L is one to one. That is, (12) has only one unique solution if it exists.

6. Discontinuity of Wave Source

Having shown the compactness of the wave maker operator, we can prove that the operator equation (12) is ill-posed in the sense of stability: that is, the solution does not depend continuously on the data of the free surface boundary measurement η on Γ_M.

Let us introduce the inverse L^{-1} of L . Its existence is guaranteed by identification in section 5. Because the above analysis states the wave maker operator is compact, it is linear invertible compact. However, the invertible compact is defined dimensional Hilbert space; therefore, its inverse must be unbounded, i.e. discontinuous(Stakgold, 1967).

Because of discontinuity of the inverse operator L^{-1}, we cannot solve (12) with a usual numerical scheme: a small error of the free surface boundary measurement would results in an arbitrary large error of solution. For this reason, it is required to introduce regularization method in order to identify the wave source of wave maker.

7. Summary and Conclusions

From this work, it is newly proposed that the inverse problem to find the wave maker velocity is ill-posed in the sense that the solution does not depend continuously on the observed elevation data. We have also proved in this paper the identifiability theorem that the wave maker source is uniquely determined. Constructing the wave maker operator, we have shown that it is compact with the respect to the topology generated by the norm in the solution space. Considering compactness of the wave maker operator, we have shown that the operator is not onto: the wave maker cannot generate all the forms of waves, in other words, there is no wave maker which generates an arbitrary wave form. For future work, it is necessary to introduce regularization method to identify or realize the wave maker velocity because of its ill-posedness.

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