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A NUMERICAL METHOD FOR SOLVING THE NONLINEAR INTEGRAL EQUATION OF THE SECOND KIND

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ABSTRACT. In this work, we use a numerical method to solve the nonlinear integral equation of the second kind when the kernel of the integral equation in the logarithmic function form or in Carleman function form. The solution has a computing time requirement of $0(N^2)$, where (2N +1) is the number of discretization points used. Also, the error estimate is computed.

1. INTRODUCTION

Many problems of mathematical physics, theory of elasticity, viscodynamics fluid and mixed problems of continues media lead to the integral equation with a kernel in the form.

$$K^{\mu\lambda}_{n,m} = \frac{x^\lambda}{y^{\varepsilon+\lambda-1}} W^{\mu}_{n,m}(x,y)$$
$$W^{\nu}_{n,m} = \int_0^\infty J_n(tx) J_m(ty) t^\nu dy \tag{1.1}$$

where $J_n(.)$ is the Bessel function of the first kind of order n. The kernel (1.1) can take different forms:
If $n = m \pm \frac{1}{2}, \lambda = \frac{1}{2}$ and $\nu = \varepsilon = 0$, we have a logarithmic function form

$$k(x,y) = \ell n \frac{1}{\mid x-y \mid}$$

for symmetric and skew symmetric respectively.
Many different methods are established for solving analytically, the Fredholm integral equation of the first kind with logarithmic kernel (see [10]).
If $n = m = \pm\frac{1}{2}, \lambda = \frac{1}{2}, \varepsilon = 0$ and $0 \leq \nu < 1$, we have Carleman function for symmetric

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and skew symmetric respectively $k(x,y) = | x - y |^{-\nu}$.
Also the Fredholm integral equation of the first kind with Carleman kernel is solved in [11], analytically, by using different methods. In the same way, different methods are used and established for solving the Fredholm integral equation of the second kind, numerically, when the kernel takes a logarithmic form or Carleman form (see [3, 5]), Arutiunian [6] has shown that the plane contact problem of the nonlinear theory of plasticitry (in its first approximation) can be reduced to a Fredholm integral equation of the first kind with Carleman kernel.
If $n = m = \varepsilon = \nu = 0$ and $\lambda = \frac{1}{2}$, we have an elliptic integral kernel

$$k(x,y) = \frac{2\sqrt{xy}}{\pi(x+y)} K\left(\frac{2\sqrt{xy}}{x+y}\right).$$

Kovelenko, in his work [2] developed the Fredholm integral equation of the first kind for the mechanics mixed problem of continuous media and obtained its approximate solution when the kernel in the elliptic integral form.
If $\varepsilon = \nu, 0, \lambda = \frac{1}{2}$ and n = m we have the potential kernel. Abdou in [7] obtained the solution of Fredholm integral equation of the second kind with potential kernel.
If $\varepsilon = 0.\lambda = \frac{1}{2}, 0 \leq \nu < 1, n = m$, we have the generalized potential kernel (see [8]). Consider the nonlinear integral equation.

$$\gamma(x,\phi(x)) - \mu \int_{-1}^{1} K(| x - y |)\phi(y)dy = f(x) \ (-1 < x < 1) \tag{1.2}$$

with the subsidiary conditions where $\gamma(x,\phi(x))$ is a monotone functional of the function $\phi(x)$, so that for every pair of admissible function
$\phi_1(x)$ and ϕ_2 ,

$$\{\phi_1(x) - \phi_2(x)\}\gamma(x,\phi_1(x)) - \gamma(x,\phi_2(x))\} \geq 0(| x |\leq 1).$$

The given function f (x) is continuous with its derivatives for $x \in [-1,1]$. The formula (1.2) when the kernel K(x - y) has a strong singularity and $\mu << 1$ is considered in the work of Nemat-Nasser [4,9]. In this paper a numerical method is used to obtain an approximate solution of Eq. (1.2) in $L_2(-1,1)$, where the kernel has weak singularities

$$k(x,y) = \ell n \mid x - y \mid \tag{1.3}$$

and

$$k(x,y) = | x - y |^{-\nu} \ , \ 0 \leq \nu < 1 \tag{1.4}$$

Our main object is to obtain a linear system of (2N + 1) equations, where (2N +1) is the number of discretization points used and the coefficients matrix is expressed as the sum of two matrics. One of them is toplitz matrix $a_{n,m} = a_{n-m}$, and the other is a matrix which has at least one non zero row (column). The error estimate is computed and some numerical examples are given.

2. The integral operator:

In order to guarantee the existance of a unique solution of Eq. (1.2) we assume through this work following conditions.
(1) $k(x,y) \in C([-1,1] \times [-1,1))$
(2) The discontinuous kernel of (1.2) satisfies

$$\{\int_{-1}^{1}\int_{-1}^{1} k^2(x-y)dx\ dy\} = c^2 < \infty$$

(3) The monotone functional $\gamma(x,\phi(x))$ satisfies the Lipschitiz condition with respect to the second argument $\phi(x)$
(4) The potential function $\phi(x)$ is continuous in $x \in [-1,1]$ and satisfies

$$\{\int_{-1}^{1} \mid \phi(y) \mid^2 dy\}^{\frac{1}{2}} \leq A \parallel \phi \parallel_2$$

where $\parallel \ \parallel_2$ denotes the L_2 norm and A is a constant.
The continuity of the integral operator

$$K\phi = \int_{-1}^{1} k(x,y)\phi(y)dy$$

in $L_2(-1,1)$ can be proved. For taking $x_1, x_2 \in L_2(-1,1)$, we have

$$\begin{aligned}
\mid k_1\phi - k_2\phi \mid &= \mid \int_{-1}^{1} k(x_1,y)\phi(y)dy - \int_{-1}^{1} k(x_2,y)\phi(y)dy \mid \\
&\leq \left(\int_{-1}^{1} \phi^2(y)dy\right)^{\frac{1}{2}} \left(\int_{-1}^{1} [K(x_1,y) - K(x_2,y)]^2 dy\right)^{\frac{1}{2}} \\
&\leq \parallel \phi \parallel_2 g(x_1,x_2)
\end{aligned}$$

where $g(x_1,x_2) \to 0$ as $x_1 \to x_2$
The normality of the integral operator can be proved as follows:

$$\begin{aligned}
\parallel k\phi \parallel_2 &= \left[\int_{-1}^{1} dx \left(\int_{-1}^{1} k(x,y)\phi(y)dy\right)^2\right]^{\frac{1}{2}} \\
&\leq \left[\int_{-1}^{1} dx \int_{-1}^{1} \phi^2(y)dy \int_{-1}^{1} k^2(x,y)dy\right]^{\frac{1}{2}} \\
&= C \parallel \phi \parallel_2
\end{aligned}$$

So $\parallel K \parallel_2 = C$ (C is a constant).

3. Method of solution

The integral term of Eq. (1.2) can be written in the form

$$\int_{-1}^{1} k(x,y)\phi(y)dy = \sum_{n=-N}^{N-1} \int_{nh}^{(n+1)h} k(x,y)\phi(y)dy \ (h = \frac{1}{N}) \tag{3.1}$$

The integral of the right hand side of Eq. (3.1) may be written in the form

$$\int_{a}^{a+h} k(x,y)\phi(y)dy = A(x)\phi(a) + B(x)\phi(a+h) + R \tag{3.2}$$

where A(x) and B(x) are arbitrary functions to be determined, R is the estimated error of order $O(h^2)$, and a = nh. A(x) and B(x) can be determined by putting $\phi(x) = 1$ and $\phi(x) = x, R = 0$, and solving the two resultant equations by putting x = mh and a = nh , $-N \leq m \leq N, -N \leq n \leq N-1$ in (3.2).
Then Eq. (1.2) takes the form

$$\gamma(\Phi(mh)) - \mu b_{n,m}\phi(mh) = F(mh) \tag{3.3}$$

where, we assume $\gamma(x,\phi(x)) = \gamma(\phi(x)), \Phi(mh) = \phi(x), F(mh) = f(x), x = mh$
Here Φ is a vector of (2N+1) elements, and the elements of the matrix $b_{n,m}$ are given by

$$b_{n,m} = \left\{ \begin{array}{cc} A_{-N}(mh) & n=-N \\ A_n(mh) + B_{n-1}(mh) & -N+1 \leq n \leq N-1 \\ B_{N-1}(mh) & n = N \end{array} \right\}$$

The square matrix $b_{n,m}$ can be written in the form

$$b_{n,m} = b_{n-m} + \left\{ \begin{array}{ccc} g_{-N,-N} & g_{-N,-N+1}\cdots\cdots & g_{-N,N} \\ 0 & 0 & 0 \\ d_{-N,-N} & d_{-N,-N+1}\cdots\cdots & d_{-N,N} \end{array} \right\}$$

The matrix b_{n-m} is the toeplitz of order (2N+1), where $-N \leq n,m \leq N$. and the lements of the second matrix are zero except the elements of the first and last rows. once the matrix $b_{n,m}$ have been obtained , we compute the approximate values $\Phi(mh)$ as solution of the system (3.3). In the linear case the system (3.3) is solved by teopliz matrix with inner product. In the nonlinear case we compute $\Phi(mh)$ from (3.3) by using the nonlinear system Visual Basic Programming.

4. Convergence analysis:

In our convergence analysis we examine the linear test equation

$$\phi(x) - \mu \int_{-1}^{1} k(x,y)\phi(y)dy = f(x) \quad -1 \leq x \leq 1 \tag{4.1}$$

and assume that the forcing function f(x) with its derivatives belong to C[-1,1], and that the kernel k(x, y) is weakly singular of the form (1.3) or (1.4). Then, the Eq. (4.1) has a unique soultion $\phi \in C[-1,1]$. If, for a given mesh $\{x_j\}_{-N}^{N}$ we apply the toepliz matrix method to the test equation (4.1), we obtain an approximat solution $\Phi_N(mh)$ in the form:

$$\phi_N(mh) = f(mh) + \mu \sum_{n=-N}^{N} b_{n,m}\Phi_N(nh) \tag{4.2}$$

where $b_{n,m}$ is given by (3.4)
The method is said to be convergent of order ℓ in [-1,1] if and only if for N sufficiently large there exists a constant $C > 0$ independent of N such that

$$\| \phi(x) - \phi_N(mh) \|_\infty \leq CN^{-\ell} \ (x = mh) \tag{4.3}$$

Also, in general the local truncation error is defined by

$$t_N(k, \phi, x) = \left| \int_{-1}^{1} k(x,y)\phi(y)dy - \sum_{n=-N}^{N} b_{n,m}\phi(nh) \right| \tag{4.4}$$

To examine the uniform convergence of the approximate solution $\phi_N(mh)$to the exact solution $\phi(x)$ of (4.1), notice that

$$\phi(x) - \phi_N(mh) = \sum_{n=-N}^{N} b_{n,m}\{\phi(nh) - \phi_N(nh)\} + t_N(k, \phi, x) \tag{4.5}$$

where $t_N(k, \phi, x)$ is given by (4.4).
Hence, we have

$$\| \Phi - \Phi_N \|_\infty \leq \| (I - D_N)^{-1} \|_\infty \| t_N \|_\infty \tag{4.6}$$

where D_N is the linear operator defined by

$$D_N : C[-1,1] \to C[-1,1]$$

$$D_N f(x) = \sum_{n=-N}^{N} b_{n,m} f(mh), \ f \in C[-1,1], x \in [-1,1] \tag{4.7}$$

To investigate the behavior of the term $\| (I - D_N)^{-1} \|_\infty$, we follows the way:
Firstly, let $L_N(f, y)$ denote the interprolating polynomial of degree $\leq (2N + 1)$ that coincide with the function f at the nodes $\{x_j\}_{j=-N}^{N}$, then for the kernel k (x,y) in the form of (1.3) or (1.4), and $f \in C[-1,1]$, we have

$$\lim_{N\to\infty} \| \int_{-1}^{1} k(x,y)f(y)dy - \int_{-1}^{1} k(x,y)L_N f(y)dy \|_\infty = 0 \tag{4.8}$$

Secondly, with the aid of [1] pp. 12014, we consider the following lemma

$$\sup_N \| \sum_{n=-N}^{N} b_{n,m} f(mh) \| \leq C \| f \|_\infty \; (\| f \|_\infty = 1) \tag{4.9}$$

Moreover, the kernel k (x, y) of Eq. (1.3) and (1.4) satisfies the following:
For $k(x,y) \in L_g, g > 0$, we have

$$\lim_{x' \to x} \| k(x',y) - k(x,y) \|_q = 0, \; x \in [-1,1] \tag{4.10}$$

Hence, we have

$$\lim_{m' \to m} \sup_N \sum_{n=-N}^{N} | b_{n,m} - b_{n,m'} | = 0 \tag{4.11}$$

In this aim, we can say, if (4.8) , (4.9) and (4.10) hold, then for all N sufficiently large there exist a constant $C > 0$ independent of N such that

$$\| (I - D_N)^{-1} \|_\infty \leq C$$

Moreover, we can say that the operator D defined by $D_N : C[-1,1] \to C[-1,1]$

$$Df(x) = \int_{-1}^{1} k(x,y) f(y) dy \tag{4.12}$$

is compact operator on C [-1,1], if the kernel k(x,y) is defined and continuous for all $x, y \in [-1,1], x \neq y$ and $| k(x,y) | \leq C | x - y |^{-\alpha}, 0 \leq \alpha < 1$ for $x, y \in [-1,1], x \neq y$. Furthermore, in the case $k(x,y) = \ell n | x - y |$, one can write.

$$\ell n | x - y | = h(x,y). | x - y |^{-\alpha}, (h(x,y) = \ell n | x - y | . | x - y |^{\alpha}, (0 \leq \alpha < 1)) \tag{4.13}$$

where $h(x,y) \in C[-1,1]$ for $x \in [-1,1]$

5. Numerical examples

Example (1) : consider the kernel $k(x,y) = ln | x - y |$, then, we have the integral equation

$$\gamma(\phi(x)) - \int_{-1}^{1} k(x,y) \phi(y) dy = f(x) \tag{5.1}$$

where

$$k(x,y) = \int_{0}^{\infty} J_{\pm\frac{1}{2}}(tx) J_{\pm\frac{1}{2}}(ty) dy \tag{5.2}$$

and $J_n(x)$ is the Bessel function of order n. The matrix b_{n-m} takes the form

$$b_{n-m} = \frac{h}{2}\{(n-m+1)^2 \ell n(| n-m+1 | h) - 2(n-m)^2 ln(| n-m | h)$$
$$+(n-m-1)^2 \ell n(| n-m-1 | h) - 3\} \tag{5.3}$$

where the two rows $g_{-N,-N+i}, d_{N,-N+i}$ take the form

$$g_{-N,-N+i} = \frac{h}{2}\{-(i+1)^2\ell n(| i+1 | h) + (2+i)i\ \ell n\ | i | h) + \frac{1}{2}(3+2i)\}$$

$$d_{N,-N+i} = \tfrac{h}{2}\{-(2N-i+1)^2\ell n(| 2N-i+1 | h)$$

$$+(2+2N-i)(2N-i)\ell n(| 2N-i | h) + \frac{1}{2}(3+2(2N-i))\}\ (0 \leq i \leq 2N) \qquad (5.4)$$

and the free term f(x) is taken in the form

$$F(mh) = 1+\mu\{(1-mh)\ell n \mid 1-mh \mid +(1+mh)\ell n \mid 1+mh \mid -2\}\ (-N \leq m \leq N) \quad (5.5)$$

Finally, the error R can be calculated using the following relation

$$| R | \leq | \frac{\phi''(0)C(m)}{2!} | h^3$$

where

$$\begin{aligned} C(m) &= \frac{1}{6}\{(3m^2 - 2m^3 - 1)ln(| 1-m | h) \\ &+ (2m^3 - 3m^2)ln(| m | h) \\ &- \frac{1}{6}(12m^2 - 10m - 5)\} \end{aligned}$$

In Fig. 1 results from the numerical solution based on the relation between $\phi(x)$ and N , when $\gamma(\phi(x)) = \phi^2(x), \lambda = 0.001$, where we take N = 10, 20, 40. The relation between the error estimate and N, when N = 10, 20, 40, is shown in Fig. 2, where we obtain the inverse relation between them i.e $R\alpha\frac{1}{N}$. If N takes large vlaues, for example N = 100, that gives more clarification of the values of ϕ which is the nearest to the extact solution.

Example (2): Consider the integral equation

$$\sqrt{\phi(x)} - (0.0001)\int_{-1}^{1} k(x,y)\phi(y)dy = f(x) \qquad (5.6)$$

where

$$k(x,y) = | y-x |^{-\nu} = \int_0^\infty t^\nu J_{\pm\frac{1}{2}}(tx)J_{\pm\frac{1}{2}}(ty)dy\ (y \geq x, 0 \leq \nu < 1) \qquad (5.7)$$

and the free term is assumed as:

$$f(x) = 1 + \frac{0.0001}{(1-\nu)}\{| 1-x |^{1-\nu} - | 1+x |^{1-\nu}\} \qquad (5.8)$$

Hence, the matrix b_{n-m} takes the form

$$b_{n-m} = \frac{h^{1-\nu}}{(1-\nu)(2-\nu))}\{(n-m+1)^{2-\nu} - 2 | n-m |^{2-\nu} + | n-m-1 |^{2-\nu}\} \quad (5.9)$$

where the trans rows $g_{-N,-N+i}, d_{N,-N,+i}$, take the form

$$g_{-N,-N+i} = \frac{h^{1-\nu}}{1-\nu}\{- \mid i \mid^{1-\nu} + \frac{1}{2-\nu} \mid i \mid^{2-\nu} - \frac{1}{2-\nu} \mid i+1 \mid^{2-\nu}\} \tag{5.10}$$

and

$$d_{N,-N+i} = \frac{h^{1-\nu}}{1-\nu}\{\mid 2N - i \mid^{1-\nu} + \frac{1}{2-\nu} \mid 2N + i \mid^{2-\nu} - \frac{1}{2-\nu} \mid 2N - i + 1 \mid^{2-\nu}\} \tag{5.11}$$

and the free term f (x) is taken in the form

$$f(mh) = 1 + \frac{0.0001}{1-\nu}\{\mid 1 - mh \mid^{1-\nu} - \mid 1 + mh \mid^{1-\nu}\} \ (-N \leq m \leq N) \tag{5.12}$$

finally the error R takes the form

$$R \leq \mid c \mid h^3$$

where

$$C = \frac{h^{-\nu}}{1-\nu}\{\frac{-1}{1-\nu}[\mid 1 - m \mid^{2-\nu} + \mid m \mid^{2-\nu}] + \frac{2}{(2-\nu)(3-\nu)}[\mid 1 - m \mid^{3-\nu} - \mid m \mid^{3-\nu}]\} \tag{5.13}$$

The relation between $\phi(x)$ and $N, \nu = 0.1$ and $y \geq x$ is established in Fig.3 for N = 10, 20, 40. The same result is obtained when $(x \geq y)$. In Fig 4 the error estimate for N = 10, 20, 40 are obtained.

References

[1] A. Palamara Orsi, Product integration for Volterra integral equations of the second kind with weakly singular kernel , Math. of Comp. Vol.65., No215 (1996) pp. 1201 - 1212.
[2] E.V. Kovalinko, Some approximate method of solving integral equation of mixed problems Appl. Math Mech. Vol. 53 (1989). pp 85-92.
[3] E.K. Atkinson, A Survey of Numerical method for The solution of Fredholm Integral Equation of The Second Kind, SIAM, Philadelphia 1979.
[4] J.R. Willis and S. Nemat -Nasser, Singular perturbatin solution of a class of singular integral equations, Quart. Appl. Math. Vol. XL V III, No. 4, December (1990) 471-753.
[5] L.M. Delves and J.L Mohamed, Compuational Methods for Integral Equations, New York 1985.
[6] N.K. Artiuniun, Plane contact problem of the theory of creef. Appl. Math. Mech. 23(1959)901-923.
[7] M.A. Abdou Integral equation of the second kind with potential kernel and its structure resolvent, Appl. Math. Comput. 107 (2000) 169-180.
[8] M.A. Abdou, Integral equation and contact problem for a system of impressing stamp, Appl. Math. Comput. 106 (1999) 36-48.
[9] Muneo Hori and S. Nemat-Nasser, Asymptotic solution of a class of strongly singular integral equations, SIAM.J. Appl. Math. Vol. 50 No. 3 June (1990) 716-725.
[10] S.M. Mkhitarian and M.A. Abdou, On different methods for solving the Fredholm integral equation of the first kind with logarithmic kernel, Dokl. Acad Nauk Armenia, Vol. 90 (1990) 1-10.
[11] S.M. Mkhitarian and M.A. Abdou, On different methods for solving the Fredholm integral equation of the first kind with Carleman kernel, Dokl. Acad Nauk Armenia, Vol. 89 (1990) 125-130.

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