J. KSIAM **Vol.7**, No.2, 23-34, 2003

CONTROLLABILITY OF PERTURBED INTEGRODIFFERENTIAL SYSTEMS WITH PRESCRIBED CONTROLS

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ABSTRACT. In this paper we establish a set of sufficient conditions for the controllability of perturbed integrodifferential systems with prescribed controls by using the Schaefer fixed point theorem.

1. INTRODUCTION

The problem of controllability of nonlinear systems by means of fixed point principles has been studied by several authors [4]. Fixed point theorem due to Schaefer, Fan, Tychonov and Schauder have been respectively applied by Anichini [1,2], Dauer [7], Kartsatos [9] and Lukes [10] for studying the controllability of nonlinear systems with prescribed controls. Balachandran [3] studied controllability of nonlinear Volterra integrodifferential systems and Balachandran and Lalitha [5] discussed the controllability of nonlinear Volterra integrodifferential systems with prescribed controls. In this paper we study the controllability of perturbed integrodifferential systems by using the Schaefer fixed point theorem.

In this work, we have to find sufficient conditions for the controllability of the perturbed integrodifferential system

$$(1)\dot{x}(t) = A(t)x(t) + \int_{t_0}^{t} H(t,s)x(s)ds + B(t)u(t) + g\left(t, x(t), \int_{t_0}^{t} K(t,s)x(s)ds\right)$$

by means of controls whose initial and final values can be prescribed in advance. That is,we want to establish conditions on $A(t)$, $B(t)$, $H(t,s)$ and $g(t,x,\int_{t_0}^{t} K(t,s)x(s)ds)$ which ensure that, for each $t_0, T \in R$; $\alpha, \beta \in R^m$; $x_0, x_T \in R^n$, there exists a control $u \in C([t_0,T];R^m)$ for (1) with $u(t_0) = \alpha$, $u(T) = \beta$ which produces a response $x(t;u)$

2000 AMS Subject Classification: 93B05
Keywords and phrases: Integrodifferential system, Schaefer fixed point theorem, Controllability, Boundary value problem

satisfying the boundary conditions $x(t_0;u) = x_0$, and $x(T;u) = x_T$. This result will be established by a fixed point argument to the linear boundary value problem

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + \int_{t_0}^{t} H(t,s)x(s)ds + B(t)u(t) + g\left(t, z(t), \int_{t_0}^{t} K(t,\tau)z(\tau)d\tau\right) \\ x(t_0) &= x_0, \quad x(T) = x_T, \quad u(t_0) = \alpha, \quad u(T) = \beta \end{aligned}$$

where $z \in C([t_0,T];R^n)$, the space of continuous functions with sup norm. For brevity let us take $t_0 = 0$, $\alpha = u_0$, $\beta = u_T$.

2. PRELIMINARIES

Consider the perturbed integrodifferential system of the form

$$(2)\ \dot{x}(t) = A(t)x(t) + \int_0^t H(t,s)x(s)ds + B(t)u(t) + g\left(t, x(t), \int_0^t K(t,s)x(s)ds\right)$$

where x is an n vector and u is an m vector functions. The matrix functions $A : J \to R^{n^2}$, $B : J \to R^{nm}$, $J = [0,T]$, $H, K = \Delta \to R^{n^2}$, $\Delta = \{(t,s) : 0 \le t \le s \le T\}$ are assumed to be continuous and the function $g : J \times R^n \times R^n \to R^n$ is such that, $g(t,x,y) \in C^1(J \times R^n \times R^n, R^n)$.

Assume that for $t \in J$, $(t,s) \in \Delta$, there exist positive constants m_1, m_2, m_3, m_4, m_5, and a continuous function $\gamma(t)$ such that, $\|A(t)\| \le m_1$, $\|B(t)\| \le m_2$, $\|H(t,s)\| \le m_3$, $\|K(t,s)\| \le m_4$ and $\|g(t,x,y)\| \le m_5\ \gamma(t)$, where the norm of a matrix is taken as in [2].

We observe that the hypothesis on $A(t)$ and $H(t,s)$ allow us to say that there exists a unique continuous matrix $E(t,s)$ such that [6],

$$\frac{\partial E(t,s)}{\partial s} + E(t,s)A(s) + \int_s^t E(t,w)H(w,s)dw = 0,$$

with $E(t,t) = I$, the identity matrix for $0 \le s \le t \le T$. The system (1) is controllable on $[0,T]$ by means of a certain set U of controls iff for every pair x_0, $x_T \in R^n$, there exists $u \in U$ such that $x(0;u) = x_0$ and $x(T;u) = x_T$. For brevity let us denote,

$$\begin{aligned} P(t;\theta) &= \int_0^t E(\theta, \theta - s)B(\theta - s)ds \\ \bar{C}(t;T) &= \int_{T-t}^{T} P(s;T)^* ds - \frac{t}{T}\int_0^T P(s;T)^* ds \\ S(t;T) &= \int_0^t E(t,s)B(s)\bar{C}(s;T)ds \end{aligned}$$

and define

$$\begin{aligned} M(0,t) &= \int_0^t B(s)B(s)^* ds \\ \bar{S}(T) &= \int_0^T P(s;\theta)P(s;\theta)^* ds - \frac{1}{T}\left[\int_0^T P(s;\theta)ds\right]\left[\int_0^T P(s;\theta)^* ds\right] \end{aligned}$$

where the star denotes the matrix transpose. We observe that $P(t;\theta)$, $\bar{C}(t;T)$, and $S(t;T)$ are continuous. To prove the main result, we use the following fixed point theorem.

Schaefer's Theorem: [8] Let S be a convex subset of a normed linear space X and assume that $0 \in S$. Let $T : S \to S$ be completely continuous operator, and let

$$\zeta(T) = \{x \in S : x = \lambda T x \text{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(T)$ is unbounded or T has a fixed point.

The following theorem is vital to the criterion of controllability.

Theorem 2.1: Assume that the control process (1) satisfies the hypotheses. If the matrix $M(0, t_1)$ is nonsingular for some $t_1 > 0$, then the set of points attainable by the trajectories of the control process (1) is all of R^n.

Proof: For fixed u, the given system has solution $x(t; u)$ which satisfies

$$\begin{aligned} (3) \qquad x(t;u) &= x_0 + \int_0^t A(s)x(s;u)ds + \int_0^t \left[\int_0^s H(s,\tau)x(\tau;u)d\tau\right] ds \\ &+ \int_0^t B(s)u(s)ds + \int_0^t g\left(s, x(s;u), \int_0^s K(s,\tau)x(\tau;u)d\tau\right) ds. \end{aligned}$$

Let x_1 be any given point in R^n. We have to find a control v such that for finite time $t_1 > 0$, $x(t_1; v) = x_1$. Consider the control of the form $v(t) = B(t)^* q$, where $q \in R^n$. Define a mapping $S : R^n \to R^n$ as

$$S(q) = M^{-1}(0, t_1)[x_1 - K(q) - x_0]$$

where

$$\begin{aligned} K(q) &= \int_0^{t_1} A(s)x(s;q)ds + \int_0^{t_1} \left[\int_0^s H(s,\tau)x(\tau;q)d\tau\right] ds \\ &+ \int_0^{t_1} g\left(s, x(s;q), \int_0^s K(s,\tau)x(\tau;q)d\tau\right) ds. \end{aligned}$$

Suppose that the mapping $q \to S(q)$ has a fixed point, then

$$q = M^{-1}(0, t_1)[x_1 - K(q) - x_0]$$

and from (3)

$$\begin{aligned} x(t_1;v) &= x_0 + \int_0^{t_1} A(s)x(s;q)ds + \int_0^{t_1}\left[\int_0^s H(s,\tau)x(\tau;q)d\tau\right]ds \\ &\quad + \int_0^{t_1} B(s)B(s)^* M^{-1}(0,t_1)[x_1 - K(q) - x_0]ds \\ &\quad + \int_0^{t_1} g\left(s, x(s;q), \int_0^s K(s,\tau)x(\tau;q)d\tau\right)ds \\ &= x_1. \end{aligned}$$

Now, we shall prove that the mapping $q \to S(q)$ has a fixed point. Since all the functions involved in the definition of the operator S are continuous, and hence this mapping is continuous. Then from (3), we have

$$\begin{aligned} \|x(t;u)\| &\le \|x_0\| + \int_0^t \|A(s)\|\|x(s;u)\|ds + \int_0^t\left[\int_s^t \|H(\tau,s)\|\|x(s;u)\|d\tau\right]ds \\ &\quad + \int_0^t \|B(s)\|\|u(s)\|ds + \int_0^t \left\|g\left(s, x(s;u), \int_0^s K(s,\tau)x(\tau;u)d\tau\right)\right\|ds \\ &\le \|x_0\| + \int_0^t m_1\|x(s;u)\|ds + \int_0^t\left[\int_s^t m_3\|x(s;u)\|d\tau\right]ds \\ &\quad + \int_0^T \sup\{m_2\|u(s)\|, s\in[0,T]\}ds + \int_0^t m_5\gamma(s)ds \\ &\le \alpha_0 + \int_0^t\left[m_1 + \int_s^t m_3 d\tau\right]\|x(s;u)\|ds \end{aligned}$$

where

$$\begin{aligned} \alpha_0 &= \|x_0\| + m_2\int_0^T \sup\{\|u(s)\|, s\in[0,T]\}ds + \gamma_0 \\ \gamma_0 &= m_5\int_0^T \sup\{\gamma(s), s\in[0,T]\}ds. \end{aligned}$$

Therefore, by Gronwall's inequality

$$\begin{aligned} \|x(t;u)\| &\le \alpha_0 exp\left(\int_0^t (m_1 + \int_s^t m_3 d\tau)ds\right) \\ &\le \alpha_0 exp\left(m_1 T + \frac{m_3 T^2}{2}\right) \end{aligned}$$

Thus if, $\|q\| < +\infty$, then $\|x(t;u)\| < +\infty$, which implies that $\|K(q)\| < +\infty$ and hence $\|S(q)\| < +\infty$. Thus $S(q)$ sends bounded sets into bounded sets. By a similar argument, we can show that the solutions of the equation $q = \lambda\ S(q)$, for $0 < \lambda < 1$, are bounded. Then, by Schaefer's theorem the mapping has a fixed point.

3. MAIN RESULT

For $z \in Z = C(J, R^n)$ consider

$$x_z(t) = E(t,0)x_0 + P(t;t)u_0 + \frac{1}{T}(u_T - u_0)\int_0^t P(s;t)ds + S(t;T)y_z(T) + \int_0^t E(t,s)g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right)ds \tag{4}$$

$$u_z(t) = (1 - \frac{t}{T})u_0 + \frac{t}{T}u_T + \bar{C}(t;T)y_z(T) \tag{5}$$

where

$$y_z(t) = [\bar{S}(T)]^{-1}[x_T - E(T,0)x_0 - P(T;t)u_0 - \frac{1}{T}(u_T - u_0)\int_0^T P(s;t)ds - \int_0^T E(T,s)g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right)ds]. \tag{6}$$

Before going to the main result, we state the following proposition without proof.

Proposition 3.1: For all $z \in Z$, we have

$$\int_0^t E(t,s)B(s)u_z(s)ds = P(t;t)u_0 + \frac{1}{T}(u_T - u_0)\int_0^t P(s;t)ds + S(t;T)y_z(T)$$

and $S(T;T) = \bar{S}(T)$.

Proposition 3.2: Consider the boundary value control process,

$$\dot{x}(t) = A(t)x(t) + \int_0^t H(t,s)x(s)ds + B(t)u(t) + g\left(t, z(t), \int_0^t K(t,\tau)z(\tau)d\tau\right)$$
$$(7) x(0) = x_0, \quad x(T) = x_T, \quad u(0) = u_0, \quad u(T) = u_T.$$

Then, if the matrix $M(0,T)$ is nonsingular, every pair $(x_z(t), u_z(t))$ defined in (4) and (5) provides a solution to the boundary value control process (7).

Proof: If the matrix $M(0,T)$ is nonsingular, then the control system (7) is controllable on $[0,T]$. Moreover, the inverse $[\bar{S}(T)]^{-1}$ exists. Thus the pair $(x_z(t), u_z(t))$ defined in (4) and (5) is well defined for all $z \in Z$. We have to show that

$$\dot{x}(t) = A(t)x(t) + \int_0^t H(t,s)x(s)ds + B(t)u(t) + g\left(t, z(t), \int_0^t K(t,\tau)z(\tau)d\tau\right)$$
$$x(0) = x_0, \quad x(T) = x_T, \quad u(0) = u_0, \quad u(T) = u_T.$$

Now we using (4),we get

$$\begin{aligned} x_z(t) &= E(t,0)x_0 + P(t;t)u_0 + \frac{1}{T}(u_T - u_0)\int_0^t P(s;t)ds \\ &\quad + S(t;T)y_z(T) + \int_0^t E(t,s)g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right) ds. \end{aligned}$$

By Proposition (3.1), we have

$$x_z(t) = E(t,0)x_0 + \int_0^t E(t,s)B(s)u_z(s)ds + \int_0^t E(t,s)g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right) ds.$$

Differentiating, we get

$$\begin{aligned} \dot{x}_z(t) &= \frac{\partial E(t,0)x_0}{\partial t} + \int_0^t \frac{\partial E(t,s)}{\partial t} B(s)u_z(s)ds \\ &\quad + E(t,t)B(t)u_z(t) + \int_0^t \frac{\partial E(t,s)}{\partial t} g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right) ds \\ &\quad + E(t,t)g\left(t, z(t), \int_0^t K(t,\tau)z(\tau)d\tau\right). \\ &= A(t)\left[E(t,0)x_0 + \int_0^t E(t,s)B(s)u_z(s)ds \right. \\ &\quad \left. + \int_0^t E(t,s)g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right) ds\right] \\ &\quad + \int_0^t H(t,\tau)\left[E(\tau,0)x_0 + \int_0^\tau E(\tau,s)B(s)u_z(s)ds \right. \\ &\quad \left. + \int_0^\tau E(\tau,s)g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right) ds\right] d\tau + B(t)u_z(t) \\ &\quad + g\left(t, z(t), \int_0^t K(t,\tau)z(\tau)d\tau\right) \\ &= A(t)x_z(t) + \int_0^t H(t,\tau)x_z(\tau)d\tau + B(t)u_z(t) + g\left(t, z(t), \int_0^t K(t,\tau)z(\tau)d\tau\right). \end{aligned}$$

Also we have

$$\begin{aligned} x_z(T) &= E(T,0)x_0 + P(T;T)u_0 + \frac{1}{T}(u_T - u_0)\int_0^T P(s;T)ds \\ &\quad + S(T;T)y_z(T) + \int_0^T E(T,s)g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right)ds \\ &= E(T,0)x_0 + P(T;T)u_0 + \frac{1}{T}(u_T - u_0)\int_0^T P(s;T)ds \\ &+ S(T;T)[\bar{S}(T;T)]^{-1}[x_T - E(T,0)x_0 - P(T;T)u_0 - \frac{1}{T}(u_T - u_0)\int_0^T P(s;T)ds \\ &\quad - \int_0^T E(T,s)g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right)ds] \\ &\quad + \int_0^T E(T,s)g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right)ds \\ &= x_T \end{aligned}$$

and $x_z(0) = x_0, \quad u_z(0) = u_0, \quad u_z(T) = u_T.$

Now we shall prove the main result of this paper.

Theorem 3.1: Assume that the nonlinear control process (1) satisfies the hypotheses and that the matrix $M(0,T)$ is nonsingular for $T > 0$. Then for every $\alpha, \beta, \gamma \in R^m$, $x_0, x_1, x_T \in R^n$ and every $w \in [0,T]$ there exists a control v, such that,

: (a) $v(0) = \alpha, \quad v(w) = \beta, \quad v(T) = \gamma$
: (b) the response of (1), for which $x(0;v) = x_0$, satisfies $x(w;v) = x_1$ and $x(T;v) = x_T$.

Proof: Consider the mapping $Q : z \in Z \to Q(z) = x_z \in Z$ where $x_z = x_z(t)$ and $x_z(t)$ and $u_z(t)$ are defined in (4) and (5) respectively. Then the proof is based upon two applications of Proposition 3.2.

First setting $w = T$, $u_0 = \alpha$, $u_T = \beta$, and $x_T = x_1$, we can obtain a response $x(t;v)$ of (1) such that $x(0;v) = x_0$ and $x(w;v) = x_1$. Then, setting $u_0 = \beta$, $x_0 = x_1$, and $u_T = \gamma$ we can obtain a response $x(t;v)$ of (1) such that $x(w;v) = x_1$ and $x(T;v) = x_T$. Thus, we extend the response $x(t;v)$ to whole interval $[0,T]$ and hence the theorem is proved.

To show that the mapping Q has a fixed point, we use Schaefer's theorem. Since $E(.,.)$, $P(.;.)$ and $\bar{S}(.)$ are continuous and $g(., z, y)$ is continuous with respect to z, y, we can say that $z \to y_z(t)$ is continuous with respect to z. Thus the map $z \to x_z$ is continuous. Assume $\|z\| \leq r$, $0 < r < +\infty$. Then,

$$\begin{aligned}
\|E(t,s)\| &\leq \|I\| + \int_s^t \|E(t,\theta)\|\|A(\theta)\|d\theta + \int_s^t \left[\int_\theta^t \|E(t,\tau)\|\|H(\tau,\theta)\|d\tau\right] d\theta \\
&\leq 1 + \int_s^t \|E(t,\theta)\|(m_1 + \int_0^\theta m_3 d\tau)d\theta
\end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned}
\|E(t,0)\| &\leq exp\left(\int_0^t (m_1 + \int_0^\theta m_3 d\tau)d\theta\right) \\
&\leq exp\left(m_1 w + \frac{m_3 w^2}{2}\right) \\
\|P(t;t)\| &\leq \int_0^t \|E(t,s)\|\|B(s)\|ds \\
&\leq m_2 w exp\left(m_1 w + \frac{m_3 w^2}{2}\right) \\
\|\bar{C}(t;T)\| &\leq \int_{T-t}^T \|P(s;T)^*\|ds + (\frac{t}{T})\int_0^T \|P(s;T)^*\|ds \\
&\leq 2m_2 w^2 exp\left(m_1 w + \frac{m_3 w^2}{2}\right) \\
\|S(t;T)\| &\leq \int_0^t \|E(t,s)\|\|B(s)\|\|\bar{C}(s;T)\|ds \\
&\leq 2w^3 m_2^2 exp 2\left(m_1 w + \frac{m_3 w^2}{2}\right) \equiv m_6. \\
\|y_z(w)\| &\leq \|[\bar{S}(T)]^{-1}\|[\|x_T\| + \|E(T,0)\|\|x_0\| + \|P(T;w)\|\|u_0\| \\
&\quad + \frac{1}{T}\|(u_T - u_0\| \int_0^T \|P(s;w)\|ds \\
&\quad + \int_0^T \|E(T,s)\|\|g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right)\|ds] \\
&\leq a_1\left[\|x_1\| + exp\left(m_1 w + \frac{m_3 w^2}{2}\right)\|x_0\|\right. \\
&\quad + \alpha m_2 w exp\left(m_1 w + \frac{m_3 w^2}{2}\right) \\
&\quad \left. + |\alpha - \beta| m_2 w exp\left(m_1 w + \frac{m_3 w^2}{2}\right) + exp\left(m_1 w + \frac{m_3 w^2}{2}\right)\gamma_0\right] \\
&\equiv m_7.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\|Q(z)\| &= \|x_z(t)\| \\
&\leq \|E(t,0)\|\|x_0\| + \|P(t;t)\|\|u_0\| + (\frac{1}{T})\|u_T - u_0\| \int_0^t \|P(s;t)\|ds \\
&\quad + \|S(t;T)\|\|y_z(T)\| + \int_0^t \|E(t,s)\|\|g\left(s, z(s), \int_0^s k(s,\tau)z(\tau)d\tau\right)\|ds \\
&\leq \|x_0\|exp\left(m_1 w + \frac{m_3 w^2}{2}\right) + \alpha m_2 w exp\left(m_1 w + \frac{m_3 w^2}{2}\right) \\
&\quad + |\alpha - \beta| m_2 w exp\left(m_1 w + \frac{m_3 w^2}{2}\right) + m_6 m_7 + exp\left(m_1 w + \frac{m_3 w^2}{2}\right)\gamma_0 \\
&\equiv m_8.
\end{aligned}$$

Let us now estimate,

$$\begin{aligned}
&\|x_z(t_1) - x_z(t_2)\| \\
&\quad \leq \|E(t_1,0) - E(t_2,0)\|\|x_0\| + \|P(t_1;t_1) - P(t_2;t_2)\||\alpha| \\
&\qquad + \frac{|\alpha - \beta|}{T}\left[\| \int_0^{t_1} (P(s;t_1) - P(s;t_2))ds\| + \| \int_{t_2}^{t_1} P(s;t_2)ds\|\right] \\
&\qquad + \|y_z(w)\|\|S(t_1;w) - S(t_2;w)\| \\
&\qquad + \| \int_0^{t_1} (E(t_1,s) - E(t_2,s))g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right) ds\| \\
&\qquad + \| \int_{t_2}^{t_1} E(t_2,s)g\left(s, z(s), \int_0^s K(s,\tau)z(\tau)d\tau\right) ds\|.
\end{aligned}$$

From the previous inequalities, we have

$$\begin{aligned}
&\|E(t_1,0) - E(t_2,0)\| \\
&\quad \leq \int_0^{t_1} \|E(t_1,\theta) - E(t_2,\theta)\|\|A(\theta)\|d\theta + \int_{t_2}^{t_1} \|E(t_2,\theta)\|\|A(\theta)\|d\theta \\
&\qquad + \int_0^{t_1}\left[\int_0^\theta \|E(t_1,\theta) - E(t_2,\theta)\|\|H(\theta,\tau)\|d\tau\right] d\theta \\
&\qquad + \int_{t_2}^{t_1}\left[\int_0^\theta \|E(t_2,\theta)\|\|H(\theta,\tau)\|d\tau\right] d\theta \\
&\quad \leq \int_0^{t_1} m_1\|E(t_1,\theta) - E(t_2,\theta)\|d\theta
\end{aligned}$$

$$
\begin{aligned}
& +|t_1 - t_2| m_1 exp\left(m_1 w + \frac{m_3 w^2}{2}\right) \\
& + \int_0^{t_1} \|E(t_1,\theta) - E(t_2,\theta)\| \left[\int_0^{\theta} m_3 d\tau\right] d\theta + \int_{t_2}^{t_1} \|E(t_2,\theta)\| \left[\int_0^{\theta} m_3 d\tau\right] d\theta \\
\leq\ & |t_1 - t_2|(m_1 + m_3 w) exp\left(m_1 w + \frac{m_3 w^2}{2}\right) \\
& +(m_1 + m_3 w) \int_0^{t_1} \|E(t_1,\theta) - E(t_2,\theta)\| d\theta
\end{aligned}
$$

and using Gronwall's inequality we get

$$
\begin{aligned}
& \|E(t_1,0) - E(t_2,0)\| \\
& \leq |t_1 - t_2|(m_1 + m_3 w) exp((m_1 + m_3 w)t_1) exp\left(m_1 w + \frac{m_3 w^2}{2}\right).
\end{aligned}
$$

Further,

$$
\begin{aligned}
& \|P(t_1;t_1) - P(t_2;t_2)\| \\
\leq\ & \int_0^{t_1} \|E(t_1,s) - E(t_2,s)\| \|B(s)\| ds + \int_{t_2}^{t_1} \|E(t_1,s)\| \|B(s)\| ds \\
\leq\ & |t_1 - t_2| m_2 (t_1(m_1 + m_3 w) exp((m_1 + m_3 w)t_1) + 1) \\
& \times exp\left(m_1 w + \frac{m_3 w^2}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left\| \int_0^{t_1} P(s;t_1) - P(s;t_2) ds \right\| \leq\ & \int_0^{t_1} \|P(s;t_1) - P(s;t_2)\| ds \\
\leq\ & |t_1 - t_2| m_2 t_1 (t_1(m_1 + m_3 w) \\
& \times exp((m_1 + m_3 w)t_1) + 1) exp\left(m_1 w + \frac{m_3 w^2}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left\| \int_{t_2}^{t_1} P(s;t_2) ds \right\| \leq\ & \int_{t_2}^{t_1} \|P(s;t_2)\| ds \\
\leq\ & |t_1 - t_2| m_2 w exp\left(m_1 w + \frac{m_3 w^2}{2}\right),
\end{aligned}
$$

$$\begin{aligned}\|S(t_1;w)-S(t_2;w)\| &\leq \int_0^{t_1}\|E(t_1,s)-E(t_2,s)\|\|B(s)\|\|\bar{C}(s;T)\|ds\\ &\quad+\int_{t_2}^{t_1}\|E(t_2,s)\|\|B(s)\|\|\bar{C}(s;T)\|ds\\ &\leq |t_1-t_2|2{m_2}^2w^2(t_1(m_1+m_3w)exp((m_1+m_3w)t_1)+1)\\ &\quad\times exp2\left(m_1w+\frac{m_3w^2}{2}\right),\end{aligned}$$

$$\begin{aligned}\|\int_0^{t_1}(E(t_1,s)-E(t_2,s))g(s,z(s),y)ds\| &\leq \int_0^{t_1}\|(E(t_1,s)-E(t_2,s))\|\|g(s,z(s),y)\|ds\\ &\leq |t_1-t_2|t_1\gamma_0(m_1+m_3w)exp((m_1+m_3w)t_1)\\ &\quad\times exp\left(m_1w+\frac{m_3w^2}{2}\right),\end{aligned}$$

and

$$\begin{aligned}\|\int_{t_2}^{t_1}E(t_2,s))g(s,z(s),y)ds\| &\leq \int_{t_2}^{t_1}\|E(t_2,s))\|\|g(s,z(s),y)\|ds\\ &\leq |t_1-t_2|\gamma_0 exp\left(m_1w+\frac{m_3w^2}{2}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\|x_z(t_1)-x_z(t_2)\|\\ &\leq \|x_0\||t_1-t_2|(m_1+m_3w)exp((m_1+m_3w)t_1)exp\left(m_1w+\frac{m_3w^2}{2}\right)\\ &+ |\alpha||t_1-t_2|m_2(t_1(m_1+m_3w)exp((m_1+m_3w)t_1)+1)exp\left(m_1w+\frac{m_3w^2}{2}\right)\\ &+ \frac{|\alpha-\beta|}{w}[|t_1-t_2|m_2t_1(t_1(m_1+m_3w)\\ &\quad exp((m_1+m_3w)t_1)+1)exp\left(m_1w+\frac{m_3w^2}{2}\right)\\ &+ |t_1-t_2|m_2w exp\left(m_1w+\frac{m_3w^2}{2}\right)]\\ &+ m_7|t_1-t_2|2{m_2}^2w^2(t_1(m_1+m_3w)exp((m_1+m_3w)t_1)+1)exp2\left(m_1w+\frac{m_3w^2}{2}\right)\\ &+ |t_1-t_2|t_1\gamma_0(m_1+m_3w)exp((m_1+m_3w)t_1)exp\left(m_1w+\frac{m_3w^2}{2}\right)\\ &+ |t_1-t_2|\gamma_0 exp\left(m_1w+\frac{m_3w^2}{2}\right).\end{aligned}$$

Thus the mapping $z \to Q(z)$ is equicontinuous and equibounded. Since the solutions of the equation $z = \lambda Q(z)$ are bounded for $0 < \lambda < 1$. Then by Schaefer's theorem, Q has a fixed point. Hence the theorem.

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