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FINITE VOLUME ELEMENT METHODS FOR NONLINEAR PARABOLIC INTEGRODIFFERENTIAL PROBLEMS

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ABSTRACT. In this paper, finite volume element methods for nonlinear parabolic integrodifferential problems are proposed and analyzed. The optimal error estimates in L^p and $W^{1,p}(2 \le p \le \infty)$ as well as some superconvergence estimates in $W^{1,p}(2 \le p \le \infty)$ are obtained. The main results in this paper perfect the theory of FVE methods.

1. INTRODUCTION

Consider the following initial boundary value problem for the nonlinear parabolic integrodifferential equation:

$$\begin{array}{ll} (a) & \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\{a(x,t,u)\dfrac{\partial u}{\partial x} + \displaystyle\int_0^t b(x,t,\tau,u(x,\tau))\dfrac{\partial u}{\partial x}(x,\tau)d\tau\} + f(x,u), \quad (x,t) \in (a,b) \times (0,T], \\ (b) & u(x,0) = u_0(x), \quad x \in I = [a,b], \\ (c) & u(a,t) = 0, \quad u(b,t) = 0, \quad t \in [0,T], \end{array} \tag{1.1}$$

where the functions a, b, f and u_0 are smooth enough to ensure the analysis validity and $a(x,u)$ is bounded from above and below:

$$0 < a_0 \le a(x,u) \le M, \quad (x,u) \in [a,b] \times R. \tag{1.2}$$

Since we shall show that the approximate solution is uniformly convergent to the exact solution of (1.1), the above assumptions only need to hold in a neighborhood of the exact solution. Here and in what follows, we will not write the independent variables x, t, τ for any function unless it is necessary.

It can be proved that (1.1) has a unique solution for any $f \in L^2(I)$ and $u_0 \in H^s(I)$ (See[4], [11-12]).

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In the past several decades, Q.Li and other authors did extensive and deep research on the theory and application of finite volume element(FVE) methods, including constructing FVE schemes on elliptic, parabolic and hyperbolic equations(See[2-3], [6-9]), establishing the optimal Sobolev norm estimates of error, and applying FVE methods to underground fluid, electromagnetic field and other fields. Theoretical researches and realistic computations show that FVE not only keep the computational simplicity of finite difference methods(FDM), but also enjoy the accuracy of finite element methods(FEM). More importantly, numerical solution generated by FVE methods usually maintains mass conservation features, which are desirable in many applications. However, the analysis for FVE methods is far behind that for FDM and FEM, and the theory of FVE methods is not perfect.

Our main goal is to discuss FVE methods of one-dimensional nonlinear parabolic integrodifferential problems. We derive the optimal error estimates in L^p and $W^{1,p}$ for $2 \leq p \leq \infty$. Moreover, some superconvergence is also obtained.

The rest of this paper is organized. In section 2, we do some preparations, including formulating FVE approximation schemes in piecewise linear finite element spaces, and introducing some important lemmas which are essential in our analysis. Some properties of the generalized Ritz-Volterra projection are established in section 3. Main results of this paper are given in section 4.

2. Preparations

In this paper we will follow the notations and symbols in [3]. For examples, $T_h = \{I_i; I_i = [x_{i-1}, x_i], 1 \leq i \leq n\}$, and $T_h^* = \{I_i^*; I_i^* = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], 1 \leq i \leq n-1,\ I_0^* = [x_0, x_{\frac{1}{2}}],\ I_n^* = [x_{n-\frac{1}{2}}, x_n]\}$denote the primal partition and its dual partition, respectively. Let $h_i = x_i - x_{i-1},\ h = max\{h_i; 1 \leq i \leq n\}$. The partitions are assumed to be regular, that is, there exists a constant $\mu > 0$ such that $h_i \geq \mu h,\ i = 1, 2, \cdots n$. The trial function space $U_h \subset H_0^1(I) \equiv \{u \in H^1(I); u(a) = u(b) = 0\}$ is defined as a piecewise linear function space over T_h and $U_h = span\{\varphi_i(x), 1 \leq i \leq n-1\}$. The test function space $V_h = span\{\psi_i(x), 1 \leq i \leq n-1\} \subset L^2(I)$ is defined as a piecewise constant function space over T_h^*.

For numerical analysis, we need to introduce the interpolation operators $\Pi_h : H_0^1(I) \bigcap C(I) \longmapsto U_h$, defined by

$$\Pi_h w = \sum_{i=1}^{n-1} w(x_i)\varphi_i(x), w \in H_0^1(I),$$

and $\Pi_h^* : H_0^1(I) \bigcap C(I) \longmapsto V_h$, defined by

$$\Pi_h^* w = \sum_{i=1}^{n-1} w(x_i)\psi_i(x), w \in H_0^1(I).$$

Using the interpolation theory, we have

$$\begin{aligned}&(a)\quad |w-\Pi_h w|_{m,p}\le Ch^{k-m}|w|_{k,p}\ ,\quad m=0,1,\ k=1,2,\ 1\le p\le\infty,\\ &(b)\quad \|w-\Pi_h^* w\|_{0,p}\le Ch|w|_{1,p}\ ,\qquad 1\le p\le\infty.\end{aligned}\tag{2.1}$$

where $|\cdot|_{m,p}$ and $\|\cdot\|_{m,p}$ stand for the semi-norm and norm of the Sobolev space $W^{m,p}(I)$ respectively, $|\cdot|_m$ and $\|\cdot\|_m$stand for the semi-norm and norm of the Sobolev space $H^m(I)=W^{m,2}(I)$ respectively, and C is a positive constant independent of h.

Let's define, for any $u,\ v\in H_0^1(I),\ u_h\in U_h$ and $v_h\in V_h$, some bilinear forms as follows:

$$\begin{aligned}&(a)\quad a(z;u,v)=\int_a^b a(x,z)u'v'dx,\qquad b(z;u,v)=\int_a^b b(x,t,\tau,z(x,\tau))u'(x,\tau)v'dx,\\ &(b)\quad a^*(z;u_h,v_h)=\sum_{j=1}^{n-1}v_ja^*(z;u_h,\psi_j),\quad b^*(z;u_h,v_h)=\sum_{j=1}^{n-1}v_jb^*(z;u_h,\psi_j),\end{aligned}\tag{2.2}$$

where $a^*(z;u_h,\psi_j)=a(z)_{j-\frac12}u_h'(x_{j-\frac12})-a(z)_{j+\frac12}u_h'(x_{j+\frac12})$, $b^*(z;u_h,\psi_j)=b(z)_{j-\frac12}u_h'(x_{j-\frac12})-b(z)_{j+\frac12}u_h'(x_{j+\frac12})$, with $u_h'(x_{j-\frac12})=\dfrac{u_j-u_{j-1}}{h_j}$, $u'=\dfrac{\partial u}{\partial x}$, $v'=\dfrac{\partial v}{\partial x}$, $u_j=u_h(x_j)$, $v_j=v_h(x_j)$, $x_{j-\frac12}=\dfrac12(x_{j-1}+x_j)$, $a(z)_{j-\frac12}=a(z(x_{j-\frac12}))$, $b(z)_{j-\frac12}=b(x_{j-\frac12},t,z(x_{j-\frac12},\tau))$, $u_0=0,\ u_n=0$.

The weak form of (1.1) is obtained by seeking a solution $u(x,t)\in H_0^1(I),\ 0\le t\le T$, such that

$$\begin{aligned}&(a)\quad (u_t,v)+a(u;u,v)+\int_0^t b(u(\tau);u(\tau),v)d\tau=(f(u),v),\quad v\in H_0^1(I),\\ &(b)\quad u(0)=u_0,\qquad\qquad x\in I.\end{aligned}\tag{2.3}$$

For error estimates, we next introduce the Ritz projection operator $R_h=R_h(t):H_0^1(I)\longmapsto U_h,\ 0\le t\le T$, defined by

$$a(u;w-R_hw,v_h)=0,\quad v_h\in U_h,\tag{2.4}$$

the Ritz-Volterra projection operator $V_h=V_h(t):H_0^1(I)\longmapsto U_h,\ 0\le t\le T$, defined by

$$a(u;w-V_hw,v_h)+\int_0^t b(u;(w-V_hw)(\tau),v_h)d\tau=0,\quad v_h\in U_h,\tag{2.5}$$

the generalized Ritz projection operator $R_h^*=R_h^*(t):H_0^1(I)\longmapsto U_h,\ 0\le t\le T$, defined by

$$a^*(u;w-R_h^*w,v_h)=0,\quad v_h\in V_h,\tag{2.6}$$

and the generalized Ritz-Volterra projection operator $V_h^*=V_h^*(t):H_0^1(I)\longmapsto U_h,\ 0\le t\le T$, defined by

$$a^*(u;w-V_h^*w,v_h)+\int_0^t b^*(u;(w-V_h^*w)(\tau),v_h)d\tau=0,\quad v_h\in V_h,\tag{2.7}$$

where u is the solution of (1.1). Obviously, $V_h(0) = R_h(0),\ V_h^*(0) = R_h^*(0)$.

Then, the semi-discrete FVE approximation scheme is to find a map $u_h(t) : H_0^1(I) \longmapsto U_h$ such that

$$\begin{aligned} &(a) \quad (u_{h,t}, v_h) + a^*(u_h; u_h, v_h) + \int_0^t b^*(u_h(\tau); u_h(\tau), v_h) d\tau = (f(t, u_h), v_h), \quad v_h \in V_h, \\ &(b) \quad u_h(0) = u_{0,h}, \end{aligned} \tag{2.8}$$

where $u_{0h} \in U_h$ is determined by

$$\begin{aligned} \mathcal{A}(\xi(0), v_h) &\equiv a^*(u_0; \xi(0), v_h) + c^*(\xi(0); R_h^* u_0, v_h) + \lambda(\xi(0), v_h) \\ &= -c^*(\eta(0); R_h^* u_0, v_h), \quad v_h \in V_h, \end{aligned}$$

here λ is a constant which will be determined by Lemma 2.9, $\xi = u_h - V_h^* u, \eta = V_h^* u - u$, and

$$c^*(z; u, v_h) = \sum_{j=1}^{n} (a_u(u_0) z)_{j-\frac{1}{2}} \frac{(u_j - u_{j-1})(v_j - v_{j-1})}{h_j} \tag{2.9}$$

Noting that for any $u_h \in U_h$, we have, by (2.2)

$$|u_h|_{1,p} = (\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |u_h'|^p dx)^{\frac{1}{p}} = \{\sum_{i=1}^{n} h_i (\frac{u_i - u_{i-1}}{h_i})^p\}^{\frac{1}{p}}.$$

Define some discrete norms in U_h:

$$\begin{aligned} \|u_h\|_{0,h} &= \{\sum_{i=1}^{n} h_i (u_i^2 + u_{i-1}^2)\}^{\frac{1}{2}}, \\ |u_h|_{1,h} = |u_h|_1 &= \{\sum_{i=1}^{n} (\frac{(u_i - u_{i-1})^2}{h_i})\}^{\frac{1}{2}}, \\ \|u_h\|_{1,h} &= (\|u_h\|_{0,h}^2 + |u_h|_{1,h}^2)^{\frac{1}{2}}, \\ |||u_h||| &= (u_h, \Pi_h^* u_h)^{\frac{1}{2}}. \end{aligned}$$

Then we can easily prove the following lemma.

Lemma 2.1(See[5, 9])There exist two positive constants C_1 and C_2, independent of h, such that for any $u_h \in U_h$,

$$\begin{aligned} &C_1 \|u_h\|_{0,h} \le \|u_h\| \le C_2 \|u_h\|_{0,h}, \\ &C_1 |||u_h||| \le \|u_h\| \le C_2 |||u_h|||, \\ &C_1 \|\Pi_h^* u_h\| \le \|u_h\| \le C_2 \|\Pi_h^* u_h\|, \\ &C_1 \|u_h\|_{1,h} \le \|u_h\|_1 \le C_2 \|u_h\|_{1,h}. \end{aligned} \tag{2.10}$$

Noting $w_0 = w_n = 0$, for any $u_h, w_h \in U_h$, we have,

$$a^*(z; u_h, \Pi_h^* w_h) = \sum_{j=1}^{n} a(z)_{j-\frac{1}{2}} \frac{(u_j - u_{j-1})(w_j - w_{j-1})}{h_j},$$
$$b^*(z; u_h, \Pi_h^* w_h) = \sum_{j=1}^{n} b(z)_{j-\frac{1}{2}} \frac{(u_j - u_{j-1})(w_j - w_{j-1})}{h_j}.$$

According to technique given in [5, 9], we easily derive the following conclusions.
Lemma 2.2 For any $u_h, w_h \in U_h$, we have,

$$\begin{aligned} &(a) \quad (u_h, \Pi_h^* w_h) = (w_h, \Pi_h^* u_h), \\ &(b) \quad a^*(z; u_h, \Pi_h^* w_h) = a^*(z; w_h, \Pi_h^* u_h). \end{aligned} \tag{2.11}$$

Lemma 2.3 There exist two positive constants M and α_0, independent of h, and $h_0 > 0$, such that, for all $0 < h \le h_0$, and any $u_h, w_h \in U_h$,

$$\begin{aligned} &(a) \quad |a^*(z; u_h, \Pi_h^* w_h)| \le M \|a(z)\|_{0,\infty} \|u_h\|_1 \|w_h\|_1, \\ &(b) \quad |a^*(z; u_h, \Pi_h^* u_h)| \ge \alpha_0 \|u_h\|_1^2, \\ &(c) \quad |b^*(z; u_h, \Pi_h^* w_h)| \le M \|b(z)\|_{0,\infty} \|u_h\|_1 \|w_h\|_1. \end{aligned} \tag{2.12}$$

Let X be a Banach space with norm $\|\cdot\|_X$ and $\phi : [0,T] \longmapsto X$. Define

$$\|\phi\|_{L^2(X)}^2 = \int_0^T \|\phi(t)\|_X^2 dt \quad \text{and } \|\phi\|_{L^\infty(X)} = ess \sup_{0 \le t \le T} \|\phi(t)\|_X.$$

Let the space $H^k(W^{s,p})$ be defined by

$$H^k(0,T; W^{s,p}) = \{u \in W^{s,p};\ \frac{\partial^j u}{\partial t^j} \in L^2(0,T; W^{s,p}),\ j = 0, 1, \cdots, k\}$$

and for any $u \in H^k(W^{s,p})$, we set

$$\|u(t)\|_{k,s,p} = \sum_{j=1}^{k} \{\|\frac{\partial^j u}{\partial t^j}\|_{s,p} + \int_0^t \|\frac{\partial^j u}{\partial t^j}\|_{s,p} d\tau\},\ t \in [0,T].$$

For R_h, R_h^*, V_h, we have the following results.
Lemma 2.4(See[6]) For $2 \le p \le \infty$, we have

$$\begin{aligned} &(a) \quad \|w - R_h w\|_{0,p} + h\|w - R_h w\|_{1,p} \le Ch^2 \|w\|_{2,p}, \\ &(b) \quad \|(w - R_h w)_t\| + h\|(w - R_h w)_t\|_1 \le Ch^2 (\|w\|_2 + \|w_t\|_2). \end{aligned} \tag{2.13}$$

Lemma 2.5(See[6]) For $k = 0, 1, 2$, we have

$$\begin{aligned} &(a) \quad \|D_t^k (w - R_h^* w)\|_1 \le Ch \sum_{l=0}^{k} \|D_t^l w\|_2, \\ &(b) \quad \|D_t^k (w - R_h^* w)\| \le Ch^2 \sum_{l=0}^{k} \|D_t^l w\|_{3,p},\ p \ge 2. \end{aligned} \tag{2.14}$$

Lemma 2.6(See[1, 10]) For $2 \le p \le \infty$, we have

$$\begin{array}{ll} (a) & \|w - V_h w\|_{0,p} + h\|w - V_h w\|_{1,p} \le Ch^2\|w\|_{0,2,p}, \\ (b) & \|(w - V_h w)_t\|_{0,p} + h\|(w - V_h w)_t\|_{1,p} \le Ch^2\|w\|_{1,2,p}. \end{array} \tag{2.15}$$

We now present a very useful lemma:

Lemma 2.7(See[3, 6]) For any $u_h, w_h \in U_h$, we can get,

$$\begin{array}{ll} (a) & |d_1(z; u - u_h, w_h)| \le C\|a(z)\|_{1,\infty} h(|u - u_h|_{1,p}|w_h|_{1,p'} + h|u|_{3,q}|w_h|_{1,q'}), \\ (b) & |d_1(z; u - u_h, w_h)| \le C\|a(z)\|_{1,\infty} h(|u - u_h|_{1,p}|w_h|_{1,p'} + |u|_{2,q}|w_h|_{1,q'}), \\ (c) & |d_2(z; u - u_h, w_h)| \le C\|b(z)\|_{1,\infty} h(|u - u_h|_{1,p}|w_h|_{1,p'} + h|u|_{3,q}|w_h|_{1,q'}), \\ (d) & |d_2(z; u - u_h, w_h)| \le C\|b(z)\|_{1,\infty} h(|u - u_h|_{1,p}|w_h|_{1,p'} + |u|_{2,q}|w_h|_{1,q'}), \end{array} \tag{2.16}$$

Where

$$\begin{array}{l} d_1(z; u - u_h, w_h) = a(z; u - u_h, w_h) + a^*(z; u - u_h, \Pi_h^* w_h), \\ d_2(z; u - u_h, w_h) = b(z; u - u_h, w_h) + b^*(z; u - u_h, \Pi_h^* w_h), \\ 1 \le p,\ q \le \infty,\ \dfrac{1}{p} + \dfrac{1}{p'} = 1,\ \dfrac{1}{q} + \dfrac{1}{q'} = 1. \end{array}$$

Remark 2.1 If $a(z)$ in the bilinear forms $a(z;\cdot,\cdot)$ and $a^*(z;\cdot,\Pi_h^*\cdot)$, and $b(z)$ in the bilinear forms $b(z;\cdot,\cdot)$ and $b^*(z;\cdot,\Pi_h^*\cdot)$ are replaced by other function, the inequalities (2.12) and (2.16) are still valid.

The following lemma gives another key character of the the bilinear forms $a^*(z;\cdot,\Pi_h^*\cdot)$ and $b^*(z;\cdot,\Pi_h^*\cdot)$.

Lemma 2.8(See[6]) For $1 < p < \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, any $u_h, w_h \in U_h$, we have

$$\begin{array}{ll} (a) & |a^*(u; v, \Pi_h^* w_h) - a^*(u_h; v, \Pi_h^* w_h)| \le C|v|_{1,\infty}(\|u - u_h\|_{0,p} + h|u - u_h|_{1,p})\|w_h\|_{1,p'}, \\ (b) & |b^*(u; v, \Pi_h^* w_h) - b^*(u_h; v, \Pi_h^* w_h)| \le C|v|_{1,\infty}(\|u - u_h\|_{0,p} + h|u - u_h|_{1,p})\|w_h\|_{1,p'}. \end{array} \tag{2.17}$$

In order to select λ in (2.8), we next introduce the following lemma:

Lemma 2.9(See[6]) $|c^*(\xi(0); R_h^* u_0, \Pi_h^* w_h)| \le C|R_h^* u_0|_{1,\infty}\|\xi(0)\|\|w_h\|_1$.

Then we now can select λ large enough to ensure the coercivity of the bilinear form $\mathcal{A}(\cdot,\Pi_h^*\cdot)$ in (2.8) over $H_0^1(I)$.

3. Some Properties of the Generalized Ritz-Volterra Projection

In this section, we will prove some properties of $V_h^* w$ defined by (2.7) for the error estimates later. For simplicity, we write $\rho = w - V_h^* w$, $\rho_t = (w - V_h^* w)_t$.

Theorem 3.1 If $V_h^* w$ and u are the solution of (2.7) and (1.1), respectively, and assume that w is sufficiently smooth and h sufficiently small, then for $0 \le t \le T$, we have

$$\|w - V_h^* w\|_{1,p} \le Ch\|w\|_{0,2,p}, \quad 2 \le p \le \infty. \tag{3.1}$$

Proof. (i) Let us first consider the case of $2 \le p < \infty$. We now introduce an auxiliary problem. Denote ϕ_x to be the derivative of ϕ and let $\Phi \in H_0^1(I)$ be the solution of

$$a(u; v, \Phi) = -(v, \phi_x), \quad v \in H_0^1(I), \tag{3.2}$$

and there is a priori estimate

$$\|\Phi\|_{1,p'} \le C\|\phi\|_{0,p'}, \quad \frac{1}{p}+\frac{1}{p'}=1. \tag{3.3}$$

By virtue of Green formula, (2.4) and (2.7), we obtain that

$$\begin{aligned}(\rho_x,\phi) &= -(\rho,\phi_x)\\ &= a(u;\rho,\Phi)\\ &= a(u;w-R_hw,\Phi-R_h\Phi)+a(u;R_hw-V_h^*w,\Phi-R_h\Phi)+a(u;\rho,R_h\Phi)\\ &= a(u;w-R_hw,\Phi)+[a(u;\rho,R_h\Phi)-a^*(u;\rho,\Pi_h^*R_h\Phi)]\\ &\quad +\int_0^t [b(u(\tau);\rho(\tau),R_h\Phi)-b^*(u(\tau);\rho(\tau),\Pi_h^*R_h\Phi)]d\tau\\ &\quad -\int_0^t b(u(\tau);\rho(\tau),R_h\Phi)d\tau\\ &= I_1+I_2+I_3+I_4.\end{aligned} \tag{3.4}$$

Noting that the boundedness of $a(z;\cdot,\cdot)$ and $b(z;\cdot,\cdot)$, (2.13a) and $\|R_h\Phi\|_{1,p'} \le C\|\Phi\|_{1,p'}$, we have

$$\begin{aligned}|I_1| &\le C\|w-R_hw\|_{1,p}\|\Phi\|_{1,p'}\\ &\le Ch\|w\|_{2,p}\|\Phi\|_{1,p'},\end{aligned}$$

and

$$\begin{aligned}|I_4| &\le C\int_0^t \|\rho\|_{1,p}d\tau\|R_h\Phi\|_{1,p'}\\ &\le C\int_0^t \|\rho\|_{1,p}d\tau\|\Phi\|_{1,p'}.\end{aligned}$$

From (2.16b) and (2.16d), we get

$$\begin{aligned}|I_2| &\le Ch(|\rho|_{1,p}+|w|_{2,p})|R_h\Phi|_{1,p'}\\ &\le Ch(\|\rho\|_{1,p}+\|w\|_{2,p})\|\Phi\|_{1,p'},\end{aligned}$$

and

$$|I_3| \le Ch\int_0^t (\|\rho\|_{1,p}+\|w\|_{2,p})d\tau\|\Phi\|_{1,p'}.$$

Combining the estimates of I_1 through I_4 with (3.4), we obtain also by (3.3) that

$$\begin{aligned}\|\rho\|_{1,p} &\le C\sup_{\phi\in L_{p'}(I)}\frac{|(\rho_x,\phi)|}{\|\phi\|_{0,p'}}\\ &\le Ch\{\|w\|_{2,p}+\int_0^t \|w\|_{2,p}d\tau\}+Ch\|\rho\|_{1,p}+C\int_0^t \|\rho\|_{1,p}d\tau.\end{aligned}$$

By letting h sufficiently small such that $Ch \le \frac{1}{2}$, the results of (3.1) for $2 \le p < \infty$ now follows by applying Gronwall's Lemma.

(ii) Let us next consider the case of $p = \infty$. For this purpose, we need to apply the Green function. Following [13], the discrete Green function $g_z^h \in U_h$ associated with $a(u;\cdot,\cdot)$ satisfies

$$a(u; w_h, g_z^h) = \partial_z w_h(z), \quad z \in I, \quad w_h \in U_h. \tag{3.5}$$

Then, by(2.4), we have

$$\begin{aligned} \partial_z(R_h w - V_h^* w) &= a(u; R_h w - V_h^* w, g_z^h) \\ &= a(u; w - V_h^* w, g_z^h) \end{aligned}$$

Consequently, upon replacing $R_h\Phi$, p and p' by g_z^h, ∞ and 1 in part(i), respectively, we can easily derive the conclusion by applying $\|g_z^h\|_{1,1} \le C$ (See [13]) and (2.13a). The proof is completed.

Now we demonstrate the estimates of $w - V_h^* w$ in $L_p(I)(2 \le p \le \infty)$.

Theorem 3.2 If, in addition the hypotheses of Theorem 3.1 , $w \in W^{3,1}(I)$, then for $0 \le t \le T$, we can obtain

$$\|w - V_h^* w\|_{0,p} \le Ch^2 \|w\|_{0,3,1}, \quad 2 \le p \le \infty. \tag{3.6}$$

Proof: For $0 \le t \le T$, the proof also proceeds in two steps.

(i) We consider the case of $p = \infty$.

Noting that the definition of Green function $G_z^h \in U_h$ (See[13]), (2.4) and (2.7), we deduce that

$$\begin{aligned} (R_h w - V_h^* w)(z) &= a(u; R_h w - V_h^* w, G_z^h) \\ &= a(u; w - V_h^* w, G_z^h) \\ &= [a(u; \rho, G_z^h) - a^*(u; \rho, \Pi_h^* G_z^h)] + \int_0^t [b(u(\tau); \rho(\tau), G_z^h) \\ &\quad -b^*(u(\tau); \rho(\tau), \Pi_h^* G_z^h)] d\tau - \int_0^t b(u(\tau); \rho(\tau), G_z^h) d\tau \\ &= d_1(u; \rho, G_z^h) + \int_0^t d_2(u(\tau); \rho(\tau), G_z^h) d\tau \\ &\quad - \int_0^t b(u(\tau); \rho(\tau), G_z^h - G_z^*) d\tau - \int_0^t b(u(\tau); \rho(\tau), G_z^*) d\tau \\ &= J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{3.7}$$

By (2.16a), (2.16b), Sobolev's imbedding inequalities and $\|G_z^h\|_{1,\infty} \le C$, We have

$$\begin{aligned} |J_1| &\le Ch(|\rho|_{1,1} + h|w|_{3,1})|G_z^h|_{1,\infty} \\ &\le Ch(\|\rho\|_1 + h\|w\|_{3,1}), \end{aligned}$$

and

$$|J_2| \le Ch \int_0^t (\|\rho\|_1 + h\|w\|_{3,1}) d\tau.$$

For J_3, it follows from $\|G_z^h - G_z^*\|_{1,1} \leq Ch$ that

$$\begin{aligned} |J_3| &\leq C\int_0^t \|\rho\|_{1,\infty} d\tau \|G_z^h - G_z^*\|_{1,1} \\ &\leq Ch\int_0^t \|\rho\|_{1,\infty} d\tau. \end{aligned}$$

Finally, it is easy to see, by integration by parts and $\|G_z^*\|_{2,1} \leq C$, that

$$\begin{aligned} |J_4| &= |\int_0^t \int_a^b b(t, u(\tau))\rho'(\tau)(G_z^*)' dx d\tau| \\ &\leq C\int_0^t \|\rho\|_{0,\infty} d\tau \|G_z^*\|_{2,1} \\ &\leq C\int_0^t \|\rho\|_{0,\infty} d\tau. \end{aligned}$$

Combining the estimates of $J_1 - -J_4$ with (3.7),we have, by Theorem 3.1,

$$\|R_h w - V_h^* w\|_{0,\infty} \leq Ch^2(\|w\|_{3,1} + \int_0^t \|w\|_{3,1} d\tau) + C\int_0^t \|\rho\|_{0,\infty} d\tau,$$

which together with (2.13a), Sobolev's imbedding inequalities, triangle inequality and Gronwall's Lemma completes the proof of $p = \infty$.

(ii) We consider the case of $2 \leq p < \infty$.
An application of part (i) and inequality

$$\|w - V_h^* w\|_{0,p} \leq C\|w - V_h^* w\|_{0,\infty}, \ 2 \leq p < \infty$$

immediately yields the results of $2 \leq p < \infty$. The proof of Theorem 3.2 is competed.

In order to obtain the estimates of $(w - V_h^* w)_t$ and $(w - V_h^* w)_{tt}$ in $L^2(I)$, we first derive the following.

Theorem 3.3 Under the conditions of Theorem 3.1 , for $0 \leq t \leq T$, we can deduce

$$\|(w - V_h^* w)_t\|_1 \leq Ch\{\|w\|_2 + \|w_t\|_2 + \int_0^t \|w\|_2 d\tau\}$$

Proof: Differentiating (2.7) with respect to t, we see that

$$a^*(u; \rho_t, v_h) + a_t^*(u; \rho, v_h) + b^*(u; \rho, v_h) + \int_0^t b_t^*(u; \rho, v_h) d\tau = 0, \quad v_h \in V_h, \qquad (3.8)$$

where the coefficients of $a_t^*(\cdot; \cdot, \cdot)$ and $b_t^*(\cdot; \cdot, \cdot)$ are obtained from differentiating the corresponding coefficients of $a^*(\cdot; \cdot, \cdot)$ and $b^*(\cdot; \cdot, \cdot)$ with respect to t, respectively.

For simplicity, we set $\theta = \Pi_h w - V_h^* w = (\Pi_h w - w) + (w - V_h^* w) = \zeta + \rho, \ \ \theta_t = \Pi_h w_t - (V_h^* w)_t$.

Then, by (2.12b), (3.8), (2.16), Theorem 3.1 and (2.1a), we can derive

$$\begin{aligned}\alpha_0\|\theta_t\|_1^2 &\le a^*(u;\theta_t,\Pi_h^*\theta_t)\\ &= a^*(u;\zeta_t,\Pi_h^*\theta_t)+a^*(u;\rho_t,\Pi_h^*\theta_t)\\ &= a^*(u;\zeta_t,\Pi_h^*\theta_t)-a_t^*(u;\rho,\Pi_h^*\theta_t)-b^*(u;\rho,\Pi_h^*\theta_t)-\int_0^t b_t^*(u;\rho,\Pi_h^*\theta_t)d\tau\\ &= \{-d_1(u;\zeta_t,\theta_t)+d_{1,t}(u;\rho,\theta_t)+d_2(u;\rho,\theta_t)\}+\int_0^t d_{2,t}(u(t,\tau);\rho(\tau),\theta_t)d\tau\\ &\quad+\{a(u;\zeta_t,\theta_t)-a_t(u;\rho,\theta_t)-b(u;\rho,\theta_t)\}-\int_0^t b_t(u(t,\tau);\rho(\tau),\theta_t)d\tau\\ &\le Ch(|\zeta_t|_1+|w|_2+|\rho|_1)|\theta_t|_1+Ch\int_0^t(|w|_2+|\rho|_1)d\tau|\theta_t|_1\\ &\quad+C(\|\zeta_t\|_1+\|\rho\|_1)\|\theta_t\|_1+C\int_0^t\|\rho\|_1)d\tau\|\theta_t\|_1\\ &\le Ch\{\|w\|_2+\|w_t\|_2+\int_0^t\|w\|_2d\tau\}\|\theta_t\|_1\end{aligned}$$

Thus, the conclusion follows from (2.1a) and triangle inequality.

We still need to proof the following lemma.

Lemma 3.1 If, in addition the hypotheses of Theorem 3.1 , V_hw is the solution of (2.5), then for $0\le t\le T$, we have

$$\|V_hw-V_h^*w\|_{1,p}\le Ch^2\|w\|_{0,3,p},\quad 2\le p\le\infty. \tag{3.9}$$

Proof: (i) For $2\le p<\infty$, we also introduce the auxiliary problem (3.2) and (3.3) used in the proof of Theorem 3.1.

Then, by (2.4), (2.5), (2.7), Lemma 2.7, (2.15a) and Theorem 3.1,

$$\begin{aligned}((V_hw-V_h^*w_x,\phi) &= a(u;V_hw-V_h^*w,\Phi)\\ &= a(u;V_hw-w,R_h\Phi)+a(u;w-V_h^*w,R_h\Phi)\\ &= \{\int_0^t b(u;w-V_hw,R_h\Phi)d\tau-\int_0^t b^*(u;w-V_hw,\Pi_h^*R_h\Phi)d\tau\}\\ &\quad+\{a(u;w-V_h^*w,R_h\Phi)-a^*(u;w-V_h^*w,\Pi_h^*R_h\Phi)\}\\ &\quad-\{\int_0^t b^*(u;w-V_h^*w,\Pi_h^*R_h\Phi)d\tau-\int_0^t b^*(u;w-V_hw,\Pi_h^*R_h\Phi)d\tau\}\\ &\le C\{\int_0^t h(|w-V_hw|_{1,p}+h|w|_{3,p})d\tau\}\|R_h\Phi\|_{1,p'}+h(|w-V_h^*w|_{1,p}\\ &\quad+h|w|_{3,p})\|R_h\Phi\|_{1,p'}\}+|\int_0^t b^*(u;V_hw-V_h^*w,\Pi_h^*R_h\Phi)d\tau|\\ &\le C\{h^2(\|w\|_{3,p}+\int_0^t\|w\|_{3,p}d\tau)+\int_0^t\|V_hw-V_h^*w\|_{1,p}d\tau\}\|\Phi\|_{1,p'}.\end{aligned} \tag{3.10}$$

Accordingly, the conclusion (3.9) for $2\le p<\infty$ is derived from the above inequality, (3.3) and Gronwall's Lemma.

(ii) The proof of $p=\infty$ is similar to that of part(ii) in Theorem 3.1. This completes the proof for $2 \le p \le \infty$.

From Theorem 3.3 and Lemma 3.1, we can obtain the following.

Theorem 3.4 Under the conditions of Theorem 3.1 , for $0 \le t \le T$, we have

$$\|D_t^k(w - V_h^* w)\| \le Ch^2 \|w\|_{k,3,p}, \quad k = 1, 2, \quad 2 \le p \le \infty. \tag{3.11}$$

Proof. Only the case of $k = 1$ will be proved.

We now apply Nitsche technique or duality argument to obtain $\|(w - V_h^* w)_t\|$. For $\psi \in L^2(I)$, let $\Psi \in H_0^1(I)$, such that

$$a(u; v, \Psi) = (v, \psi), \quad v \in H_0^1(I), \tag{3.12}$$

and there is the regularity estimate

$$\|\Psi\|_2 \le C\|\psi\|. \tag{3.13}$$

We differentiate (2.5) to get

$$a(u; (w - V_h w)_t, v_h) + a_t(u; w - V_h w, v_h) + b(u; w - V_h w, v_h) + \int_0^t b_t(u; w - V_h w, v_h)d\tau = 0, \ v_h \in U_h, \tag{3.14}$$

where the coefficients of $a_t(\cdot;\cdot,\cdot)$ and $b_t(\cdot;\cdot,\cdot)$ are obtained from differentiating the corresponding coefficients of $a(\cdot;\cdot,\cdot)$ and $b(\cdot;\cdot,\cdot)$ with respect to t, respectively. Then, using (3.8) and (3.14), we write

$$\begin{aligned}
(\rho_t, \psi) &= a(u; \rho_t, \Psi) \\
&= a(u; \rho_t, \Psi - V_h\Psi) + a(u; \rho_t, V_h\Psi) \\
&= a(u; \rho_t, \Psi - V_h\Psi) + d_1(u; \rho_t, V_h\Psi) + a^*(u; \rho_t, \Pi_h^* V_h\Psi) \\
&= a(u; \rho_t, \Psi - V_h\Psi) + d_1(u; \rho_t, V_h\Psi) - a_t^*(u; \rho, \Pi_h^* V_h\Psi) \\
&\quad -b^*(u; \rho, \Pi_h^* V_h\Psi) - \int_0^t b_t^*(u; \rho, \Pi_h^* V_h\Psi)d\tau \\
&= \{a(u; \rho_t, \Psi - V_h\Psi) + d_1(u; \rho_t, V_h\Psi) + d_{1,t}(u; w - V_h w, V_h\Psi) \\
&\quad +d_2(u; w - V_h w, V_h\Psi) + \int_0^t d_{2,t}(u; w - V_h w, V_h\Psi)d\tau\} \\
&\quad +a(u; (w - V_h w)_t, V_h\Psi) - \{a_t^*(u; V_h w - V_h^* w, \Pi_h^* V_h\Psi) \\
&\quad +b^*(u; V_h w - V_h^* w, \Pi_h^* V_h\Psi) + \int_0^t b_t^*(u; V_h w - V_h^* w, \Pi_h^* V_h\Psi)d\tau\} \\
&= Q_1 + Q_2 + Q_3.
\end{aligned} \tag{3.15}$$

Applying Lemmas 2.7, 2.6 and Theorem 3.3, we get

$$\begin{aligned}
|Q_1| &\le Ch\{\|\rho_t\|_1\|\Psi\|_2 + (\|\rho_t\|_1 + h\|w_t\|_3)\|V_h\Psi\|_1\} + Ch\{h\|w_t\|_{3,p} + \|w - V_h w\|_{1,p} \\
&\quad +h\|w\|_{3,p} + \int_0^t (\|w - V_h w\|_{1,p} + h\|w\|_{3,p})d\tau\}\|V_h\Psi\|_{1,p'} \\
&\le Ch^2\{\|w\|_{3,p} + \|w_t\|_{3,p} + \int_0^t \|w\|_{3,p}d\tau\}\|\Psi\|_2, \ 2 \le p \le \infty, \ \frac{1}{p} + \frac{1}{p'} = 1.
\end{aligned}$$

where $\|V_h\Psi\|_{1,p'} \le C\|\Psi\|_{1,p'}$ and Sobolev's imbedding inequalities have been used. Also we know, from Green formula , Lemmas 2.6 and 3.1, that

$$\begin{aligned}|Q_2| &= |a(u;(w-V_hw)_t, V_h\Psi-\Psi)+a(u;(w-V_hw)_t,\Psi)| \\ &\le Ch\|(w-V_hw)_t\|_1\|\Psi\|_2 + C\|(w-V_hw)_t\|_{0,p}\|\Psi\|_{2,p'} \\ &\le Ch^2\{\|w\|_{2,p}+\|w_t\|_{2,p}+\int_0^t(\|w\|_{2,p}+\|w_t\|_{2,p})d\tau\}\|\Psi\|_2,\end{aligned}$$

and

$$\begin{aligned}|Q_3| &\le C\{\|V_hw-V_h^*w\|_{1,p}+\int_0^t\|V_hw-V_h^*w\|_{1,p}d\tau\}\|V_h\Psi\|_{1,p'} \\ &\le Ch^2\{\|w\|_{3,p}+\int_0^t\|w\|_{3,p}d\tau\}\|\Psi\|_2.\end{aligned}$$

Combining the estimates of $Q_1 - -Q_3$ with (3.15), we obtain also by (3.13) that

$$|(\rho_t,\psi)| \le Ch^2\{\|w\|_{3,p}+\|w_t\|_{3,p}+\int_0^t(\|w\|_{3,p}+\|w_t\|_{3,p})d\tau\}\|\psi\|,\ \psi\in L^2(I),$$

which implies the conclusion of $k=1$.

Similarly, the case of $k=2$ can be proved. The proof is completed.

4. Main Results

We let $\xi = u_h - V_h^*u, \eta = V_h^*u - u$ as in section 2. Noting that $V_h^*(0) = R_h^*(0)$, from [6], we have the following lemma.

Lemma 4.1 Assume that u_0 and u_{0h} are the initial values of problems (1.1) and (2.8), respectively, then we have

$$\begin{aligned}&(a) \quad \|\xi(0)\|_1 \le Ch^2\|u_0\|_{3,1} \\ &(b) \quad \|\xi_t(0)\| \le Ch^2\{\|u_0\|_{3,p}+\|u_t(0)\|_{3,p}\},\ p>1.\end{aligned} \tag{4.1}$$

where $u_t(0) = \frac{\partial}{\partial x}(a(x,u_0)\frac{\partial u_0}{\partial x}) + f(x,0,u_0)$.

Now, let us consider estimates of ξ and ξ_t.

Lemma 4.2 Assume that u and u_h are the solutions of problems (1.1) and (2.8), respectively, then, for h sufficiently small, we have

$$\|\xi_t\|+\|\xi\|_1+(\int_0^t\|\xi_t\|_1^2d\tau)^{\frac12} \le Ch^2\{\|u_0\|_{3,p}+\|u_t(0)\|_{3,p}+\|u\|_{2,3,p}\},\ 2\le p\le\infty,\ 0\le t\le T. \tag{4.2}$$

Proof. To show (4.2), apply (2.3a), (2.8a) and (2.7) to get the error equation

$$\begin{aligned}&(\xi_t,v_h)+a^*(u_h;\xi,v_h)+\int_0^t b^*(u_h;\xi,v_h)d\tau \\ &=(f(u_h)-f(u)-\eta_t,v_h)+a^*(u;V_h^*u,v_h)-a^*(u_h;V_h^*u,v_h) \\ &\quad+\int_0^t b^*(u;V_h^*u,v_h)d\tau-\int_0^t b^*(u_h;V_h^*u,v_h)d\tau.\end{aligned} \tag{4.3}$$

We differentiate (4.3) with respect to t to get

$$
\begin{aligned}
&(\xi_{tt}, v_h) + a^*(u_h; \xi_t, v_h) \\
&= -a_t^*(u_h; \xi, v_h) - b^*(u_h; \xi, v_h) - \int_0^t b_t^*(u_h; \xi, v_h) d\tau \\
&+((f(u_h) - f(u) - \eta_t)_t, v_h) + \{a^*(u; (V_h^* u)_t, v_h) - a^*(u_h; (V_h^* u)_t, v_h)\} \\
&+\{a_t^*(u; V_h^* u, v_h) - a_t^*(u_h; V_h^* u, v_h)\} + \{b^*(u; V_h^* u, v_h) \\
&-b^*(u_h; V_h^* u, v_h)\} + \int_0^t \{b_t^*(u; V_h^* u, v_h) - b_t^*(u_h; V_h^* u, v_h)\} d\tau.
\end{aligned}
\tag{4.4}
$$

Setting $v_h = \Pi_h^* \xi_t$ and using Lemmas 2.3 and 2.8, we have , by the boundedness of $\|V_h^* u\|_{1,\infty}$ and $\|(V_h^* u)_t\|_{1,\infty}$ and ε-inequality,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|||\xi_t|||^2 + \alpha_0 \|\xi_t\|_1^2 \;&\leq C\{\|\xi\|_1 + \int_0^t \|\xi\|_1 d\tau + \|\xi_t\| + \|\eta_t\| + \|\eta_{tt}\| \\
&+\|\xi\| + \|\eta\| + h(\|\xi\|_1 + \|\eta\|_1 + \|\xi_t\|_1 + \|\eta_t\|_1) \\
&+ \int_0^t (\|\xi_t\| + \|\eta_t\| + h(\|\xi_t\|_1 + \|\eta_t\|_1) d\tau\} \\
&\leq (\varepsilon + Ch)\|\xi_t\|_1^2 + C\{\|\xi\|_1 + \int_0^t \|\xi\|_1 d\tau + \|\xi_t\|\}^2 \\
&+C\{\|\eta\| + \|\eta_t\| + \|\eta_{tt}\| + h(\|\eta\|_1 + \|\eta_t\|_1) + \int_0^t (\|\eta_t\| + h\|\eta_t\|_1) d\tau\}^2 \\
&+C\int_0^t \|\xi_t\|^2 d\tau + Ch^2 \int_0^t \|\xi_t\|_1^2 d\tau.
\end{aligned}
$$

Hence, letting h sufficiently small and using Gronwall's Lemma to eliminate the first and last terms on the right hand side, respectively, and applying Theorems 3.1–3.4 and Lemma 4.1, we obtain

$$
\|\xi_t\|^2 + \int_0^t \|\xi_t\|_1^2 d\tau \leq Ch^2\{\|u_0\|_{3,p} + \|u_t(0)\|_{3,p} + \|u\|_{2,3,p}\}^2 + C\int_0^t (\|\xi\|_1^2 + \|\xi_t\|^2) d\tau.
$$

Observing that

$$
\begin{aligned}
\|\xi\|_1^2 \;&= \|\xi(0)\|_1^2 + \int_0^t \frac{d}{dt}[(\xi, \xi) + (\xi', \xi')] d\tau \\
&\leq \|\xi(0)\|_1^2 + \varepsilon \int_0^t \|\xi_t\|_1^2 d\tau + C\int_0^t \|\xi\|_1^2 d\tau,
\end{aligned}
$$

then, by (4.1a), we have

$$
\|\xi_t\|^2 + \|\xi\|_1^2 + \int_0^t \|\xi_t\|_1^2 d\tau \leq Ch^2\{\|u_0\|_{3,p} + \|u_t(0)\|_{3,p} + \|u\|_{2,3,p}\}^2 + C\int_0^t (\|\xi\|_1^2 + \|\xi_t\|^2) d\tau.
$$

The result of this lemma follows by applying Gronwall's Lemma.

We next demonstrate a superconvengence result of ξ.

Theorem 4.1 Under the conditions of Lemmas 4.1 and 4.2, for $0 \le t \le T$, we have

$$\|\xi\|_{1,p} \le Ch^2\{\|u_0\|_{3,p} + \|u_t(0)\|_{3,p} + \|u\|_{2,3,p}\}, \quad 2 \le p \le \infty. \tag{4.5}$$

Proof: (i) Consider the case of $2 \le p < \infty$.

In order to show (4.5), we also introduce the auxiliary problem (3.2) and (3.3). We then know from (4.2), Lemmas 2.7, 2.8 and the inverse property, that

$$\begin{aligned}
(\xi_x, \phi) &= a(u; \xi, \Phi) \\
&= a(u; \xi, R_h\Phi) \\
&= d_1(u; u_h - u, R_h\Phi) + d_1(u; u - V_h^* u, R_h\Phi) + a^*(u; \xi, \Pi_h^* R_h\Phi) \\
&= d_1(u; u_h - u, R_h\Phi) + d_1(u; u - V_h^* u, R_h\Phi) + (f(u_h) - f(u) - \eta_t - \xi_t, \Pi_h^* R_h\Phi) \\
&\quad + [a^*(u; V_h^* u, \Pi_h^* R_h\Phi) - a^*(u_h; V_h^* u, \Pi_h^* R_h\Phi)] + \int_0^t [b^*(u; V_h^* u, \Pi_h^* R_h\Phi) \\
&\quad - b^*(u_h; V_h^* u, \Pi_h^* R_h\Phi)]d\tau - \int_0^t b^*(u_h; \xi, \Pi_h^* R_h\Phi)d\tau \\
&\quad + [a^*(u; \xi, \Pi_h^* R_h\Phi) - a^*(u_h; \xi, \Pi_h^* R_h\Phi)] \\
&\le C\{h(\|\xi\|_{1,p} + \|\eta\|_{1,p} + h\|u\|_{3,p}) + \|\xi\|_{0,p} + \|\eta\|_{0,p} + \|\xi_t\| + \|\eta_t\| \\
&\quad + \int_0^t [\|\xi\|_{0,p} + \|\eta\|_{0,p} + h(\|\xi\|_{1,p} + \|\eta\|_{1,p}) + \|\xi\|_{1,p}]d\tau\}\|\Phi\|_{1,p'} \\
&\quad + Ch^{-\frac{1}{2}}\|\xi\|_1\{\|\xi\|_{0,p} + \|\eta\|_{0,p} + h(\|\xi\|_{1,p} + \|\eta\|_{1,p})\}\|\Phi\|_{1,p'}.
\end{aligned}$$

Then, by (3.3), Lemma 4.2 and the imbedding property $W^{1,2}(I) \hookrightarrow L^p(I)$,

$$\begin{aligned}
\|\xi\|_{1,p} &\le C|\xi|_{1,p} = C \sup_{\phi \in L_{p'}} \frac{|(\xi_x, \phi)|}{\|\phi\|_{0,p'}} \\
&\le Ch\|\xi\|_{1,p} + C\{h\|\eta\|_{1,p} + h^2\|u\|_{3,p} + \|\eta\|_{0,p} + \|\eta_t\| + \|\xi\|_1 + \|\xi_t\| \\
&\quad + \int_0^t (\|\xi\|_1 + \|\eta\|_{0,p} + h\|\eta\|_{1,p})d\tau\} + C\int_0^t \|\xi\|_{1,p}d\tau.
\end{aligned}$$

After eliminating the first term for h sufficiently small and the last term by Gronwall's Lemma on the right hand side, the results (4.5) for $2 \le p < \infty$ now follows by Theorems 3.1, 3.2, 3.4 and Lemma 4.2.

(ii) Consider the case of $p = \infty$.

By (3.5),

$$\xi_z(z) = a(u; \xi, g_z^h).$$

Therefore, similar to part(ii) of Theorem 3.1, the proof is easily completed.

$W^{1,p}$ and L_p norms error estimates for $u - u_h$ are then an immediate consequence of Theorem 4.1 combined with Theorems 3.1 and 3.2.

Theorem 4.2 Under the same conditions of Theorem 4.1, for $0 \le t \le T$, we have

$$\|u - u_h\|_{0,p} + h\|u - u_h\|_{1,p} \le Ch^2\{\|u_0\|_{3,p} + \|u_t(0)\|_{3,p} + \|u\|_{2,3,p}\}, \quad 2 \le p \le \infty.$$

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