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THE STRONG STABILITY OF ALGORITHMS FOR SOLVING THE SYMMETRIC EIGENPROBLEM

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ABSTRACT. The concepts of stability of algorithms for solving the symmetric and generalized symmetric–definite eigenproblems are discussed. An algorithm for solving the symmetric eigenproblem $Ax = \lambda x$ is stable if the computed solution z is the exact solution of some slightly perturbed system $(A + E)z = \lambda z$. We use both normwise approach and componentwise way of measuring the size of the perturbations in data. If E preserves symmetry we say that an algorithm is strongly stable (in a normwise or componentwise sense, respectively). The relations between the stability and strong stability are investigated for some classes of matrices.

1. INTRODUCTION

In [11] D.J.Higham and N.J.Higham introduced new definitions of structured backward error and condition number for linear systems. They stated: "When perturbations to a symmetric matrix are measured using the 2–norm it makes little difference to the backward error or to the condition number whether symmetry is enforced or not".

In this work we show that it holds for the symmetric eigenproblem and symmetric–definite pencils . Our result is similar to obtained by J.R.Bunch, J.W.Demmel and C.V Loan [4] but we prove it in another way. We consider also a componentwise way of measuring the size of the perturbations in data. For a recent account of the perturbation theory in numerical linear algebra see [12].

An algorithm for solving the symmetric eigenproblem $Ax = \lambda x$ is said to be **numerically stable** if it gives a computed solution $z \neq 0$ satisfying a relation $(A + E)z = \lambda z$ with $\| E \|$ of order $\rho \| E \|$, where ρ is the relative computer precision. If all $| e_{i,j} |$ are of order $\rho \mid a_{i,j} \mid$ then an algorithm is said to be numerically stable in a componentwise sense.

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When we solve the symmetric eigenvalue problem $Ax = \lambda x$, it is often important to force symmetry of the perturbed matrix $A + E$. It can help us to obtain more realistic bounds for computed eigenvalues and eigenvectors ([1], [8], [12]).

We will say an algorithm for solving the symmetric eigenproblem is **strongly stable** for a class of matrices $\mathcal{A}$ if for each $A \in \mathcal{A}$, the computed solution z to $Ax = \lambda x$ satisfies $\tilde{A}z = \lambda z$, where $\tilde{A} \in \mathcal{A}$ and $\tilde{A}$ is close to A (Cf. [3]).

We summarize the contents of the paper. In Section 2 we prove that any stable algorithm for solving the symmetric eigenproblem $Ax = \lambda x$ on the class symmetric matrices is also strongly stable on the same matrix class. Next we extend these results to symmetric–definite pencils $Ax = \lambda Bx$. No such result holds for componentwise approach (Cf. [11] for similar problem for linear systems). However, we prove in Section 3 that it is true for some classes of matrices. We stress that only a little is known about componentwise stability of algorithms for solving the symmetric eigenproblem (Cf. [1], [8]). In Section 4 we give a generalization of Weyl's inequality.

2. The strong normwise stability

In [9] J.E.Dennis and J.J.Morè proved the following theorem.

THEOREM 2.1 (Dennis, Morè). *Let $A = A^T \in \mathrm{IR}^{\mathrm{nxn}}$ and let $r = b - Az$, where $z \neq 0$. Then*

$$\delta A = \frac{rz^T + zr^T}{z^T z} - \frac{r^T z}{(z^T z)^2} zz^T$$

is the smallest symmetric matrix in the Frobenius norm for which the vector z satisfies $(A + \delta A)z = b$.

J.R.Bunch, J.W.Demmel and C.V Loan ([4]) showed that if $A = A^T$ and E is any matrix for which $(A + E)z = b$, $z \neq 0$, then there exists a symmetric ΔA such that $(A + \Delta A)z = b$, $\| \Delta A \|_2 \leq \| E \|_2$ and $\| \Delta A \|_F \leq \sqrt{2} \| E \|_F$.

If we take $b = \lambda z$ in the above then we obtain the similar result for the symmetric eigenvalue problem $Ax = \lambda x$. However here is another way to show this.

THEOREM 2.2. *Let $A = A^T \in \mathrm{IR}^{\mathrm{nxn}}$ and let $r = \lambda z - Az$, where $\lambda \in \mathrm{IR}$ and $0 \neq z \in \mathrm{IR}^{\mathrm{n}}$. Then*

$$\delta A = \frac{rz^T + zr^T}{z^T z} - \frac{r^T z}{z^T z} I \tag{1}$$

is the smallest symmetric matrix in the 2– norm for which the vector z satisfies $(A + \delta A)z = \lambda z$.

Moreover, if $(A + E)z = \lambda z$ where $\| E \|_2 \leq \epsilon \| A \|_2$, then we have $\| \delta A \|_2 \leq \| E \|_2 \leq \epsilon \| A \|_2$.

Here I denotes the n by n identity matrix.

Proof. It is clear that $\delta A z = r$, hence $(A+\delta A)z = \lambda z$. Note that if $(A+E)z = \lambda z$ for any matrix E , then $r = Ez$, so $\| r \|_2 \leq \| E \|_2 \| z \|_2$ and $\| E \|_2 \geq \frac{\|r\|_2}{\|z\|_2}$. What is left is to show that $\| \delta A \|_2 = \frac{\|r\|_2}{\|z\|_2}$.

Let $H = H^T \in \mathbb{R}^{nxn}$ be any orthogonal matrix such that $Hz = \| z \|_2 e_1$, where $e_1 = [1, 0, \ldots, 0]^T$. For example, we can take Householder's transformation (Cf. [10],[15]).

For simplicity of notation we write $y = [y_1, y_2, \ldots, y_n]^T$, where

$$Y = H\delta A H \ , \ y = \frac{Hr}{\| z \|_2} \ . \tag{2}$$

It is easily seen that $Y = Y^T$ and

$$Y = y e_1^T + e_1 y^T - y_1 I \ . \tag{3}$$

The characteristic polynomial of Y is equal to

$$\det(Y - \lambda I) = (-1)^{n+1}(y_1 + \lambda)^{n-2}\{(y_1^2 + \ldots + y_n^2) - \lambda^2\} \ .$$

From this it follows that $\| Y \|_2 = \| y \|_2$. Therefore

$$\| \delta A \|_2 = \| Y \|_2 = \frac{\| r \|_2}{\| z \|_2} \ .$$

This finishes the proof. ∎

By a similar argument we can prove the following theorem.

THEOREM 2.3. *Let $A \in \mathbb{R}^{nxn}$ and $B \in \mathbb{R}^{nxn}$ be symmetric matrices with B positive definite. Assume that $(A+E)z = \lambda(B+F)z$ where $E \in \mathbb{R}^{nxn}$, $F \in \mathbb{R}^{nxn}$, $\lambda \in \mathbb{R}$ and $0 \neq z \in \mathbb{R}^n$. Let $r_A = Ez$ and $r_B = Fz$. Then*

$$\delta A = \frac{r_A z^T + z r_A^T}{z^T z} - \frac{r_A^T z}{z^T z} I$$

and

$$\delta B = \frac{r_B z^T + z r_B^T}{z^T z} - \frac{r_B^T z}{z^T z} I$$

are symmetric and satisfy $(A+\delta A)z = \lambda(B+\delta B)z$ where $\| \delta A \|_2 \leq \| E \|_2$ and $\| \delta B \|_2 \leq \| F \|_2$.

Proof. Note that $\delta A z = r_A$ and $\delta B z = r_B$. Hence $(A+\delta A)z = (A+E)z$ and $(B+\delta B)z = (B+F)z$. This implies that $(A+\delta A)z = \lambda(B+\delta B)z$.

Because

$$\| \delta A \|_2 = \frac{\| r_A \|_2}{\| z \|_2} \ , \ \| \delta B \|_2 = \frac{\| r_B \|_2}{\| z \|_2} \tag{4}$$

it is evident that $\| \delta A \|_2 \leq \| E \|_2$ and $\| \delta B \|_2 \leq \| F \|_2$. ∎

3. The strong componentwise stability

In order to obtain a sharper perturbation bounds we use componentwise analysis. Note that matrix $| A |$ is the matrix whose elements are $| a_{i,j} |$ and we write $| A | \leq | B |$ to mean that inequalities between matrices hold componentwise.

In [16] it is shown that any stable algorithm in a componentwise sense for solving symmetric linear systems is strongly stable on the following classes of matrices:

: (1) $\mathcal{A}_1 = \{$ symmetric positive definite matrices $\}$;
: (2) $\mathcal{A}_2 = \{$ symmetric positive definite band matrices with bandwidth ω $\}$;
: (3) $\mathcal{A}_3 = \{$ symmetric diagonally dominant matrices $\}$;
: (4) $\mathcal{A}_4 = \{$ symmetric matrices with $| a_{i,j} | \leq \gamma | a_{i,i} |$ $\}$;
: (5) $\mathcal{A}_5 = \{$ symmetric band matrices with $| a_{i,j} | \leq \gamma | a_{i,i} |$ and with bandwith ω $\}$.

THEOREM 3.1. *Assume that $A \in \mathbb{R}^{nxn}$ is symmetric and $A \in \bigcup_{i=1}^{5} \mathcal{A}_i$. If $(A+E)z = b$, where $| E | \leq \epsilon | A |$ and $z \neq 0$, then there exists a symmetric matrix $\Delta A \in \mathbb{R}^{nxn}$ such that $(A + \Delta A)z = b$ and $| \Delta A | \leq K_i \epsilon | A |$ with*

$$K_i = \begin{cases} 2n-1 & i=1 \\ 2\omega - 1 & i=2 \\ 3 & i=3 \\ 2(n-1)\gamma + 1 & i=4 \\ 2(\omega-1)\gamma+1 & i=5\ . \end{cases} \tag{5}$$

Taking $b = \lambda z$ in the above we obtain the following theorem.

THEOREM 3.2. *Assume that $A \in \mathbb{R}^{nxn}$ is symmetric and $A \in \bigcup_{i=1}^{5} \mathcal{A}_i$. If $(A+E)z = \lambda z$, where $| E | \leq \epsilon | A |$ and $z \neq 0$, then there exists a symmetric matrix $\Delta A \in \mathbb{R}^{nxn}$ such that $(A + \Delta A)z = \lambda z$ and $| \Delta A | \leq K_i \epsilon | A |$ with K_i defined in (5).*

From Theorem 3.1 by taking $b = (A + E)z$ and then $b = (B + F)z$ we obtain the following theorem .

THEOREM 3.3. *Let $A \in \bigcup_{i=1}^{5} \mathcal{A}_i$ and B be symmetric positive definite . If $(A+E)z = \lambda(B + F)z$, where $| E | \leq \epsilon | A |$, $| F | \leq \epsilon | B |$ and $z \neq 0$, then there exist symmetric matrices ΔA and ΔB such that $(A + \Delta A)z = \lambda(B + \Delta B)z$ and $| \Delta A | \leq K_i \epsilon | A |$ and $| \Delta B | \leq K_1 \epsilon | B |$ where K_i is defined in (5).*

REMARK It is clear that if $A \in \bigcup_{i=1}^{3} \mathcal{A}_i$ then also $DAD \in \bigcup_{i=1}^{3} \mathcal{A}_i$ for any non-singular diagonal matrix D. See [1] where such classes of matrices were considered in order to determine eigenvalues with high relative accuracy.

4. Perturbation theorems for symmetric–definite pencils

For a fuller treatment we refer the reader to [1] ,[5] and [8] and [15]. It is well known that each eigenvalue of symmetric–definite pencil $Ax = \lambda Bx$ is real and satisfies the following a max–min characterization ([15]):

THEOREM 4.1 (Fischer). *Assume that $A \in \mathbb{R}^{nxn}$ is a symmetric matrix and $B \in \mathbb{R}^{nxn}$ is symmetric positive definite. Let the eigenvalues of (A, B) be ordered so that*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Then for $i = 1, 2, \ldots, n$

$$\lambda_i = \max_{\dim(\mathcal{X})=i} \min_{0 \neq x \in \mathcal{X}} \frac{x^T A x}{x^T B x} \quad .$$

The following theorem was proved in [1] (Cf. [8]).

THEOREM 4.2 (Barlow and Demmel). *Let A and B be symmetric matrices with B positive definite. Let the pencil $A - \lambda B$ have eigenvalues λ_i. Let δA and δB be symmetric perturbations and let λ_i' be the (properly ordered) eigenvalues of $(A + \delta A) - \lambda(B + \delta B)$. Suppose that*

$$\mid x^T \delta A x \mid \leq \eta_A \mid x^T A x \mid \quad \textit{and} \quad \mid x^T \delta B x \mid \leq \eta_B \mid x^T B x \mid$$

for all vectors x and some $\eta_A < 1$ and $\eta_B < 1$. Then either $\lambda_i = \lambda_i' = 0$ or

$$\frac{1 - \eta_A}{1 + \eta_B} \leq \frac{\lambda_i'}{\lambda_i} \leq \frac{1 + \eta_A}{1 - \eta_B}$$

for all i.

Note that for singular matrix A the assumption $\mid x^T \delta A x \mid \leq \eta_A \mid x^T A x \mid$ forces some special correlations between elements of δA . The following example illustrates this. For

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \delta A = \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_2 & \delta_3 \end{pmatrix}$$

and $x = [1, -1]^T$ we have $Ax = 0$ so $x^T \delta A x = 0$ hence $\delta_2 = (\delta_1 + \delta_3)/2$.

Using very similar proof to [1] we obtain the following theorem.

THEOREM 4.3. *Let A and B be symmetric matrices with B positive definite. Let the pencil $A - \lambda B$ have eigenvalues λ_i. Let δA and δB be symmetric perturbations and let λ_i' be the (properly ordered) eigenvalues of $(A + \delta A) - \lambda(B + \delta B)$. Assume that $\rho(B^{-1}\delta B) < 1$, where $\rho()$ denotes the spectral radius. Denoting the i-the largest eigenvalues of $B^{-1}\delta A$ and $B^{-1}\delta B$ by $\lambda_i(B^{-1}\delta A)$ and $\lambda_i(B^{-1}\delta B)$, we have*

$$\frac{\lambda_i + \lambda_n(B^{-1}\delta A)}{1 + \lambda_1(B^{-1}\delta B)} \leq \lambda_i' \leq \frac{\lambda_i + \lambda_1(B^{-1}\delta A)}{1 + \lambda_n(B^{-1}\delta B)} \quad . \tag{6}$$

Proof. The Fischer theorem implies that

$$\lambda_i{}' = \max_{\dim(\mathcal{X})=i} \min_{0\neq x\in\mathcal{X}} \frac{x^T(A+\delta A)x}{x^T(B+\delta B)x} \ . \tag{7}$$

Because $x^T Bx > 0$ for $0 \neq x \in \mathcal{X}$, we have

$$x^T(B+\delta B)x = x^T Bx(1+e_B) \ , \text{ where } e_B = \frac{x^T \delta Bx}{x^T Bx} \ .$$

It is easy to check that

$$\frac{x^T(A+\delta A)x}{x^T(B+\delta B)x} = \frac{\frac{x^T Ax}{x^T Bx} + e_A}{1+e_B} \ , \text{ where } e_A = \frac{x^T \delta Ax}{x^T Bx} \ . \tag{8}$$

From the Fischer max–min theorem for the symmetric–definite pencils $\delta A - \lambda B$ and $\delta B - \lambda B$ we have

$$\lambda_n(B^{-1}\delta A) \leq e_A \leq \lambda_1(B^{-1}\delta A) \tag{9}$$

and

$$\lambda_n(B^{-1}\delta B) \leq e_B \leq \lambda_1(B^{-1}\delta B) \ . \tag{10}$$

We conclude from (7) that

$$\frac{\lambda_i + \lambda_n(B^{-1}\delta A)}{1+\lambda_1(B^{-1}\delta B)} \leq \frac{x^T(A+\delta A)x}{x^T(B+\delta B)x} \leq \frac{\lambda_i + \lambda_1(B^{-1}\delta A)}{1+\lambda_n(B^{-1}\delta B)} \ .$$

which completes the proof.■

A weaker form of the Theorem 4.3 is stated in the following corollary.
COROLLARY.

$$|\ \lambda_i{}' - \lambda_i\ | \leq \frac{\rho(B^{-1}\delta A) + |\ \lambda_i\ |\ \rho(B^{-1}\delta B)}{1-\rho(B^{-1}\delta B)} \ . \tag{11}$$

This result follows from the observation that $|\ \lambda_i(B^{-1}\delta B)\ | \leq \rho(B^{-1}\delta B)$ and $|\ \lambda_i(B^{-1}\delta A)\ | \leq \rho(B^{-1}\delta A)$. ■

COROLLARY. In Theorem 4.3 suppose that $B = I$ and $\delta B = 0$. Let λ_i and $\lambda_i{}'$ denote the i-th largest eigenvalue of symmetric matrices A and $A + \delta A$. Then

$$\lambda_i + \lambda_n(\delta A) \leq \lambda_i{}' \leq \lambda_i + \lambda_1(\delta A) \ .$$

It is well–known Weyl's inequality (Cf. [15]). ■

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