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MULTIGRID METHOD FOR AN ACCURATE SEMI-ANALYTIC FINITE DIFFERENCE SCHEME

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ABSTRACT. Compact schemes are shown to be effective for a class of problems including convection-diffusion equations when combined with multigrid algorithms [7, 8] and V-cycle convergence is proved[5]. We apply the multigrid algorithm for an semi-analytic finite difference scheme, which is desinged to preserve high order accuracy despite of singularities.

1. INTRODUCTION

High order finite difference schemes are very effectively working for solving elliptic partial differential equations[1, 3, 6]. But in the presence of singularities their high order convergence rate deteriorates[1, 6]. Singular points may arise on the boundary from an abrupt change of boundary conditions or a re-entrant corner. To preserve the high order accuracy, one needs to use a highly refined mesh in the vicinity of singularities. But it cost is high. To overcome this difficulty, an accurate semi-analytic finite difference scheme is proposed by Yosibash *et al.* in [11]. This method is based on the use of known expansions of the solution near the singular points and of high order scheme away from the singular points. This method produces large and sparse linear system away from singularities, and small but dense linear system near the singularities. So we propose that we use multigrid method for he linear system that arises from high order finite difference scheme on the smooth region and direct solvers, for example Gauss elimination, for the linear system that arises from the vicinity of singularities.

The rest of this paper is organized as follows: In section 2, we briefly present the multigrid algorithm and results of high order finite difference scheme. In section 3, we introduce the semi-analytic finite difference scheme and "Motz problem" for example to explain the main idea of this scheme. In section 4, we propose algorithm that uses multigrid algorithm and direct solvers.

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2. Multigrid method for high order finite difference method

In this section, we briefly describe an accurate semi-analytic scheme and introduce multigrid algorithm for this scheme. We first consider the following Poisson Equation :

$$\begin{cases} -\Delta u(x,y) & = f(x,y) \quad \text{in} \quad \Omega \\ \quad u(x,y) & = g(x,y) \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

Here, Ω can be any region in $\mathbb{R}^2$ covered by squares. For simplicity, we assume Ω is the unit square. For $k = 1, 2, \ldots, J$, let $h_k = 2^{-k}$ be a mesh size of level k. Define Ω_k be a space of points $(x_i, y_j) = (ih_k, jh_k)$ for $i, j = 0, 1, \ldots, 2^k$ and V_k be a vector space of function evaluated at Ω_k. The stencil of the high-order semi-analytic finite difference scheme is written as follows[3]:

$$\begin{aligned} &\frac{1}{6h^2}\left[20u_{i,j} - 4(u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1})\right. \\ &\quad \left. -(u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1})\right] \\ &= -\left[1 + \frac{2h^2}{4!}\Delta + \frac{2h^4}{6!}\left(\Delta^2 + 2\frac{\partial^4}{\partial x^2 \partial y^2}\right)\right] f_{i,j} \end{aligned} \tag{2.2}$$

where $h = h_k$. This nine-point discretization gives a truncation error of $O(h^8)$ over a square mesh(i.e., a convergence rate of $O(h^6)$). We obtain a system of linear equation of the form

$$A_k u = f, \tag{2.3}$$

where A_k is a sparse, $n \times n$, symmetric, positive definite matrix and u is the vector whose entries are $u_{i,j}$, and f is the vector whose entries are $f(x_i, y_j)$.

To describe the multigrid algorithm for this problem, we need certain intergrid transfer operators between two grids. Assuming we are given a certain prolongation operator $I_{k-1}^k : V_{k-1} \to V_k$, we define the restriction operator $I_k^{k-1} : V_k \to V_{k-1}$ as its adjoint with respect to $(\cdot, \cdot)$:

$$(I_k^{k-1}u, v)_{k-1} = (u, I_{k-1}^k v)_k \quad \forall u \in V_k, \forall v \in V_{k-1}.$$

Now the multigrid algorithm for solving (2.3) is defined as follows :

Multigrid Algorithm. Set $B_1 = A_1^{-1}$. For $1 < k \leq J$, assume that B_{k-1} has been defined and define $B_k f$ for $f \in V_k$ as follows:

(1) Set $x^0 = 0$.

(2) Define x^l for $l = 1, \ldots, m$ by

$$x^l = x^{l-1} + R_k^{(l+m)}(f - A_k x^{l-1}).$$

(3) Define $y^m = x^m + I_k q$, where q is defined by

$$q = q + B_{k-1}\left[P_{k-1}^0(f - A_k x^m)\right].$$

(4) Define y^l for $l = m+1, \dots, 2m$ by

$$y^l = y^{l-1} + R_k^{(l+m)}(f - A_k y^{l-1}).$$

(5) Set $B_k f = y^{2m}$.

Let the bilinear form $A_k(\cdot,\cdot)$ and discrete L^2-inner product be defined as follows:

$$\begin{aligned} A_k(u,v) =& (A_k u, v) \\ =& \frac{1}{6}\sum_{i,j} [\, 20u_{i,j} - 4(u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1}) \\ & -(u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1})\,]\, v_{i,j} \end{aligned} \tag{2.4}$$

and

$$(u,v)_k = \sum_{i,j} u_{i,j} v_{i,j} h_k^2. \tag{2.5}$$

Fix k. Let $u_{i,j}, u_{i+1,j}, u_{i,j+1}, u_{i+1,j+1}$ be points of level $k-1$ and $E_{i,j}$ be a cell having them as its vertex. (We borrow the term "cell" from cell-centered method.) Let $u_{i,j}^1, u_{i,j}^2, u_{i,j}^3, u_{i,j}^4$ and $u_{i,j}^5$ be points of level k defined as in Figure 1. Note that $u_{i,j}^1 = u_{i,j-1}^3, u_{i,j}^2 = u_{i+1,j}^4, u_{i,j}^3 = u_{i,j+1}^1$ and $u_{i,j}^4 = u_{i-1,j}^2$. Divide $E_{i,j}$ into four subcells, labeling them counterclockwise as $e_{i,j}^1, e_{i,j}^2, e_{i,j}^3, e_{i,j}^4$ at level k. (See Figure 1.)

Now, we define the prolongation operator I_{k-1}^k be bilinear interpolation of four points $u_{i,j}, u_{i+1,j}, u_{i,j+1}$ and $u_{i+1,j+1}$. First, $u_{i,j}, u_{i+1,j}, u_{i,j+1}$ and $u_{i+1,j+1}$ of level k are the same value of level $k-1$ respectively. The mid points $u_{i,j}^1, u_{i,j}^2, u_{i,j}^3$ and $u_{i,j}^4$ can be written as follows:

$$\begin{aligned} u_{i,j}^1 &= \frac{u_{i,j} + u_{i+1,j}}{2}, & u_{i,j}^2 &= \frac{u_{i,j} + u_{i+1,j+1}}{2} \\ u_{i,j}^3 &= \frac{u_{i,j+1} + u_{i+1,j+1}}{2}, & u_{i,j}^4 &= \frac{u_{i,j} + u_{i,j+1}}{2}. \end{aligned} \tag{2.6}$$

The value of center $u_{i,j}^5$ is the average of $u_{i,j}, u_{i+1,j}, u_{i,j+1}$ and $u_{i+1,j+1}$:

$$u_{i,j}^5 = \frac{u_{i,j} + u_{i+1,j} + u_{i,j+1} + u_{i+1,j+1}}{4}. \tag{2.7}$$

Then we have the following lemmas 2.1, 2.2, Theorem 2.1, V-cycle convergence by employing the framework presented in [2].

Lemma 2.1. [5] *We have*

$$A_k(I_{k-1}^k u, I_{k-1}^k u) \le A_{k-1}(u,u), \quad \forall u \in V_{k-1}. \tag{2.8}$$

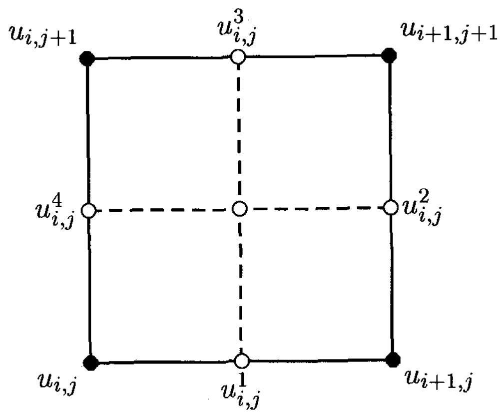

FIGURE 1. cell $E^k_{i,j}$ and its subcells

Lemma 2.2. *Let the operator P_{k-1} be defined by (2.10). There exist a number $0 < \alpha \le 1$ and a constant C_α such that for all $k = 1, \cdots, J$,*

$$A_k((I - I^k_{k-1}P_{k-1})u, u) \le C_\alpha \left(\frac{\|A_k u\|^2_k}{\lambda_k}\right)^\alpha A_k(u,u)^{1-\alpha}, \quad \forall u \in V_k. \tag{2.9}$$

holds for $\alpha = \frac{1}{2}$. Here, λ_k is the largest eigenvalue of A_k and P_{k-1} is the elliptic projection defined by

$$A_{k-1}(P_{k-1}u, v) = A_k(u, I^k_{k-1}v), \quad \forall u \in V_k, v \in V_{k-1}. \tag{2.10}$$

Theorem 2.1. *Let $E_k = I - B_k A_k$ in algorithm V(m,m). Then we have*

$$A_k(E_k u, u) \le \delta_k A_k(u,u), \quad \forall u \in V_k,$$

where $\delta_k = \frac{Ck}{Ck+\sqrt{m}}$.

3. AN ACCURATE SEMI-ANALYTIC FINITE DIFFERENCE SCHEME

An accurate semi-analytic finite difference scheme is proposed by Yosibash *et al.* in [11] to overcome degradation of numerical solution for two-dimensional elliptic problems with singularities. The scheme applies an explicit functional representation of the exact solution near the singularities and a convectional finite difference scheme on the remaining domain. For example, we consider the Laplace equation, so called, "Motz

Problem", on the region $\Omega = \{\, (x,y) \mid -1 \le x \le 1, 0 \le y \le 1\,\}$(See Fig 2).

$$\begin{cases} \Delta u = 0, \text{ in } \Omega \\ u = 0 \text{ on } \quad -1 \le x \le 0,\ y = 0 \\ u = 500 \text{ on } \quad x = 1,\ 0 \le y \le 1 \\ \partial u/\partial n = 0 \text{ elsewhere on the boundary.} \end{cases} \tag{3.1}$$

The discontinuity at the origin in the boundary conditions along the line $y = 0$ result in a singularity at this point. The asymptotic expansion of the solution about the origin is of the form [10]

$$u = \sum_{i=1}^{\infty} A_i r^{i-1/2} \cos[(i - 1/2)\theta]. \tag{3.2}$$

As mentioned earlier, we first apply the compact scheme(high order scheme) on the region Ω/Ω_s. Note that grid points also lies on Γ_s but do not say how to treat them yet. Now, we consider the grids points only on Γ_s and denote grid points by $u_{i,j}$. We add N algebraic equations of the form

$$u_{i,j} = \sum_{k=1}^{N} A_k r_{i,j}^{i-1/2} \cos[(i - 1/2)\theta_{i,j}]. \tag{3.3}$$

Here, $r_{i,j}$ and $\theta_{i,j}$ represents the radius vector and the angle respectively from the vertex to the grid point i,j. It is essential that the number of the grid points along Γ_s be greater than $N + i$ where i is either 0,1, or 2 depending on the boundary condition in the vicinity of the singularities(See [11] for the details). Then an over-determined set of equations is obtained, such that a procedure of least squares is applied. Define the quadratic functional as follows:

$$Q = \sum_{i,j} \left[u_{i,j} - \left(\sum_{k=1}^{N} A_k r_{i,j}^{i-1/2} \cos[(i - 1/2)\theta_{i,j}] \right) \right], \quad (x_i, y_j) \in \Gamma_s. \tag{3.4}$$

To find a minimum of Q, we differentiate Q by A_k and have

$$\frac{\partial Q}{\partial A_k} = 0, \quad k = 1, 2, \ldots, N. \tag{3.5}$$

4. Multigrid for an semi-analytic finite difference scheme

The system of linear equations, obtained in previous section is nonsymmetric and non positive definite. It is difficult to solve the linear system, so we propose an iterative algorithm using the effective multigrid algorithm.

Algorithm. Let u^i be the discrete solution after i-th iteration and $u^i_{\Gamma_s}$ be the restriction of the discrete solution on Γ_s. Given initial $A^0_1, \ldots, A^0_N$,

1. Interpolate $u^i_{\Gamma_s}$ using $A^i_1, \ldots, A^i_N$.

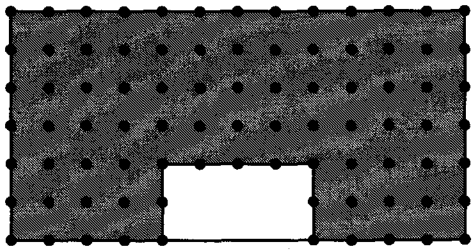

FIGURE 2. Grid for the Motz problem

2. Solve u^i with the boundary condition $u^i_{\Gamma_s}$ using multigrid algorithm.
3. Solve the linear system that arises from minimizing the quadratic functional Q and update $A_1^{i+1}, \ldots, A_N^{i+1}$.
4. Compute the residual of the linear system. If the residual is small enough, then stop. Otherwise, $i \leftarrow i+1$ and go to step 1.

This algorithm is similar to block Gauss-Seidel. The linear system that arises from high order finite difference scheme on the smooth region is large and sparse and we use multigrid algorithm. On the other hand, the linear system that arises from minimizing Q is small and dense but we can solve using direct solver, for example, Gauss elimination.

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