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AN EFFICIENT IMPLEMENTATION OF BDM MIXED METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS

J.H. KIM

ABSTRACT. BDM mixed methods are obtained for a good approximation of velocity for flow equations. In this paper, we study an implementation issue of solving the algebraic system arising from the BDM mixed finite elements. First we discuss post-processing based on the use of Lagrange multipliers to enforce interelement continuity. Furthermore, we establish an equivalence between given mixed methods and projection finite element methods developed by Chen. Finally, we present the implementation of the first order BDM on rectangular grids and show it is as simple as solving the pressure equation.

1. INTRODUCTION

Consider the homogeneous Dirichlet problem :

$$(1.1)\qquad \begin{cases} -\mathrm{div}(a(x)\mathbf{grad}\ p) = f(x), & \forall x \in \Omega, \\ p = 0, & \forall x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and $f \in L^2(\Omega)$ is a given real-valued function and $a(x)$ is a positive, smooth function on the closure of Ω.

For solving this problem, we use a mixed finite element method. so that we have a good approximation for the velocity. We note that other methods, such as cell-centered finite difference method and finite volume method can be derived from mixed methods. To introduce the mixed method for the above problem, we first define the spaces :

$$V := H(\mathrm{div}\ ;\ \Omega) = \{\ \mathbf{v} \in (L^2(\Omega))^2 \mid \ \mathrm{div}\ \mathbf{v} \in L^2(\Omega)\ \},$$
$$W := L^2(\Omega).$$

Let $\mathbf{u} = -a(x)\mathbf{grad}\ p$ in Ω and $c(x) = a(x)^{-1}$. Then the mixed formulation of (1.1) is obtained by

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$$
(1.2) \qquad \begin{cases} c(x)\mathbf{u} + \mathbf{grad}\ p = 0, & \forall x \in \Omega, \\ \operatorname{div} \mathbf{u} = f, & \forall x \in \Omega, \\ p = 0, & \forall x \in \partial\Omega. \end{cases}
$$

The mixed weak formulation of (1.1) is given by [1] :
Find $(\mathbf{u}, p) \in V \times W$ such that

$$
(1.3) \qquad \begin{cases} (c\mathbf{u},\ \mathbf{v}) - (\operatorname{div} \mathbf{v},\ p) = 0, & \forall \mathbf{v} \in V, \\ (\operatorname{div} \mathbf{u},\ q) = (f,\ q), & \forall q \in W, \end{cases}
$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$.

We discretize this continuous formulation in the usual way : establish a triangulation $\mathcal{T}_h$ and construct finite-dimensional subspaces $V_h \subset V$ and $W_h \subset W$. But, as different from the standard method, there are some compatibility conditions, i.e., $\mathbf{v} \in V$ if and only if for all $K \in \mathcal{T}_h$, $\mathbf{v}|_K \in V(K)$ and for any pair of adjacent elements $K_1, K_2 \in \mathcal{T}_h$, we have $\mathbf{v}|_{K_1} \cdot n_{K_1} + \mathbf{v}|_{K_2} \cdot n_{K_2} = 0$ where n_{K_i} is the unit outer normal vector to K_i, $i = 1, 2$. Then, the mixed finite element approximation is the solution $(\mathbf{u}_h,\ p_h) \in V_h \times W_h$ of

$$
(1.4) \qquad \begin{cases} (c\mathbf{u}_h,\ \mathbf{v}) - (\operatorname{div} \mathbf{v},\ p_h) = 0, & \forall \mathbf{v} \in V_h, \\ (\operatorname{div} \mathbf{u}_h,\ q) = (f,\ q), & \forall q \in W_h. \end{cases}
$$

This discretization by mixed finite element methods leads to linear system of the form

$$
(1.5) \qquad \begin{cases} A\mathbf{u}_h - Bp_h = 0, \\ B^t\mathbf{u}_h = f. \end{cases}
$$

It was shown in [2] that this system is generally indefinite. For this reason we cannot use any well-known numerical method such as conjugate gradient method to solve the system (1.5). However, we can algebraically reduce this system to a symmetric and positive definite system as follows : Inverting A in the first equation of (1.5) gives

$$
(1.6) \qquad \mathbf{u}_h = A^{-1}Bp_h.
$$

Substituting (1.6) into the second equation in (1.5) yields

$$
(1.7) \qquad B^tA^{-1}Bp_h = f.
$$

We know that it is difficult to invert the matrix A, since A is not block diagonal. This is due to the continuity constraints from the finite element space V_h.

To overcome this drawback and to enforce the continuity of the normal component of $\mathbf{u}_h$ across interelement boundaries, we will change the problem (1.4) to the hybrid

version using Lagrange multipliers introduced in [3], [4].
Let e denotes the edge or face of element. We define the spaces :

$$\begin{aligned} V &:= \{ \mathbf{v} \in (L^2(\Omega))^2 \mid \mathbf{v}|_T \in H(\text{div}\ ;\ T),\ \ \forall T \in \mathcal{T}_h \}, \\ W &:= L^2(\Omega), \\ L &:= \{ \lambda |\ \lambda \in L^2(e) \text{ and } \lambda = 0 \text{ on } \partial\Omega \}, \end{aligned}$$

and construct finite dimensional subspaces $V_h \subset V,\ W_h \subset W,\ L_h \subset L$.
Then we obtain the extended problem :
Find $(\mathbf{u}_h, p_h, \lambda_h) \in V_h \times W_h \times L_h$ such that

$$(1.8)\quad \begin{cases} (c\mathbf{u}_h,\ \mathbf{v}) - \sum_{T\in\mathcal{T}_h} (\text{div}\ \mathbf{v},\ p_h)_T + \sum_{T\in\mathcal{T}_h} \langle\ \mathbf{v}\cdot n_T,\ \lambda_h\ \rangle_{\partial T} = 0, & \forall \mathbf{v} \in V_h, \\ \sum_{T\in\mathcal{T}_h} (\text{div}\ \mathbf{u}_h,\ q)_T = (f,\ q), & \forall q \in W_h, \\ \sum_{T\in\mathcal{T}_h} \langle\ \mathbf{u}_h\cdot n_T,\ m\ \rangle_{\partial T} = 0, & \forall m \in L_h, \end{cases}$$

where $(\cdot,\cdot)_T$ and $\langle\cdot,\cdot\rangle_T$ indicate the inner product in $L^2(T)$ and $L^2(\partial T)$, respectively and n_T is the unit outer normal vector.
Let us consider the linear system generated by hybridization (1.8). It can be written in the matrix form

$$(1.9)\quad \begin{cases} A\mathbf{u}_h - Bp_h + C\lambda_h = 0, \\ B^t\mathbf{u}_h = f, \\ C^t\mathbf{u}_h = 0. \end{cases}$$

The advantage of the system (1.9) is that the matrix A is block diagonal, with positive definite diagonal of size $\dim(V_h(T))$. So A may be inverted easily and inexpensively at the element level. We can algebraically compute this system by eliminating

$$(1.10)\quad \mathbf{u}_h = A^{-1}Bp_h - A^{-1}C\lambda_h$$

to obtain

$$(1.11)\quad \begin{cases} B^tA^{-1}Bp_h - B^tA^{-1}C\lambda_h = f, \\ C^tA^{-1}Bp_h - C^tA^{-1}C\lambda_h = 0. \end{cases}$$

The well-known Arnold and Brezzi [5] reduction of the saddle-point problem in (1.8) to a symmetric, positive-definite linear system is to solve for p_h in terms of λ_h using the first equation of (1.11) and to substitute this relation into the second equation of (1.11) to obtain an equation for λ_h. Hence we have

$$(1.12)\quad \{C^tA^{-1}C - (C^tA^{-1}B)(B^tA^{-1}B)^{-1}(B^tA^{-1}C)\}\lambda_h = f'.$$

We may solve this system and recover $\mathbf{u}_h$ from (1.10) by simple element-by-element post process.

The technique described above applies to the Brezzi-Douglas-Marini space [6]. And we show that the multipliers obtained from (1.12) can be used in the reconstruction of a new approximation p_h^* which is more rapidly convergent to p than p_h.

We also present a general theory of the equivalence between mixed and nonconforming method. A nonconforming method for some finite element space M_h is a Galerkin method with the addition of some special projection operator, and so we call it a projection finite element method. We consider two conditions on M_h that are sufficient to imply that the equivalence of the projection method to a given meixed method. Then we will construct the space M_h for BDM space. And we deal with one method of solving the lowest order BDM mixed method.

This paper is organized as follows : In §2, we introduce BDM spaces devised by Brezzi-Douglas-Marini[6] and the error analysis of BDM mixed method is made. In §3, we consider the post-processing suggseted by Arnold and Brezzi[5]. We can obtain better approximation from this process. §4 is devoted to the derivation of projection finite element method. And also space M_h associated with projection method is constructed. Finally, we introduce a way to solve the first order BDM mixed method over rectangular elements.

2. Error Analysis for BDM space

2.1. Brezzi-Douglas-Marini space. The Brezzi-Douglas-Marini (BDM) space $V_h \times W_h$ on triangles of order k is defined by [6], [7] :

$$V_h = \{ \mathbf{v} \in (L^2(\Omega))^2 \mid \mathbf{v}|_T \in (P_k(T))^2, \quad \forall\, T \in \mathcal{T}_h \},$$
$$W_h = \{ q \in L^2(\Omega) \mid q|_T \in P_{k-1}(T), \quad \forall\, T \in \mathcal{T}_h \},$$
$$L_h = \{ \lambda \in L^2(e) \mid \lambda|_e \in P_k(e) \quad \text{if } e \subset \Omega \quad \text{and} \quad \lambda|_e = 0 \quad \text{if } e \subset \partial\Omega \}.$$

We have by a simple count, dim $(V_h(T)) = (k+1)(k+2)$ and dim $(W_h(T)) = \frac{1}{2}k(k+1)$. For the choice of degrees of freedom, we have

Lemma 2.1. *For $k \geq 1$ and for any $\mathbf{v} \in V_h$, the following relations imply* $\mathbf{v} = 0$.
(1) $\int_{e_i} \mathbf{v} \cdot n\varphi_i \, ds = 0, \quad \forall\, \varphi_i \in P_k(e_i),$
(2) $\int_T \mathbf{v} \cdot \mathbf{grad}\psi_{i-1} \, dx = 0, \quad \forall\, \psi_{i-1} \in P_{k-1}(T),$
(3) $\int_T \mathbf{v} \cdot \phi_i \, dx = 0, \quad \forall\, \phi_i \in \{ \phi_i \in (P_k)^2 \mid \text{div } \phi_i = 0,\ \phi_i \cdot n|_e = 0 \}.$

The BDM space on rectangles of order k is given by [6], [7]:

$$V_h = \{ \mathbf{v} \in (L^2(\Omega))^2 \mid \mathbf{v}|_R \in (P_k(R))^2 \oplus \text{Span}(\text{curl } x^{k+1}y,\ \text{curl } xy^{k+1}), \quad \forall\, R \in \mathcal{T}_h \},$$
$$W_h = \{ q \in L^2(\Omega) \mid q|_R \in P_{k-1}(R), \quad \forall\, R \in \mathcal{T}_h \},$$
$$L_h = \{ \lambda \in L^2(e) \mid \lambda|_e \in P_k(e) \ \text{ if } e \subset \Omega \ \text{ and } \lambda|_e = 0 \ \text{ if } e \subset \partial\Omega\},$$

where curl $w = (-\frac{\partial w}{\partial y}, \frac{\partial w}{\partial x})$. By a simple computation, we get $\dim(V_h(R)) = k^2+3k+4$ and $\dim(W_h(R)) = \frac{1}{2}k(k+1)$.

For the three-dimensional case, the BDM space over rectangular parallelepiped of order k is defined by [8] :

$$V_h = \{ \ \mathbf{v} \in (L^2(\Omega))^3 \mid \mathbf{v}|_R \in (P_k(R))^3 \oplus \text{Span} [\ \text{curl}(0,0,xy^{i+1}z^{k-i}), \text{curl}(0,x^{i+1}y^{k-i}z,0), \ \text{curl}(x^{k-i}yz^{i+1},0,0), \ i=1,2,\cdots k \ ] \ \},$$

$$W_h = \{ \ q \in L^2(\Omega) \mid q|_R \in P_{k-1}(R), \quad \forall \ R \in \mathcal{T}_h \ \},$$

$$L_h = \{ \ \lambda \in L^2(e) \mid \lambda|_e \in P_k(e) \ \text{ if } e \subset \Omega \ \text{ and } \ \lambda|_e = 0 \ \text{ if } e \subset \partial\Omega \},$$

where curl $F = (\ \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}, \ -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \ )$. The dimension of V_h is that of $(P_k(R))^3$ plus $3k+3$ i.e., $\frac{1}{2}(k+1)(k+2)(k+3) + 3(k+1)$.
This space have been carefully defined in order to have

$$\begin{cases} \text{div } \mathbf{v} \in P_{k-1}(R), \\ \mathbf{v} \cdot n|_{e_i} \in P_k(e_i) \end{cases}$$

The degrees of freedom is determined by the following result.

Lemma 2.2. *For $k \geq 1$ and for any $\mathbf{v} \in V_h$, the relation*
(1) $\int_{e_i} \mathbf{v} \cdot n\varphi_i \, ds = 0, \qquad \forall \ \varphi_i \in P_k(e_i),$
(2) $\int_R \mathbf{v} \cdot \phi_{i-2} \, dx = 0, \qquad \forall \ \phi_{i-2} \in (P_{k-2})^2,$
imply $\mathbf{v} = 0$.

Remark 2.1. *The above definitions for* BDM *space have been designed in order to keep* div $\mathbf{v}$ *in $P_{k-1}(R)$ by adding divergence-free functions to $(P_k(R))^n$ while providing terms with a normal component in $P_k(e_i)$ on each side or face e_i. In the three-dimensional case, there is no unique way to give such a definition. For example, we could have used,*

$$V_h = \{ \ \mathbf{v} \in (L^2(\Omega))^3 \mid \mathbf{v}|_R \in (P_k(R))^3 \oplus \text{Span} [\ \text{curl}(0,0,x^{i+1}yz^{k-i}), \text{curl}(0,xy^{k-i}z^{i+1},0), \ \text{curl}(x^{k-i}y^{i+1}z,0,0), \ i=1,2,\cdots k \ ] \ \}.$$

2.2. **Error estimates.** The analysis of our mixed methods will be simplified by the existence of a projection $\Pi_h \times P_h : V \times W \longrightarrow V_h \times W_h$ having the properties :
(1) P_h is $L^2(\Omega)$-projection.
(2) div $\Pi_h = P_h$ div : $V \longrightarrow W_h$ is surjective.
(3) the following approximation properties hold :

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_0 \leq C\|\mathbf{u}\|_r h^r, \qquad (1 \leq r \leq k+1), \tag{2.1}$$

$$\|\text{div}(\mathbf{u} - \Pi_h \mathbf{u})\|_0 \leq C\|\text{div } \mathbf{u}\|_r h^r, \qquad (0 \leq r \leq k+1), \tag{2.2}$$

$$\|p - P_h p\|_0 \leq C\|p\|_r h^r, \qquad (0 \leq r \leq k). \tag{2.3}$$

(4) orthogonality :

$$(\text{div}(\mathbf{u} - \Pi_h \mathbf{u}), q) = 0, \qquad \forall \ q \in W_h, \tag{2.4}$$

$$(\text{div } \mathbf{v}, \ p - P_h p) = 0, \qquad \forall \ \mathbf{v} \in V_h. \tag{2.5}$$

Let us turn to the analysis of the error in the procedure of (1.4). Subtracting (1.4) from (1.3) and applying (2.5) leads to the error equation

$$\text{(2.6)} \qquad \begin{cases} (c(\mathbf{u}-\mathbf{u}_h),\ \mathbf{v}) - (\operatorname{div}\mathbf{v},\ P_h p - p_h) = 0, \\ (\operatorname{div}(\mathbf{u}-\mathbf{u}_h),\ q) = 0. \end{cases}$$

To obtain error estimates, we need a duality lemma described by Douglas and Robert[9]. First, we will define that Ω is (s+2)-regular if the Dirichlet problem

$$\begin{cases} L^*\phi = \psi, & \forall\ x \in \Omega, \\ \phi = 0, & \forall\ x \in \partial\Omega, \end{cases}$$

is uniquely solvable for $\psi \in L^2(\Omega)$ and if $\|\phi\|_{s+2} \le C\|\psi\|_s$, for all $\psi \in H^s(\Omega)$. The duality lemma is as follows.

Lemma 2.3. *Let Ω be 2-regular. Then for sufficiently small h,*

$$\|P_h p - p_h\|_0 \le C(\|\mathbf{u}-\mathbf{u}_h\|_0 h + \|\operatorname{div}(\mathbf{u}-\mathbf{u}_h)\|_0 h^{2-\delta_{k,0}}).$$

Proof. Let $\psi \in H^0(\Omega)$ and $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ be such that $L^*\phi = \psi$. Then, by (2.4),

$$\begin{aligned}(P_h p - p_h,\ \psi) &= (P_h p - p_h,\ -\operatorname{div}\Pi_h(a\,\mathbf{grad}\phi)) \\ &= (P_h p - p_h,\ \operatorname{div}(a\,\mathbf{grad}\phi - \Pi_h a\,\mathbf{grad}\phi)) - (P_h p - p_h,\ \operatorname{div}(a\,\mathbf{grad}\phi)).\end{aligned}$$

By the first equation of (2.6), we have

$$(P_h p - p_h,\ \psi) = (c(\mathbf{u}-\mathbf{u}_h),\ a\,\mathbf{grad}\phi - \Pi_h(a\,\mathbf{grad}\phi)) - (c(\mathbf{u}-\mathbf{u}_h),\ a\,\mathbf{grad}\phi).$$

Using Green's formula, we can obtain

$$(P_h p - p_h,\ \psi) = (c(\mathbf{u}-\mathbf{u}_h),\ a\,\mathbf{grad}\phi - \Pi_h(a\,\mathbf{grad}\phi)) + (\operatorname{div}(\mathbf{u}-\mathbf{u}_h),\ \phi - P_h\phi),$$

since $\phi \in H_0^1(\Omega)$.
By (2.1), (2.3) and using $\sup_{\psi\in H^0} \frac{(P_h p - p_h,\psi)}{\|\psi\|_0} = \|P_h p - p_h\|_0$, we can obtain the desired result.

Theorem 2.1. *Assume that the Dirichlet problem* (1.1) *has a unique solution and that Ω is 2-regular. Then for h sufficiently small there exists a unique solution $(\mathbf{u}_h, p_h) \in V_h \times W_h$ of the mixed method equation* (1.4) *. Moreover, the error $\mathbf{u}-\mathbf{u}_h$, $p - p_h$ can be estimated by the inequalities*

$$\text{(2.7)} \qquad \|p - p_h\|_0 \le C\|p\|_k h^k,$$

$$\text{(2.8)} \qquad \|\mathbf{u}-\mathbf{u}_h\|_0 \le C\|p\|_{k+1} h^{k+1}.$$

Proof. Since $p - p_h = p - P_h p + P_h p - p_h$, we have

$$\|p - p_h\|_0 \le \|p - P_h p\|_0 + \|P_h p - p_h\|_0.$$

We apply the duality lemma to the error equation (2.6). Then

$$\text{(2.9)} \qquad \|P_h p - p_h\|_0 \le C(\ h\|\mathbf{u}-\mathbf{u}_h\|_0 + \|\operatorname{div}(\mathbf{u}-\mathbf{u}_h)\|_0 h^{\min(2,k)}\ ).$$

Since $\mathbf{u}-\mathbf{u}_h=\mathbf{u}-\Pi_h\mathbf{u}+\Pi_h\mathbf{u}-\mathbf{u}_h$,

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le \|\mathbf{u}-\Pi_h\mathbf{u}\|_0+\|\Pi_h\mathbf{u}-\mathbf{u}_h\|_0.$$

Now we take the test function $\mathbf{v}=\Pi_h\mathbf{u}-\mathbf{u}_h$ in (2.6) to see that

$$\begin{aligned}(c(\Pi_h\mathbf{u}-\mathbf{u}_h),\Pi_h\mathbf{u}-\mathbf{u}_h) &= (\mathrm{div}(\Pi_h\mathbf{u}-\mathbf{u}_h),P_hp-p_h)-(c(\mathbf{u}-\Pi_h\mathbf{u}),\Pi_h\mathbf{u}-\mathbf{u}_h)\\ &= -(c(\mathbf{u}-\Pi_h\mathbf{u}),\Pi_h\mathbf{u}-\mathbf{u}_h).\end{aligned}$$

Then,

$$\|\Pi_h\mathbf{u}-\mathbf{u}_h\|_0 \le C\|\mathbf{u}-\Pi_h\mathbf{u}\|_0.$$

By (2.1) , we have

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le C\|\mathbf{u}\|_t h^t, \qquad (1\le t\le k+1). \tag{2.10}$$

Since $(\mathrm{div}(\mathbf{u}-\mathbf{u}_h),q)=0$ and $(\mathrm{div}(\mathbf{u}-\mathbf{u}_h),q)=(\mathrm{div}(\Pi_h\mathbf{u}-\mathbf{u}_h),q)$, $\forall\ q\in W_h$, we know that $\mathrm{div}(\Pi_h\mathbf{u}-\mathbf{u}_h)=0$. Hence,

$$\|\mathrm{div}(\mathbf{u}-\mathbf{u}_h)\|_0 \le C\|\mathrm{div}(\mathbf{u}-\Pi_h\mathbf{u})\|_0 \le C\|\mathrm{div}\ \mathbf{u}\|_s h^s,\ (0\le s\le k+1). \tag{2.11}$$

If (2.10) and (2.11) are substituted into (2.9) , then

$$\|P_hp-p_h\|_0 \le C(\|\mathbf{u}\|_t h^{t+1}+\|\mathrm{div}\ \mathbf{u}\|_s h^{\min(2,k)+s}).$$

By (2.3), we obtain,

$$\|p-p_h\|_0 \le C(\|p\|_r h^r+\|\mathbf{u}\|_t h^{t+1}+\|\mathrm{div}\ \mathbf{u}\|_s h^{\min(2,k)+s}), \quad (0\le r\le k).$$

Thus for small h and the choice $r=t+1=s+\min(2,k)$

$$\|p-p_h\|_0 \le C\|p\|_k h^k,$$

because of $\|\mathbf{u}\|_{r-1}+\|\mathrm{div}\ \mathbf{u}\|_{r-2}\le C\|p\|_r$. It then follows immediately that

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le C\|p\|_{k+1}h^{k+1}.$$

2.3. Comparison with Raviart-Thomas space. The Raviart-Thomas (RT) space $V_h\times W_h$ on triangles of order k is defined by [1] :

$$\begin{aligned} V_h &= \{\ \mathbf{v}\in (L^2(\Omega))^2\ |\ \mathbf{v}|_T\in (P_k(T))^2\ \oplus\ \mathrm{Span}\ (xP_k(T)),\quad \forall\ T\in\mathcal{T}_h\ \},\\ W_h &= \{\ q\in L^2(\Omega)\ |\ q|_T\in P_k(T),\quad \forall\ T\in\mathcal{T}_h\ \},\\ L_h &= \{\ \lambda\in L^2(e)\ |\ \lambda|_e\in P_k(e)\ \text{ if } e\subset\Omega\ \text{ and } \lambda|_e=0 \text{ if } e\subset\partial\Omega\ \}.\end{aligned}$$

It can easily be checked that the dimensions of V_h and W_h are given by $\dim(V_h(T))=(k+1)(k+3)$ and $\dim(W_h(T))=\frac{1}{2}(k+1)(k+2)$, respectively.
To give a more precise definition for rectangular elements, we shall define

$$P_{k,l}(R)=\{\ p(x,y)\ |\ p(x,y)=\sum_{\substack{i\le k\\ j\le l}} a_{ij}x^iy^j,\ a_{ij}\in\mathbb{R}\ \},$$

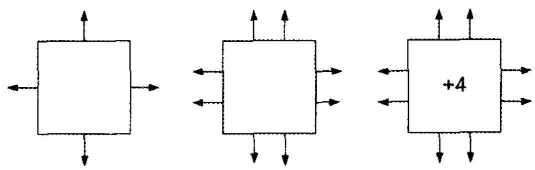
the space of polynomials of degree $\leq k$ in x and $\leq l$ in y.
Let $Q_k(R) = P_{k,k}(R)$. Then the RT space on rectangles of order k is given by [1], [9] :

$$V_h = \{ \mathbf{v} \in (L^2(\Omega))^2 \mid \mathbf{v}|_R \in P_{k+1,k} \times P_{k,k+1}, \quad \forall R \in \mathcal{T}_h \},$$
$$W_h = \{ q \in L^2(\Omega) \mid q|_R \in Q_k(R), \quad \forall R \in \mathcal{T}_h \},$$
$$L_h = \{ \lambda \in L^2(e) \mid \lambda|_e \in P_k(e) \text{ if } e \subset \Omega \text{ and } \lambda|_e = 0 \text{ if } e \subset \partial\Omega \}.$$

We have by a simple count, $\dim(V_h(R)) = 2(k+1)(k+2)$ and $\dim(W_h(R)) = (k+1)^2$. The error estimates are given by [9], [10] :

$$\|p - p_h\|_0 \leq C\|p\|_{k+1+\delta_{k,0}} h^{k+1},$$
$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C\|p\|_{k+1} h^{k+1}.$$

Remark 2.2. BDM *space lies between corresponding* RT *spaces, i.e.,* $\mathrm{RT}_{k-1} \subset \mathrm{BDM}_k \subset \mathrm{RT}_k$. *For instance, in the case of* $k = 1$ *we have*



Remark 2.3. BDM *space have smaller dimension than the* RT *space of the same index. For the case of rectangular elements, the dimension of* V_h *for* BDM *space is* k^2+3k+4, *whereas the dimension of* V_h *for* RT *space is* $2k^2+6k+4$, *which is essentially twice as great. Also* $\dim(W_h(R))$ *of* BDM *scalar part is* $\frac{1}{2}k(k+1)$ *versus* $(k+1)^2$ *for the* RT *scalar part. So the solution of the linear algebraic system associated with* BDM *space is simpler than that associated with* RT *space.*

Remark 2.4. BDM *space and* RT *space provide asymptotic error estimates for the velocity but different error estimates for the pressure. After post-processing, which is mentioned at the next section, they also provide same error estimates for the pressure.*

3. Analysis of the hybrid form and post-processing

3.1. Error estimates for the Lagrange multiplier.

Defining the norms on $P_k(e)$

$$|\lambda_h|_{0,h}^2 = \sum_{e\subset\Omega} \|\lambda_h\|_{0,e}^2,$$
$$|\lambda_h|_{-\frac{1}{2},h}^2 = \sum_{e\subset\Omega} |e|\|\lambda_h\|_{0,e}^2.$$

We now compare λ_h with $Q_h p$, where Q_h be the orthogonal projection defined locally by $L^2(e)$-projection onto $P_k(e)$ for $e \subset \Omega$.

Theorem 3.1. *If* $(\mathbf{u}_h, p_h, \lambda_h) \in V_h \times W_h \times L_h$ *is the solution of* (1.8) *then*

$$\|\lambda_h - Q_h p\|_{0,e} \leq C(h_T^{\frac{1}{2}} \|\mathbf{u} - \mathbf{u}_h\|_{0,T} + h_T^{-\frac{1}{2}} \|P_h p - p_h\|_{0,T}), \tag{3.1}$$

$$|\lambda_h - Q_h p|_{-\frac{1}{2},h} \leq C(h\|\mathbf{u} - \mathbf{u}_h\|_0 + \|P_h p - p_h\|_0). \tag{3.2}$$

Proof. Clearly, (3.2) is an immediate consequence of (3.1). In order to prove (3.1), let us consider $e \subset \Omega \cap T$. It is easy to prove that there exists a unique $\mathbf{v} \in V_h$ having support in T such that

$$\mathbf{v} \cdot n_e = \lambda_h - Q_h p, \qquad \text{on e,} \tag{3.3}$$

$$\mathbf{v} \cdot n_T = 0, \qquad \text{on } \partial T \setminus e, \tag{3.4}$$

$$(\mathbf{v}, \mathbf{grad}\ \phi)_T = 0, \qquad \phi \in P_{k-1}(T), \tag{3.5}$$

$$(\mathbf{v}, \text{curl}\ \psi)_T = 0, \qquad \psi \in B_{k+1}(T). \tag{3.6}$$

Then a simple scaling argument shows that

$$h_T\|\mathbf{v}\|_{1,T} + \|\mathbf{v}\|_{0,T} \leq C h_T^{\frac{1}{2}} \|\lambda_h - Q_h p\|_{0,e}. \tag{3.7}$$

We may choose $\mathbf{v}$ as the test function in the first equation of (1.8) which gives using (3.3),

$$(c\mathbf{u}_h,\ \mathbf{v})_T - (\text{div}\ \mathbf{v},\ p_h)_T + \langle\ \lambda_h,\ \lambda_h - Q_h p\ \rangle_e = 0. \tag{3.8}$$

Since $c\mathbf{u} = -\mathbf{grad}\ p$, Green's formula implies

$$(c\mathbf{u},\ \mathbf{v})_T - (\text{div}\ \mathbf{v},\ p)_T + \langle\ p,\ \lambda_h - Q_h p\ \rangle_e = 0. \tag{3.9}$$

Subtracting (3.9) from (3.8), we have

$$\begin{aligned} \|\lambda_h - Q_h p\|_{0,e}^2 &= \langle\ \lambda_h - p,\ \lambda_h - Q_h p\ \rangle_e \\ &= (c(\mathbf{u} - \mathbf{u}_h),\ \mathbf{v})_T - (\text{div}\ \mathbf{v},\ P_h p - p_h). \end{aligned} \tag{3.10}$$

Finally, (3.7) and (3.10) gives (3.1).

3.2. **Post-processing of the BDM mixed methods.** We have two pieces of information about λ_h which is a polynomial of degree $\leq k$ on each e, and p_h which is a polynomial of degree $\leq k-1$ in each element. We shall use λ_h and p_h to define better approximation p_h^*.
First, consider the triangular case.

Lemma 3.1. *Let* $T \in \mathcal{T}_h$ *be a triangle with edges* e_1, e_2, e_3 . *Then for all* $\lambda_h \in L^2(e_i)$ $(i = 1, 2, 3)$ *and* $p_h \in L^2(T)$ *there exists a unique* $p_h^* \in P_{k+1}(T)$ *such that*

(1) k = *nonnegative even integer*

$$\int_{e_i} (p_h^* - \lambda_h)\phi \; ds = 0, \qquad \forall\; \phi \in P_k(e_i), \quad i = 1,2,3,$$

$$\int_T (p_h^* - p_h)\psi \; dx = 0, \qquad \forall\; \psi \in P_{k-2}(T),$$

(2) $k = 1$

$$\int_{e_i} (p_h^* - \lambda_h) \; ds = 0, \qquad i = 1,2,3,$$

$$\int_T (p_h^* - p_h)\psi \; dx = 0, \qquad \forall\; \psi \in P_1(T),$$

(3) $k = 3$

$$\int_{e_i} (p_h^* - \lambda_h)\phi \; ds = 0, \qquad \forall\; \phi \in P_2(e_i), \quad i = 1,2,3,$$

$$\int_T (p_h^* - p_h)\psi \; dx = 0, \qquad \forall\; \psi \in P_2(T).$$

Moreover,

$$\|p_h^*\|_{0,T} \le C(\; \|p_h\|_{0,T} + h_T^{\frac{1}{2}} \sum_{i=1}^{3} \|\lambda_h\|_{0,e_i} \;). \tag{3.11}$$

Other *ad hoc* choices may be made for each particular odd k. However we did not find an elegant general structure.

On the other hand, it is possible to give an analogue of above lemma for all $k \ge 1$ in the rectangular case.

Lemma 3.2. *Let $R \in \mathcal{T}_h$ be a rectangle with edges e_1, e_2, e_3, e_4. Then for all $\lambda_h \in L^2(e_i)$ $(i = 1,2,3,4)$ and $p_h \in L^2(R)$, there exists a unique $p_h^* \in P_{k+1}(R) \oplus$* Span $(x^{k+1}y, xy^{k+1}, q^{k+1})$ *where*

$$q^{k+1}(x,y) = \begin{cases} x^{k+2}y - xy^{k+2} & \text{if } k = \text{odd} \\ x^{k+2} - y^{k+2} & \text{if } k = \text{even}, \end{cases}$$

such that

$$\int_{e_i} (p_h^* - \lambda_h)\phi \; ds = 0, \qquad \forall\; \phi \in P_k(e_i), \quad i = 1,2,3,4,$$

$$\int_R (p_h^* - p_h)\psi \; dx = 0, \qquad \forall\; \psi \in P_{k-3}(R) \; + \; \text{Span}\; (l_{k-1}(x) l_{k-1}(y)),$$

where l_{k-1} is the Laguerre polynomial of degree $k-1$.
Moreover,

$$\|p_h^*\|_{0,R} \le C(\; \|p_h\|_{0,R} + h_R^{\frac{1}{2}} \sum_{i=1}^{4} \|\lambda_h\|_{0,e_i} \;). \tag{3.12}$$

We now prove that p_h^* indeed approximates p with a higher order of accuracy than p_h .

Theorem 3.2. *Let p be the solution of* (1.1) *and* $(\mathbf{u}_h, p_h, \lambda_h)$ *the solution of* (1.8) . *Define* p_h^* *by* Lemma 3.1 *in triangular case and by* Lemma 3.2 *in rectangular case. Then,*

$$\|p - p_h^*\|_0 \leq C\|p\|_{k+1-\delta_{k,1}} h^{k+2-\delta_{k,1}}.$$

Proof. Let us consider the case of positive, even k on the triangular elements. We first define $p_h^\sharp \in P_{k+1}(T)$ by

$$\int_{e_i} (p_h^\sharp - p)\phi \, ds = 0, \qquad \forall \ \phi \in P_k(e_i),$$

$$\int_T (p_h^\sharp - p)\psi \, dx = 0, \qquad \forall \ \psi \in P_{k-2}(T).$$

Then Lemma 3.1 implies existence and uniqueness of $p_h^\sharp$. By standard arguments, we have

$$\|p - p_h^\sharp\|_0 \leq C\|p\|_{k+2} h^{k+2}.$$

Let $q_h^* = p_h^* - p_h^\sharp$. Since $q_h^* = p_h^* - \lambda_h + \lambda_h - Q_h p + Q_h p - p_h^\sharp$, we know that

$$\int_{e_i} q_h^* \phi \, ds = \int_{e_i} (\lambda_h - Q_h p)\phi \, ds, \quad \forall \ \phi \in P_k(e_i), \quad i = 1, 2, 3.$$

Similarly, we can obtain,

$$\int_T q_h^* \psi \, dx = \int_T (p_h - P_h p)\psi \, dx, \quad \forall \ \psi \in P_{k-2}(T).$$

Using formula (3.11) with $p_h := p_h - P_h p$ and $\lambda_h := \lambda_h - Q_h p$, we obtain for each $T \in \mathcal{T}_h$ that

$$\|p_h^* - p_h^\sharp\|_{0,T} \leq C(\ \|p_h - P_h p\|_{0,T} + h_T^{\frac{1}{2}} \|\lambda_h - Q_h p\|_{0,\partial T} \ ). \tag{3.13}$$

Combining (3.13) with (3.1) and then using (2.3), (2.8), we obtain

$$\|p - p_h^*\|_0 \leq C\|p\|_{k+1} h^{k+2}.$$

We can apply a similar argument to the other case.

The resulting superconvergent approximation of the pressure is asymptotically of the same order as that for the similarly modified version of the Raviart-Thomas method.

4. Projection finite element method

In this chapter, we show that for BDM mixed method, if the projection method's finite element space M_h satisfies two conditions, then the two approximation methods are equivalent. And also we construct M_h for the triangular and rectangular elements.

4.1. **Equivalent projection finite element method.** In order to introduce the projection finite element method, let M_h be some as yet unspecified finite dimensional space defined over $\mathcal{T}_h$. For $\phi \in L^2(\Omega)$, let $R_h : (L^2(\Omega))^n \longrightarrow V_h$ be the weighted $(L^2(\Omega))^n$-projection defined by

$$(c(\phi - R_h\phi),\ \mathbf{v}) = 0, \quad \forall\ \mathbf{v} \in V_h. \tag{4.1}$$

Then the projection finite element method is to seek $m_h \in M_h$ such that

$$\sum_{K\in\mathcal{T}_h} (R_h(a\nabla m_h),\ \nabla n)_K = (f,\ P_h n), \quad \forall\ n \in M_h. \tag{4.2}$$

Theorem 4.1. *For a given mixed method* (1.8), *if* M_h *satisfies*

(C1) *For* $n \in M_h$, *if* $(\nabla n,\ \mathbf{v})_K = 0$ *for all* $\mathbf{v} \in V_h(K)$ *and all* $K \in \mathcal{T}_h$ *then* $n = 0$.

(C2) *For* $n \in M_h$, *its projection* $Q_h n$ *can be uniquely defined on each edge and for any* $(p_h, \lambda_h) \in W_h \times L_h$, *there exist* $n_1, n_2 \in M_h$, *such that*

$$\begin{cases} P_h n_1 = p_h \\ Q_h n_1 = 0 \end{cases} \quad \textit{and} \quad \begin{cases} P_h n_2 = 0 \\ Q_h n_2 = \lambda_h, \end{cases}$$

then the projection finite element method (4.2) *is well-defined and equivalent to it by the following relations :*

$$\mathbf{u}_h = -R_h(a\nabla m_h), \tag{4.3}$$

$$p_h = P_h m_h, \tag{4.4}$$

$$\lambda_h = Q_h m_h. \tag{4.5}$$

Proof. To show that M_h give rise to a reasonable finite element method defined by (4.2), we require that there exists a unique solution to the problem. It suffices to show that if $m_h \in M_h$ satisfies

$$\sum_{K\in\mathcal{T}_h} (\ R_h(a\nabla m_h), \nabla n\ )_K = 0, \quad \forall\ n \in M_h$$

then $m_h = 0$. Taking $n = m_h$ and using (4.1), we obtain

$$\begin{aligned} (\ R_h(a\nabla m_h), \nabla m_h\ )_K &= (\ a^{-1}R_h(a\nabla m_h), a\nabla m_h\ )_K \\ &= (\ a^{-1}R_h(a\nabla m_h), R_h(a\nabla m_h)\ )_K \\ &= 0. \end{aligned}$$

Then the R_h-projection of $a\nabla m_h$ is zero. By $(C1)$, we have $m_h = 0$.
Now, we will show that two schemes (1.8) and (4.2) are equivalent. If $m_h \in M_h$ satisfies

(4.2) then by relations (4.3) – (4.5) and integration by parts, we have

$$
\begin{aligned}
&(c\mathbf{u}_h, \mathbf{v}) - \sum_{K\in\mathcal{T}_h} (\text{div } \mathbf{v}, p_h)_K + \sum_{K\in\mathcal{T}_h} \langle \mathbf{v}\cdot n_K, \lambda_h \rangle_{\partial K} \\
&= -(cR_h(a\nabla m_h), \mathbf{v}) - \sum_{K\in\mathcal{T}_h} (\text{div } \mathbf{v}, P_h m_h)_K + \sum_{K\in\mathcal{T}_h} \langle \mathbf{v}\cdot n_K, Q_h m_h \rangle_{\partial K} \\
&= -\sum_{K\in\mathcal{T}_h} (\nabla m_h, \mathbf{v})_K - \sum_{K\in\mathcal{T}_h} (\text{div } \mathbf{v}, m_h)_K + \sum_{K\in\mathcal{T}_h} \langle \mathbf{v}\cdot n_K, m_h \rangle_{\partial K} \\
&= 0,
\end{aligned}
$$

this is the first equation of (1.8).
Conversely, for any $n \in M_h$,

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_h} (R_h(a\nabla m_h), \nabla n)_K &= \sum_{K\in\mathcal{T}_h} (-\mathbf{u}_h, \nabla n)_K \\
&= \sum_{K\in\mathcal{T}_h} (\text{div } \mathbf{u}_h, n)_K - \sum_{K\in\mathcal{T}_h} \langle \mathbf{u}_h\cdot n_K, n\rangle_{\partial K}.
\end{aligned}
$$

By introducing two projection operators P_h and Q_h, using $(C2)$, and finally definition of (1.8), we see that

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_h} (R_h(a\nabla m_h), \nabla n)_K &= \sum_{K\in\mathcal{T}_h} (\text{div } \mathbf{u}_h, P_h n)_K - \sum_{K\in\mathcal{T}_h} \langle \mathbf{u}_h\cdot n_K, Q_h n\rangle_{\partial K} \\
&= (f, P_h n), \qquad \forall\, n \in M_h.
\end{aligned}
$$

Corollary 4.1. *If a given mixed finite element method* (1.8) *is equivalent to the projection finite element method* (4.2) *by the relations* (4.3) – (4.5), *then* $\dim W_h + \dim L_h \leq \dim M_h \leq \dim V_h + 1$.

This result can be used to bound the dimension of M_h.

4.2. **Construction of M_h for the BDM spaces.** Now, we discuss the problem of how to construct an appropriate nonconforming space M_h. Consider the localization of the condition $(C1)$:
(C1') For $n \in M_h(K)$, if $(\nabla n, \mathbf{v})_K = 0$ for all $\mathbf{v} \in V_h(K)$ then n is constant on K.

Theorem 4.2. *Suppose that $V_h \times W_h$ is a mixed finite element space such that $W_h = \nabla\cdot V_h$, $1 \in W_h(K)$ for each $K \in \mathcal{T}_h$, and $1 \in L_h(e)$ for each edges. If M_h satisfies $(C1')$ for each $K \in \mathcal{T}_h$ and $(C2)$ then M_h satisfies $(C1)$.*

We derive a local criterion that guarantees the equivalence in the case of mixed spaces possessing the usual vector projection operator.

Theorem 4.3. *Suppose that K is convex and that $V_h(K) \times W_h(K)$ is a mixed finite element space such that $W_h(K) = \nabla\cdot V_h(K)$, $1 \in W_h(K)$, $1 \in L_h(e)$ and there exists* *an operator* $\Phi : (H^1(K))^n \longrightarrow V_h(K)$ *satisfying*

$$\begin{cases} \text{div } \Phi_h \mathbf{v} &= P_h(\text{div } \mathbf{v}), & (4.6)\\ (\Phi_h \mathbf{v}) \cdot n &= Q_h(\mathbf{v} \cdot n). & (4.7) \end{cases}$$

If $M_h(K)$ *is a space of functions such that*

$$\dim\,(M_h(K)) = \dim\,(W_h(K)) + \dim\,(L_h(\partial K)),$$

with unisolvent degrees of freedom described by
$(DF1)$ $(n, p)_K$ *for all* p *in a basis of* $W_h(K)$,
$(DF2)$ $(n, \lambda)_{\partial K}$ *for all* λ *in a basis of* $L_h(\partial K)$
and if $M_h(K)$ *contains the constant functions then* $M_h(K)$ *satisfies* $(C1')$ *and* $(C2)$.

We are now in a position to construct space M_h that gives rise to projection finite element methods that are equivalent to BDM mixed methods defined over triangular or rectangular elements. These mixed spaces satisfy the condition of Theorem 4.2.2, it remains only to define space M_h of the correct dimension and prove the unisolvence of (DF).

First, consider the BDM spaces on triangles.
Let us define

$$M_h(T) = \begin{cases} \{\ v \in P_{k+2}(T) \mid v|_e \in P_{k+1}(e)\ \}, & \text{if } k \text{ is even },\\ \{\ v \in P_{k+2}(T) \mid v|_e \in P_k(e)\ \oplus\ (\ P_{k+2}(e) \setminus P_{k+1}(e)\ )\}, & \text{if } k \text{ is odd.} \end{cases}$$

We first show that $M_h(T)$ has the correct dimension. $\dim(P_{k+2}(T)) = \frac{1}{2}(k+4)(k+3)$ is exactly three more than $\dim(W_h(T)) + 3\dim(L_h(\partial T)) = \frac{1}{2}(k+1)(k+6)$. For simplicity, assume that k is even. Let $l_i,\ i = 1, 2, 3$ be the barycentric coordinates defined on T to be the unique affine functions that take the value one at vertex i, and the value zero on the opposite edge. For any $\xi \in P_{k+2}(T)$, we can write that

$$\xi(x) = \sum_{0 \le i+j \le k+2} a_{ij} l_1^i(x) l_2^j(x), \quad a_{ij} \in \mathbb{R}.$$

If $\xi \in M_h(T)$, $\xi|_{e_i} \in P_{k+1}(e_i)$ implies that $a_{0,k+2} = a_{k+2,0} = 0$ for $i = 1, 2$. Since $l_2 = 1 - l_1$ on e_3,

$$\xi|_{e_3} = \sum_{0 \le i+j \le k+2} a_{ij} l_1^i (1 - l_1)^j \in P_{k+1}(e_3)$$

implies $\sum_{i+j=k+2} (-1)^j a_{ij} = 0$. So M_h has the correct dimension.
Now we consider the unisolvence of (DF). Suppose that $\xi \in M_h(T)$ has degrees of freedom (DF) equal to zero. The $(DF2)$ imply that on each edge e, ξ is a polynomial of degree $k+1$ if k is even and $k+2$ if k is odd. Since ξ is a polynomial of odd degree and the odd degree polynomials are odd functions, traversing ∂T, we see that ξ must vanish identically on the the boundary. As a consequence, we can write that $\xi = l_1 l_2 l_3 p$ for some $p \in P_{k-1}(T)$. Now $(DF1)$ shows that $(l_1 l_2 l_3 p,\ p)_T = 0$, which gives that $\xi = 0$.

For the BDM spaces over rectangular elements, the space M_h can be defined by $M_h(R) = P_{k-1}(R) \oplus A^k(R) \oplus B^k(R)$, where

$$A^k(R) = \{ \sum_{i=0}^{k} [\, a_{i,1}p_{k+1}(x) + a_{i,2}p_{k+2}(x) \,]p_i(y),\ a_{i,j} \in \mathbb{R} \},$$

$$B^k(R) = \{ \sum_{i=0}^{k} p_i(x)[\, b_{i,1}p_{k+1}(y) + b_{i,2}p_{k+2}(y) \,],\ b_{i,j} \in \mathbb{R} \}.$$

Note that $\dim(A^k(R)) = \dim(B^k(R)) = 2(k+1)$, so it is trivial to verify that $\dim(M_h(R)) = \dim(W_h(R)) + 4\dim(L_h(\partial R))$. Since the proof of unisolvence is similar to that given above, we omit it.

5. One way to solve the first order BDM mixed method

Consider the lowest order BDM space over rectangular elements.
From section 2.1, we have

$$V_h(R) = \{ \mathbf{v} \mid \mathbf{v} = \mathbf{P}_1(x,y) + r\ \mathrm{curl}(x^2y) + s\ \mathrm{curl}(xy^2) \}$$

and $\dim(V_h(R)) = 8$.
A function $\mathbf{v} \in V_h(R)$ is of the form

$$\begin{cases} v_1 = a_1 + b_1x + c_1y + rx^2 + 2sxy, \\ v_2 = a_2 + b_2x + c_2y - 2rxy - sy^2, \end{cases}$$

and the degrees of freedom of $\mathbf{v}$ may be chosen by Lemma 2.2. Following Brezzi and Fortin [7], we begin by constructing basis functions of the V_h on reference element $R = [0,1] \times [0,1]$. We will take test functions $\{1, x - \frac{1}{2}\}$ or $\{1, y - \frac{1}{2}\} \in P_1(\partial R)$. For example,

$$\int_{e_1} \phi_1 \cdot n_1\, ds = 1,$$

$$\int_{e_1} \phi_1 \cdot n_1(y - \frac{1}{2})\, ds = \int_{e_3} \phi_1 \cdot n_3\, ds = \int_{e_3} \phi_1 \cdot n_3(y - \frac{1}{2})\, ds = 0,$$

$$\int_{e_i} \phi_1 \cdot n_i\, ds = \int_{e_i} \phi_1 \cdot n_i(x - \frac{1}{2})\, ds = 0, \quad i = 2,4.$$

Then we see that,

$$-\int_0^1 (a_1 + c_1y)\, dy = 1, \qquad -\int_0^1 (a_1 + c_1y)(y - \frac{1}{2})\, dy = 0,$$

$$-\int_0^1 (a_2 + b_2x)\, dx = 0, \qquad -\int_0^1 (a_2 + b_2x)(x - \frac{1}{2})\, dx = 0,$$

$$\int_0^1 (a_1+b_1+r)+(c_1+2s)y\,dy=0,$$
$$\int_0^1 [(a_1+b_1+r)+(c_1+2s)y](y-\frac{1}{2})\,dy=0,$$
$$\int_0^1 (a_2+c_2-s)+(b_2-2r)x\,dx=0,$$
$$\int_0^1 [(a_2+c_2-s)+(b_2-2r)x](x-\frac{1}{2})\,dx=0.$$

Thus $a_1=-1,\ b_1=1,\ a_2=b_2=c_1=c_2=r=s=0$ and so $\phi_1=\binom{-1+x}{0}$.
By similar calculation, we can obtain $\phi_i,\ i=2,\cdots 8$. Therefor, the basis functions of BDM_1 are given by

$$\phi_1^n=\begin{pmatrix}-1+x\\0\end{pmatrix},\qquad \phi_1^c=\begin{pmatrix}6(1-x)(1-2y)\\6y(1-y)\end{pmatrix},$$
$$\phi_2^n=\begin{pmatrix}0\\-1+y\end{pmatrix},\qquad \phi_2^c=\begin{pmatrix}6x(1-x)\\6(1-2x)(1-y)\end{pmatrix},$$
$$\phi_3^n=\begin{pmatrix}x\\0\end{pmatrix},\qquad \phi_3^c=\begin{pmatrix}-6x(1-2y)\\6y(1-y)\end{pmatrix},$$
$$\phi_4^n=\begin{pmatrix}0\\y\end{pmatrix},\qquad \phi_4^c=\begin{pmatrix}6x(1-x)\\-6(1-2x)y\end{pmatrix}.$$

Remark 5.1. $\phi_i^n,\ i=1,\cdots,4$, *do not contain curl part of continuous quadratic functions and be the same basis as* RT_0. *And we also know that* div $\phi_i^c=0,\ i=1,\cdots,4$.

Let $\mathbf{u}_h=\mathbf{u}_h^n+\mathbf{u}_h^c$, where $\mathbf{u}_h^n$ is the part of non curl and $\mathbf{u}_h^c$ is the curl part. Then we have discretization formula by mixed finite element methods

$$\text{(5.1)}\qquad \begin{cases} (c\mathbf{u}_h^n,\ \mathbf{v})+(c\mathbf{u}_h^c,\ \mathbf{v})-(\mathrm{div}\ \mathbf{v},\ p_h)=0, & \forall \mathbf{v}\in V_h,\\ (\mathrm{div}\ \mathbf{u}_h^n,\ q)+(\mathrm{div}\ \mathbf{u}_h^c,\ q)=(f,\ q), & \forall q\in W_h. \end{cases}$$

First, take $\mathbf{v}=\phi_i^c$ and then $\phi_i^n,\ i=1,\cdots,4$. Since div $\phi_i^c=0$, we have $(c\mathbf{u}_h^n,\phi_i^c)+(c\mathbf{u}_h^c,\phi_i^c)=0$ and $(\mathrm{div}\ \mathbf{u}_h^n,q)=(f,\ q)$. This formula leads to linear system of the form

$$\text{(5.2)}\qquad \begin{cases} A_1\mathbf{u}_h^n+A_2\mathbf{u}_h^c=0,\\ B^t\mathbf{u}_h^n=f. \end{cases}$$

For the case of $\mathbf{v}=\phi_i^n$, we obtain $(c\mathbf{u}_h^n,\phi_i^n)+(c\mathbf{u}_h^c,\phi_i^n)-(\mathrm{div}\ \phi_i^n,p_h)=0$ and so

$$\text{(5.3)}\qquad A_3\mathbf{u}_h^n+A_1^t\mathbf{u}_h^c-Bp_h=0.$$

Since $\mathbf{u}_h^c=-A_2^{-1}A_1\mathbf{u}_h^n$, we put it to (5.3) to obtain

$$(A_3-A_1^tA_2^{-1}A_1)\mathbf{u}_h^n-Bp_h=0.$$

Hence,

$$\mathbf{u}_h^n = (A_3 - A_1^t A_2^{-1} A_1)^{-1} B p_h. \tag{5.4}$$

Substituting (5.4) into the second equation in (5.2) yields

$$B^t (A_3 - A_1^t A_2^{-1} A_1)^{-1} B p_h = f.$$

It can be shown that this system is also symmetric and positive-definite. In particular, when $\mathcal{T}_h$ is the partition of Ω into squares and node-numbering is appropriately given, matrices A_2 and A_3 are block diagonal, so that $(A_3 - A_1^t A_2^{-1} A_1)$ is invertible in a very simple way. But we found that the coefficient matrix for p_h is rather complicated for the general cases.

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Department of Mathematics,
KAIST, Taejon, Korea, 305-701,
email: lm97jh@kaist.ac.kr