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GENERALIZED SYSTEMS OF RELAXED g-γ-r-COCOERCIVE NONLINEAR VARIATIONAL INEQUALITIES AND PROJECTION METHODS

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ABSTRACT Let K be a nonempty closed convex subset of a real Hilbert space H. Approximation solvability of a system of nonlinear variational inequality (SNVI) problems, based on the convergence of projection methods, is given as follows: find elements

$$x^*, y^* \in H \text{ such that } g(x^*), g(y^*) \in K \text{ and}$$

$$< \rho T(y^*) + g(x^*) - g(y^*), g(x) - g(x^*) \geq 0 \ \forall \ g(x) \in K \text{ and for } \rho > 0$$

$$< \eta T(x^*) + g(y^*) - g(x^*), g(x) - g(y^*) \geq 0 \ \forall \ g(x) \in K \text{ and for } \eta > 0,$$

where T: H→ H is a relaxed g-γ-r-cocoercive and g-μ-Lipschitz continuous nonlinear mapping on H and g: H → H is any mapping on H. In recent years general variational inequalities and their algorithmic applications have assumed a central role in the theory of variational methods. This two-step system for nonlinear variational inequalities offers a great promise and more new challenges to the existing theory of general variational inequalities in terms of applications to problems arising from other closely related fields, such as complementarity problems, control and optimizations, and mathematical programming.

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1. Introduction

Verma [5] presented a new system of nonlinear strongly monotone variational inequalities and initiated the approximation solvability of this system based on the convergence of projection methods. Projection methods have been applied widely to problems arising from mathematical and physical sciences, especially from complementarity problems, convex quadratic programming, and variational inequality problems. More research developments on the approximation solvability of a system of nonlinear variational inequalities are followed by Nie, Liu, Kim and Kang [4], Verma [10] and others. In this paper, we intend to consider the approximation solvability of a system of nonlinear relaxed g-γ-r-cocoercive variational inequalities in a Hilbert space setting. The obtained results complement the investigations of Verma [5, 10], Nie et al. [4] and others. For a better account on general variational inequality problems and related mappings, we refer to [1-20]. The notion of the relaxed cocoercivity is more general than the well-known concepts of cocoercivity and strong monotonicity.

Let H be a real Hilbert space with the inner product $< x, y >$ and norm $\| x \|$ for $x,y \in H$. Let $T,g: H \to H$ be mappings on H and K be a nonempty closed convex subset of H. We consider a system of nonlinear variational inequality (abbreviated as SNVI) problems as follows: determine elements $x^*, y^* \in H$ such that $g(x^*), g(y^*) \in K$ and

$$< \rho T(y^*) + g(x^*) - g(y^*), g(x) - g(x^*) > \geq 0 \ \forall\ g(x) \in K \text{ and for } \rho > 0 \tag{1.1}$$

$$< \eta T(x^*) + g(y^*) - g(x^*), g(x) - g(y^*) > \geq 0 \ \forall\ g(x) \in K \text{ and for } \eta > 0. \tag{1.2}$$

The SNVI (1.1)-(1.2) problem is equivalent to the following projection formulas

$$g(x^*) = P_K [g(y^*) - \rho T(y^*)] \text{ for } \rho > 0$$

$$g(y^*) = P_K [g(x^*) - \eta T(x^*)] \text{ for } \eta > 0 ,$$

where P_K is the projection of H onto K.

We note that for $\eta = 0$, the SNVI (1.1)-(1.2) problem reduces to the NVI problem: determine an element $x^* \in H$ such that $g(x^*) \in K$ and

$$< T(x^*), g(x) - g(x^*) > \geq 0 \ \forall\ g(x) \in K. \tag{1.3}$$

Let K be a closed convex cone of H. The SNVI (1.1)-(1.2) problem is equivalent to a system of nonlinear complementarities (abbreviated as SNC): find the elements $x^*, y^* \in H$ such that $g(x^*)$, $g(y^*) \in K$, $T(y^*) \in K^*$, $T(x^*) \in K^*$ and,

$$< \rho T(y^*) + g(x^*) - g(y^*), g(x^*) > = 0 \text{ for } \rho > 0, \tag{1.4}$$
$$< \eta T(x^*) + g(y^*) - g(x^*), g(y^*) > = 0 \text{ for } \eta > 0, \tag{1.5}$$

where K^* is a polar cone to K defined by

$$K^* = \{f \in H : < f, g(x) > \geq 0 \ \forall \ g(x) \in K\}.$$

Now we need to recall the following auxiliary result, most commonly used in the context of approximation solvability of nonlinear variational inequality problems based on iterative procedures.

Lemma 1.1. [3] For an element $z \in H$, we have

$$x \in K \text{ and } < x - z, y - x > \geq 0 \ \forall \ y \in K \text{ if and only if } x = P_K(z).$$

Definition 1.1. A mapping $T: H \to H$ is called:

(i) *monotone* if for each $x,y \in H$, we have

$$< T(x) - T(y), x - y > \geq 0.$$

(ii) *r-strongly monotone* if for each $x,y \in H$, we have

$$< T(x) - T(y), x - y > \geq r \ ||x - y||^2 \text{ for a constant } r > 0.$$

(iii) *r-expansive* if

$$||T(x) - T(y)|| \geq r \ ||x - y||.$$

(iv) *expansive* if

$$||T(x) - T(y)|| \geq ||x - y||.$$

(v) s-*Lipschitz continuous* (or *Lipschitzian*) if there exists a constant $s \geq 0$ such that

$$\|T(x) - T(y)\| \le s \|x - y\| \ \forall \ x,y \in H.$$

(vi) *μ-cocoercive* [1, 5] if for each x,y $\in$ H, we have

$$< T(x) - T(y), x - y > \ge \mu \|T(x) - T(y)\|^2 \text{ for a constant } \mu > 0.$$

Proposition 1.1. If a mapping T: H $\to$ H is $(1/\mu)$-Lipschitz continuous and $(1/\mu)$-strongly monotone, then T is μ-cocoercive.

Proposition 1.2. If a mapping T: H$\to$ H is nonexpansive, then I – T is (1/2)-cocoercive, where I denotes the identity mapping.

Clearly, every μ-cocoercive mapping T is $(1/\mu)$-Lipschitz continuous.

We can easily see that the following implications on monotonicity, strong monotonicity and expansiveness hold:

$$\begin{array}{ccc} \text{strong monotonicity} & \Rightarrow & \text{monotonicity} \\ & & \Downarrow \\ & & \text{expansiveness} \end{array}$$

Definition 1.2. A mapping T: H $\to$ H is said to be:

(i) *g-r-strongly monotone* if there exists a constant r > 0 such that

$$< T(x) - T(y), g(x) - g(y) > \ge r \| g(x) - g(y) \|^2 \text{ for all } x, y \in H,$$

where g: H $\to$ H is any mapping on H.

(ii) *g-s-Lipschitz continuous* if for all x, y$\in$H, we have

$$\| T(x) - T(y) \| \le s \| g(x) - g(y) \| \text{ for } s > 0.$$

(iii) *relaxed g-γ-cocoercive* if there exists a constant $\gamma > 0$ such that

$$< T(x) - T(y), g(x) - g(y) > \ge (-\gamma) \| T(x) - T(y) \|^2 \ \forall \ x,y \in H.$$

(iv) *relaxed g-γ-r-cocoercive* if there exist constants γ, r > 0 such that

$$< T(x) - T(y), g(x) - g(y) > \ge (-\gamma) \| T(x) - T(y) \|^2 + r \| g(x) - g(y) \|^2 \text{ for all } x,y \in H.$$

For $\gamma = 0$, T is g-r-strongly monotone, and for $r = 0$, T is relaxed g-γ-cocoercive . This class of mappings are more general than the class of strongly monotone mappings. We have the following implication:

g-r-strong monotonicity

$\Downarrow$

relaxed g-γ-r-cocoercivity

2. Projection Methods

In this section we present the convergence analysis for projection methods in the context of the approximation solvability of the SNVI (1.1)-(1.2) problem.

Algorithm 2.1. For arbitrarily chosen initial points x^0, $y^0 \in H$ with $g(x^0)$, $g(y^0) \in K$, compute sequences $\{g(x^k)\}$ and $\{g(y^k)\}$ such that

$$g(x^{k+1}) = (1 - a^k)g(x^k) + a^k P_K[g(y^k) - \rho T(y^k))],$$

$$g(y^k) = P_K[g(x^k) - \eta T(x^k)],$$

where P_K is the projection of H onto K, ρ, $\eta > 0$ are constants, and the sequence $\{a^k\}$ satisfies

$$0 \leq a^k \leq 1 \text{ and } \Sigma_{k=0}^{\infty} a^k = \infty.$$

For $\eta = \rho$ in Algorithm 2.1, we have

Algorithm 2.2. For arbitrarily chosen initial points x^0, $y^0 \in H$ with $g(x^0)$, $g(y^0) \in K$, compute sequences $\{g(x^k)\}$ and $\{g(y^k)\}$ such that

$$g(x^{k+1}) = (1 - a^k)g(x^k) + P_K[g(y^k) - \rho T(y^k)],$$

$$g(y^k) = P_K[g(x^k) - \rho T(x^k)],$$

where

$$0 \leq a^k \leq 1 \text{ and } \Sigma_{k=0}^{\infty} a^k = \infty$$

For $\eta = 0$ in Algorithm 2.1, we arrive at

Algorithm 2.3. For an arbitrarily chosen initial point $x^0 \in H$ with $g(x^0) \in k$, compute the sequence $\{g(x^k)\}$ such that

$$g(x^{k+1}) = (1 - a^k)g(x^k) + P_K[g(x^k) - \rho T(x^k)],$$

where

$$0 \leq a^k \leq 1 \text{ and } \Sigma_{k=0}^{\infty} a^k = \infty.$$

We now present, based on Algorithm 2.1, the approximation solvability of the SNVI (1.1)-(1.2) problem involving relaxed g-γ-r-cocoercive and g-μ-Lipschitz continuous mappings in a Hilbert space setting.

Theorem 2.1. Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let T: $H \rightarrow H$ be relaxed g-γ-r-cocoercive and g-μ-Lipschitz continuous. Suppose that $x^*, y^* \in H$ (with $g(x^*)$, $g(y^*) \in K$) form a solution to the SNVI (1.1)-(1.2) problem, sequences $\{g(x^k)\}$ and $\{g(y^k)\}$ are generated by Algorithm 2.1, and the sequence $\{a^k\}$ satisfies

$$0 \leq a^k \leq 1 \text{ and } \Sigma_{k=0}^{\infty} a^k = \infty.$$

Then sequences $\{g(x^k)\}$ and $\{g(y^k)\}$, respectively, converge to $g(x^*)$ and $g(y^*)$ for

$$0 < \rho < 2(r - \gamma\mu^2)/\mu^2 \text{ and } 0 < \eta < 2(r - \gamma\mu^2)/\mu^2.$$

If, in addition, g is expansive, then sequences $\{x^k\}$ and $\{y^k\}$, respectively, converge to x^* and y^*.

For $\eta = 0$, we have

Theorem 2.2. Let H be a real Hilbert space and K a nonempty closed convex subset H. Let T: $H \to H$ be a relaxed g-γ-r-cocoercive and g-μ-Lipschitz continuous mapping. In addition, if $x^* \in H$ is a solution to the NVI (1.3) problem, the sequence $\{g(x^k)\}$ is generated by Algorithm 2.3, and the sequence $\{a^k\}$ satisfies

$$0 \le a^k \le 1 \text{ and } \Sigma_{k=0}^{\infty} a^k = \infty,$$

then the sequence $\{g(x^k)\}$ converges to $g(x^*)$ for

$$0 < \rho\ 2(r - \gamma\mu^2)/\mu^2.$$

In addition, if g: $H \to H$ is expansive, then $x^k \to x^*$.

Proof of Theorem 2.1. Since x^* is a solution of the SNVI (1.1)-(1.2) problem, it follows that

$$g(x^*) = P_K[g(y^*) - \rho T(y^*)] \text{ and } g(y^*) = P_K[g(x^*) - \eta T(x^*)].$$

Applying Algorithm 2.1, we have

$$||g(x^{k+1}) - g(x^*)|| = ||(1 - a^k)\, g(x^k) + a^k P_K[g(y^k) - \rho T(y^k)]$$

$$- (1 - a^k) g(x^*) - a^k P_K[g(y^*) - \rho T(y^*)]||$$

$$\le (1 - a^k)\, ||g(x^k) - g(x^*)|| + a^k\, ||P_K[g(y^k) - \rho T(y^k)] - P_K[g(y^*) - \rho T(y^*)]||$$

$$\le (1 - a^k)\, ||g(x^k) - g(x^*)|| + a^k\, ||g(y^k) - g(y^*) - \rho[T(y^k) - T(y^*)]||. \tag{2.1}$$

Since T is relaxed g-γ-r-cocoercive and g-μ-Lipschitz continuous, we have

$$||g(y^k) - g(y^*) - \rho[T(y^k) - T(y^*)]||^2$$

$$= ||g(y^k) - g(y^*)||^2 - 2\rho < T(y^k) - T(y^*), g(y^k) - g(y^*) > + \rho^2 ||T(y^k) - T(y^*)||^2$$

$$\leq ||g(y^k) - g(y^*)||^2 + 2\rho\gamma ||T(y^k) - T(y^*)||^2 - 2\rho r ||g(y^k) - g(y^*)||^2$$
$$+ (\rho^2 \mu^2) ||g(y^k) - g(y^*)||^2$$

$$\leq ||g(y^k) - g(y^*)||^2 + 2\rho\gamma\mu^2 ||g(y^k) - g(y^*)||^2 + (\rho\mu)^2 ||g(y^k) - g(y^*)||^2$$
$$- 2\rho r ||g(y^k) - g(y^*)||^2$$

$$= [1 - 2\rho r + 2\rho\gamma\mu^2 + (\rho\mu)^2] ||g(y^k) - g(y^*)||^2.$$

As a result, we have

$$||g(x^{k+1}) - g(x^*)|| \leq (1 - a^k) ||g(x^k) - g(x^*)|| + a^k \theta ||g(y^k) - g(y^*)||, \tag{2.2}$$

where $\theta = [1 - 2\rho r + 2\rho\gamma\mu^2 + (\rho\mu)^2]^{1/2}$.

Similarly, we have

$$||g(y^k) - g(y^*)||^2 = ||P_K[g(x^k) - \eta T(x^k)] - P_K[g(x^*) - \eta T(x^*)]||^2$$

$$\leq ||g(x^k) - g(x^*) - \eta[T(x^k) - T(x^*)]||^2$$

$$\leq [1 - 2\eta r + 2\eta\gamma\mu^2 + (\eta\mu)^2] ||g(x^k) - g(x^*)||^2.$$

Hence, we have

$$||g(y^k) - g(y^*)|| \leq \sigma ||g(x^k) - g(x^*)||,$$

$$\leq ||g(x^k) - g(x^*)|| \text{ for } \sigma < 1, \tag{2.3}$$

where $\sigma = [1 - 2\eta r + 2\eta\gamma\mu^2 + (\eta\mu)^2]^{1/2}$.

It follows from (2.2) and (2.3) that

$$||g(x^{k+1}) - g(x^*)|| \leq (1 - a^k)||g(x^k) - g(x^*)|| + a^k\theta||g(x^k) - g(x^*)||$$

$$= [1 - (1 - \theta)a^k]||g(x^k) - g(x^*)||$$

$$\leq \Pi_{j=0}{}^k [1 - (1 - \theta)a^j]||g(x^0) - g(x^*)||, \tag{2.4}$$

where $\theta = [1 - 2\rho r + 2\rho\gamma\mu^2 + (\rho\mu)^2]^{1/2} < 1$

Since $\theta < 1$ and $\Sigma_{k=0}{}^{\infty} a^k$ is divergent, it implies in light of [17] that

$\lim_{\to\infty} \Pi_{j=0}{}^k [1 - (1 - \theta)a^j] = 0.$

Hence, the sequence $\{g(x^k)\}$ converges to $g(x^*)$ by (2.4), and the sequence $\{g(y^k)\}$ converges to $g(y^*)$ by (2.3) for

$$0 < \rho < 2(r - \gamma\mu^2)/\mu^2.$$

Since g is expansive, it implies that sequences $\{x^k\}$ and $\{y^k\}$ converge to x^* and y^*, respectively. □

Corollary 2.1. Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let T: H → H be g-r-strongly monotone and g-μ-Lipschitz continuous. Suppose that $x^*, y^* \in H$ form a solution to the SNVI (1.1)-(1.2) problem, sequences $\{g(x^k)\}$ and $\{g(y^k)\}$ are generated by Algorithm 2.1, and

$$0 \leq a^k \leq 1 \text{ and } \Sigma_{k=0}{}^{\infty} a^k = \infty .$$

Then sequences $\{g(x^k)\}$ and $\{g(y^k)\}$, respectively, converge to $g(x^*)$ and $g(y^*)$ for

$$0 < \rho < 2r/\mu^2 .$$

If, in addition, g is expansive, then sequences $\{x^k\}$ and $\{y^k\}$ converge, respectively, to x^*, y^*.

Corollary 2.2. Let H be a real Hilbert space and K be its nonempty closed convex subset. If $T: K \rightarrow H$ is an g-r-strongly monotone and g-μ-Lipschitz continuous mapping, x^* is a solution to the NVI (1.3) problem, the sequence $\{g(x^k)\}$ is generated by Algorithm 2.3, and

$$1 \leq a^k \leq 1 \text{ and } \Sigma_{k=0}^{\infty} a^k = \infty,$$

then the sequence $\{g(x^k)\}$ converges to $g(x^*)$ for

$$0 < \rho < 2r/\mu^2.$$

If, g is expansive, then the sequence $\{x^k\}$ converges to x^*.

Remark

The SNVI (1.1)-(1.2) problem can be extended as follows: let H_1 and H_2 be two real Hilbert spaces, and K_1 and K_2, respectively, be nonempty closed convex subsets of H_1 and H_2. Let $S: K_1 \times K_2 \rightarrow H_1$, and $T: K_1 \times K_2 \rightarrow H_2$ be two nonlinear mappings. Then the problem of finding $(x^*, y^*) \in K_1 \times K_2$ such that

$$< S(x^*, y^*), x - x^* > \geq 0 \ \forall \ x \in K_1$$

$$< T(x^*, y^*), y - y^* > \geq 0 \ \forall \ y \in K_2 ,$$

is said to be a *system of nonlinear variational inequalities.*

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