Journal of Natural Science
Pai Chai University, Korea
Vol. 12, No. 1:1~10,

Pebbling Numbers on Graphs

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그래프 위에서의 Pebbling 수

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Let G be a connected graph on n vertices. The pebbling number of a graph G, $f(G)$, is the least m such that, however m pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex.

In this paper, we compute the pebbling number of the Petersen Graph. We also show that the pebbling number of the categorical Product $G \cdot H$ is $(m+n)h$ where G is the complete bipartite graph $K_{m,n}$ and H is the complete graph with $h(\geq 4)$ vertices.

연결 그래프의 꼭지점에 자갈이 분포되어 있다고 하자. 한 꼭지점에서 두 개의 자갈을 취하여 한 개의 자갈만을 인접한 꼭지점에 보내는 이동을 할 때, 자갈이 분포될 수 있는 모든 경우에서 임의의 꼭지점에 한 개의 자갈을 보내기 위해 필요한 최소의 자갈의 수를 그 그래프의 pebbling number 라고 한다.

이 논문에서 Petersen Graph의 pebbling 수를 계산하였고 complete bipartite 그래프 $K_{m,n}$과 꼭지점의 수 h가 4개 이상인 complete 그래프의 categorical product 의 pebbling number가

$(m+n)h$ 이 됨을 보였다.

Key Words : connected graph, pebbling number, Petersen Graph, categorical Product.

I . Introduction

Suppose 2^n pebbles are arbitrarily distributed onto the vertices of an n-cube. Does there exist a method that allows us to make sequence of moves, each move taking two pebbles off one vertex and placing one pebble on an adjacent vertex, in such a way that we can end up with a pebble on any desired vertex? This question was proposed by M. Saks and J. Lagarias. The concept of pebbling numbers has been studied by Fan R. K. Chung in.(chung, 1989)

Let G be a connected graph on n vertices. Suppose p pebbles are distributed onto the vertices of a graph G. We define a pebbling move(step) to consist of removing two pebbles from one vertex and placing one pebble on adjacent vertex. We say that we can pebble to a vertex v, the target vertex, if we can apply pebbling moves repeatedly so that it is possible to reach a configuration with at least one pebble at v. We define the pebbling number of a vertex v in a graph G, denoted f(G, v), to be the smallest integer m which guarantees that any starting pebble configuration with m pebbles allows pebbling to v. We define the pebbling number of G, denoted f(G) as the maximum of f(G, v), over all vertices v.

Theorem 1.1. (Xavier and Lourdusamy, 1996) Every connected graph has a pebbling number.

Theorem 1.2. (clarke et al, 1996) Let C_n denote a cycle with n vertices. Then

for $k \geq 1$, $f(C_{2k}) = 2^k$ and $f(C_{2k+1}) = 2[\frac{2^{k+1}}{3}]+1$.

Theorem 1.3. (Xavier and Lourdusamy, 1996) Let P_n be a simple path of length $n-1$. Then its pebbling number is 2^{n-1}.

It is conjectured by Ronald Graham (chung, 1989) that for all graphs G and H,

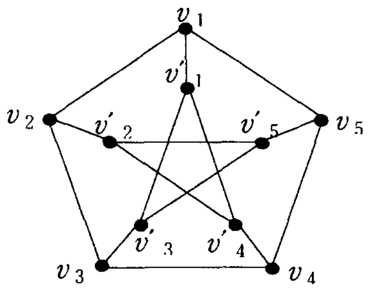
$$f(G \times H) \leq f(G)f(H)$$

In this paper, we compute the pebbling number of the Petersen Graph. We also show that the pebbling number of the categorical Product $G \cdot H$ is $(m+n)h$, where G is the complete bipartite graph $K_{m,n}$ and H is the complete graph with $h(\geq 4)$ vertices.

II. Petersen Graph

The Petersen graph is the graph possessing ten nodes, all of whose nodes have degree 3 .

Theorem 2.1 The pebbling number of Peterson graph is 10.



Proof. Suppose that 10 pebbles assigned to the vertices of the Petersen graph. We may label the vertices as the above figure. Suppose that target vertex v is one of $\{v_1, v_2, v_3, v_4, v_5\}$. Let $p(v_i)$ denote the number of pebbles on v_i. Without Loss of generality, we may assume that $v = v_1$. We consider the following three cases.

Case 1. $\sum p(v_i) \geq 5$, where $i \in \{1,2,3,4,5\}$.

We consider the five vertices v_1, v_2, v_3, v_4, v_5 as the vertices of the cycle C_5. By theorem 1, the pebbling number of C_5 is 5. Therefore if $\sum p(v_i) \geq 5$, then one pebble can be moved to v_1.

Case 2. $\sum p(v_i)=4$, where $i \in \{1,2,3,4,5\}$.

$\sum p(v'_i)=6$. So $p(v'_i) \geq 2$ for some i, $i \in \{1,2,3,4,5\}$, and one pebble can be moved to v_i from v'_i. Thus $\sum p(v_i)=5$, and one pebble can be moved to v_1 by case 1.

Case 3. $\sum p(v_i) < 4$, where $i \in \{1,2,3,4,5\}$.

$\sum p(v'_i) \geq 7$. If $p(v'_1) \geq 2$, then one pebble can be moved to v_1 from v'_1. Suppose $p(v'_1) < 2$. Now we can consider the following two possibilities.

(3.1) In case $p(v'_1)=0$.

If $p(v'_2)+p(v'_5)=4$, one pebble can be moved to v'_2 or v'_5 from v'_3 or v'_4 because $p(v'_3)+p(v'_4) \geq 3$. Similarly, if $p(v'_3)+p(v'_4)=4$, one pebble can be moved to v'_3 or v'_4 from v'_2 or v'_5 because $p(v'_2)+p(v'_5) \geq 3$. Thus we can always make $p(v'_2)+p(v'_5) \geq 5$ or $p(v'_3)+p(v'_4) \geq 5$.

(3.1.1) If $p(v'_2)+p(v'_5) \geq 5$, we can move 4 pebbles to v'_2 (or v'_5). Then two pebbles can be moved to v_2 (or v_5) from v'_2 (or v'_5), successively one pebble can be moved to v_1 from v_2 (or v_5).

(3.1.2) If $p(v'_3)+p(v'_4) \geq 5$, then two pebbles can be moved to v'_1 from v'_3 (or v'_4) So one pebble can be moved to v_1 from v'_1.

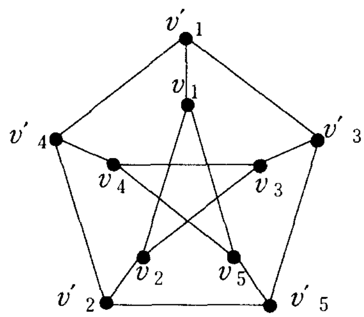
(3.2) In case $p(v'_1)=1$.

(3.2.1) If $p(v'_2)+p(v'_5) \geq 5$, we are done in (3.1.1).

(3.2.2) If $p(v'_2)+p(v'_5)=4$, then at least one pebble can be moved to v'_3 or v'_4 from v'_2 or v'_5. So one pebble can be moved to v'_1 from v'_3 or v'_4. Then $p(v'_1)=2$, one pebble can be moved to v_1 from v'_1.

(3.2.3) If $p(v'_2)+p(v'_5)<4$, then $p(v'_3)+p(v'_4)\geq 3$. So one pebble can be moved to v'_1 from v'_3 (or v'_4). Then $p(v'_1)=2$, and one pebble can be moved to v_1 from v'_1.

If target vertex v is one of $\{v'_1, v'_2, v'_3, v'_4, v'_5,\}$, then we exchange $\{v_1, v_2, v_3, v_4, v_5, v'_1, v'_2, v'_3, v'_4, v'_5\}$ with $\{v'_1, v'_4, v'_2, v'_5, v'_3, v_1, v_4, v_2, v_5, v_3\}$ in order, where $i\in\{1,2,3,4,5\}$, and apply same process.



Ⅲ. Categorical Product

We now define the categorical product of two graphs.

Definition 3.1. Given two graphs $G=(V_G, E_G)$ and $H=(V_H, E_H)$ the categorical product of G and H is the graph $G\cdot H$, whose vertex set is the Cartesian product.

$V_{G\cdot H} = V_G\times V_H = \{(x,y) : x\in V_G, y\in V_H\}$

and whose edge are given by

$E_{G\cdot H} = \{((x,y),(x',y')): (x,x')\in E_G \text{ and } (y,y')\in E_H\}$

Theorem 3.2. The pebbling number of $K_{m,n} \cdot H$ is $(m+n)h$, where $K_{m,n}$ is the complete bipartite graph ($m, n \geq 2$), and H is the complete graph with $h(\geq 4)$ vertices.

Proof. Let M and N be the disjoint vertex sets of $K_{m,n}$ such that the vertices in M are mutually nonadjacent and the vertices in N are mutually nonadjacent, and every vertex of M is adjacent to every vertex of N. Let v_i and w_j be denote the vertex of M and N, respectively, where $i \in \{1,2,\cdots,m\}$, $j \in \{1,2,\cdots,n\}$.

Let x_k, $k \in \{1,2,\cdots,h\}$, denote the vertex of H. Without loss of generality, we assume that (v_1, x_1) is target vertex. If there exists an adjacent vertex of (v_1, x_1) with more than one pebble, then we are done. Thus we may assume that there is no more than one pebble on all of the adjacent vertices of (v_1, x_1).

We consider three cases for the adjacent vertices (w_j, x_k) of (v_1, x_1).

Case 1. There are at least two adjacent vertices (w_{j_1}, x_{k_1}), (w_{j_2}, x_{k_2}), where $k_1 \neq k_2$, which have one pebble.

(1.1) Let some vertex (v_i, x_{k_3}) have more than one pebble. Since $k_3 \neq k_1$ or $k_3 \neq k_2$, Without loss of generality., we may assume that $k_3 \neq k_1$. Then $\{(v_i, x_{k_3}),(w_{j_1}, x_{k_1}),(v_1, x_1)\}$ forms a transmitting subgraph of $K_{m,n} \cdot H$. So we are done.

(1.2) Suppose that at least two vertices (v_{i_1}, x_{k_3}), (v_{i_2}, x_{k_4}), where $k_3 \neq k_4$, have one pebble. Since $(m+n)h-(nh-n)-(mh-1) = n+1$, some vertex (w_{j_3}, x_1) have more than one pebble. Since $k_3 \neq 1$ or $k_4 \neq 1$, we may assume that $k_3 \neq 1$ without loss of generality.
Then $\{(w_j, x_1)(v_{i_1}, x_{k_3}),(w_{j_1}, x_{k_1}),(v_1, x_1)\}$ or $\{(w_j, x_1)(v_{i_1}, x_{k_3}),(w_{j_2}, x_{k_2}),(v_1, x_1)\}$ forms a transmitting subgraph of $K_{m,n} \cdot H$.

(1.3) Suppose that at most m vertices (v_i, x_k) have one pebble. Since $(m+n)h-(nh-n)-m=(h-1)m+n$, $(h-1)m+n$ pebbles are assigned to the n vertices (w_j, x_1). Since $m\geq 2$, $h\geq 4$, two pebbles can be moved to some (v_i, x_k), where $i\in\{1,2,\cdots,m\}$, $k\in\{1,2,\cdots,h\}$. By (1.1), one pebble can be moved to a target vertex.

Case 2. All of adjacent vertices of (v_1, x_1) with one pebble are (w_j, x_{k_1}) for some fixed $k_1\neq 1$.

(2.1) If some vertex (v_i, x_{k_2}), where $k_1\neq k_2$, have more than one pebble, then $\{(v_i, x_{k_2}),(w_j, x_{k_1}),(v_1, x_1)\}$ forms a transmitting subgraph of $K_{m,n}\cdot H$.

(2.2) If all of vertices (v_i, x_{k_1}) have more than $m+2$ pebbles, then two pebbles can be moved to (w_j, x_{k_2}), where $k_2\neq k_1$, $k_2\neq 1$. and successively one pebble can be moved to (v_1, x_1) from (w_j, x_{k_2}).

(2.3) For all $k_2(\neq k_1)$, let (w_i, x_{k_2}), have at most one pebble, and all of vertices (v_i, x_{k_1}) have at most $m+2$ pebbles.

(2.3.1) Suppose that there are at least three vertices (v_{i_1}, x_{k_3}), (v_{i_2}, x_{k_4}), (v_{i_3}, x_{k_5}) with a pebble, where k_3, k_4, k_5 are mutually distinct.

Since $(m+n)h-n-(mh-m)-(m+2)=(h-1)n-2=n+(h-2)n-2$, at least $n+(h-2)n-2$ pebbles are assigned to the n vertices (w_j, x_1), for $j\in\{1,2,\cdots,n\}$. Without loss of generality, we may assume that $k_3\neq 1$, $k_3\neq k_1$. Since $(h-2)n-2\geq 2$, there is a vertex (w_j, x_1), where $j\in\{1,2,\cdots,n\}$ with more than one pebble. $\{(w_j, x_1),(v_{i_1}, x_{k_3})$, $(w_j, x_{k_1}),(v_1, h_1)\}$ forms a transmitting subgraph of $K_{m,n}\cdot H$.

(2.3.2) Suppose that there are only two distinct x_{k_3}, x_{k_4} such that all of vertices with a pebble among (v_i, x_k) vertices are (v_{i_1}, x_{k_3}) or (v_{i_2}, x_{k_4}), where $i_1, i_2 \in \{1,2,\cdots,m\}$, $k_3 \neq k_4$. Since

$$\begin{aligned}(m+n)h-n-m-(m+2) &= (h-2)m+(h-1)n-2 \\ &= n+(h-2)(m+n)-2\end{aligned}$$

at least $n+(h-2)(m+n)-2$ pebbles are assigned to the vertices (w_j, x_1), $j \in \{1,2,\cdots,n\}$. Without loss of generality, we may assume that $k_3 \neq 1$. Since $(h-2)(m+n)-2 \geq 6$, three pebbles can be moved to (v_{i_1}, x_{k_3}). Thus the vertex (v_{i_1}, x_{k_3}) have four pebbles. Therefore two pebbles can be moved to a vertex (w_j, x_{k_5}), where $k_3 \neq k_5$.

(2.3.3) Suppose that there is only one x_{k_2} such that all of vertices with a pebble are (v_{i_2}, x_{k_2}), where $i_2 \in \{1,2,\cdots,m\}$.

Since $(m+n)h-n-(m+2) = n+(h-1)m+(h-2)n-2$ at least $n+(h-1)m+(h-2)n-2$ pebbles are assigned to vertices (w_j, x_1), $j \in \{1,2,\cdots,n\}$. Since $(h-1)m+(h-2)n-2 \geq 8$, four pebbles can be moved to the vertex (v_i, x_{k_3}), where $k_3 \neq 1$. Then two pebbles can be moved to the adjacent vertex (w_j, x_{k_4}), where $k_3 \neq k_4$.

Case 3. There are no pebbles on all of the adjacent vertices of (v_1, x_1).

(3.1) Suppose that there are more than $mh+1$ pebbles on all of the vertices (v_i, x_k). Then two pebbles can be moved from some vertices (v_i, x_k) to a common adjacent vertex (w_j, x_{k_1}).

(3.2) Suppose that there are at most $mh+1$ pebbles on the (v_i, x_k) vertices.

(3.2.1) Suppose that there are at least two vertices with a pebble, (v_{i_1}, x_{k_1}), (v_{i_2}, x_{k_2}), where $i_1, i_2 \in \{1, 2, \cdots, m\}$, $k_1 \neq k_2$. Since $(m+n)h-(mh+1)=nh-1=n+(h-1)n-1$, at least $n+(h-1)n-1$ pebbles are assigned to the vertices (w_j, x_1), for $j \in \{1, 2, \cdots, n\}$. Without loss of generality, we may assume that $k_2 \neq 1$. Since $(h-1)n-1 \geq 5$, three pebbles can be moved to (v_{i_2}, x_{k_2}) from vertices (w_j, x_1). Then (v_{i_2}, x_{k_2}) have four pebbles, and two pebbles can be moved to some adjacent vertex of (v_1, x_1).

(3.2.2) Suppose that there is only x_{k_1} such that all of vertices with a pebble are (v_i, x_{k_1}), where $i \in \{1, 2, \cdots, m\}$. Then at least $(m+n)h-m=n+(h-1)(m+n)$ pebbles are assigned to the vertices (w_j, x_1), $j \in \{1, 2, \cdots, n\}$. Since $(h-1)(m+n) \geq 12$, four pebbles can be moved to some vertex (v_i, x_{k_2}), where $k_2 \neq 1$. Then two pebbles can be moved to the adjacent vertex (w_j, x_{k_3}) of (v_1, x_1), where $k_2 \neq k_3$.

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