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The Total Ranking Method from Multi-Categorized Voting Data Based on Analytic Hierarchy Process

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Abstract. It is important to evaluate the performance of candidates mathematically from various aspects, and reflect it on decision making. In decision making, we judge the candidates through two steps, classification of objects and comparison of objects or candidates with plural elements. In the former step, Analytic Hierarchy Process (AHP) is useful method to evaluate candidates from plural viewpoints, and in the latter step, Data Envelopment Analysis (DEA) is also useful method to evaluate candidates with plural categorized data. In fact, decision-maker uses usually only DEA or AHP, so it is hard to evaluate candidates from multi purposes when each candidate has plural elements, nevertheless it has been more important to evaluate from various aspects in IT society. So, we propose a new procedure complementing AHP with DEA.

Keywords: Data Envelopment Analysis (DEA), Analytic Hierarchy Process (AHP), Categorized Voting Data

1. INTRODUCTION

As human preferences become diversity, it is important to evaluate alternatives or candidates from multi purposes when we rank alternatives or candidates. At this situation, judgment should be based on the mathematical method. Borda (1781) proposed the "Method of Marks" as a means of deriving a consensus of opinions. This method amounts to determining the advantage of ranks assigned by voters to each alternative, with the winning alternative being the one with the lowest average. Benchmarking has been a popular method to compare objects relatively, to reflect decision making. Many mathematical methods such as Principal Component Analysis (PCA), Data Envelopment Analysis (DEA), Conjoint Analysis, and Analytic Hierarchy Process (AHP) are used usually in the situation of decision making.

To deal with categorized data such as ranked voting data, we use Data Envelopment Analysis (DEA), developed by Charnes *et al.* (1978). DEA is a linear programming originally developed for the estimation of the relative efficiency of a set of units (called decision making units, DMUs) producing a set of outputs from common inputs. DEA seeks set of weights for each unit that maximizes a weighted sum of variables, with the constraint that no units have a weighted sum larger than one. As a result, each unit receives a score between 0 and 1. DEA is applied to the problem of evaluating the relative efficiency of a set of decision making units, but DEA is very sensitive to data. Cook and Kress (1990) presented general model for aggregating votes from preferential ballot. On the other hand, to evaluate alternatives from multi criteria, we also use Analytic Hierarchy Process (AHP), by Saaty (1977). AHP is useful method to find priorities of alternatives through the stratification of the relation of each criterion and pair-wise comparison from each criterion.

Recently it becomes important to evaluate the performance of candidates mathematically from various aspects, and reflect it on decision making. But the structure of decision making problem is too complex to apply many mathematical methods to decision making problem as it is.

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So, we propose a new procedure complementing AHP with DEA that enables us to rank candidates with plural elements from multi objects. In section 2, we explain frameworks of AHP and DEA. These methods are fundamentals of our proposing method. In section 3, we propose our methods, how to apply decision making problem, and in section 4, we explain the validity of our methods through an example of marketing model. Finally in section 5, we conclude our discuss and state unsolved subjects.

2. DEA AND AHP

In this section we explain frameworks of two methods, AHP and DEA, consisted of our method. These are famous methods to analyze decision making or evaluate candidates.

2.1 AHP

When we face to alternative situation, we make decision based in own criterion. It is natural that the number of objects is plural, and each object conflicts with another. In this case, we use Analytic Hierarchy Process (AHP) developed by Saaty (1977).

Steps to calculate priority in AHP are as follows.

1. Put the problem in broad context-embed if necessary in a larger system including other actors, their objects, and outcomes.
2. Identify the criteria that influence the behavior of the problem.
3. Structure a hierarchy of the criteria, sub-criteria, properties of candidates, and the candidates themselves.
4. In a many party problem the levels may relate to the environment, actors, actor objects, actor policies, and outcomes, from which one derives the composite outcome (state of the word).
5. To remove ambiguity carefully define every element in the hierarchy.
6. Priorities the primary objects with respect to their impact on the overall objective called the focus.
7. State the question for pair-wise comparisons clearly above each matrix.
8. Prioritize the sub-objects with respect to their objects.
9. Enter pair-wise comparison judgments and force their reciprocals.
10. Calculate priorities by adding the elements of each column and dividing each entry by the total of the column. Average over the rows of the resulting matrix and you have the priority vector.

We make pair wise comparisons between p purposes or items or objects for ranking and set each element of $p \times p$ matrix a_{ij}, the evaluation value indicating how important decision maker regard object i (i =1, 2, $\cdots$, p), comparing to object j (j = 1, 2, $\cdots$, p), when the importance of object j is regarded to 1. We allocate a_{ij} as element of pair-wise comparison matrix as (1).

$$A = \begin{pmatrix} 1 & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{p1} & \cdots & 1 \end{pmatrix} \tag{1}$$

The entries a_{ij} are defined by the following entry rules.

Rule 1. If relative importance of object i to object j a_{ij} is considered to α, relative importance of object j to object i a_{ij} is $1/\alpha$.

Rule 2. If object i is judged to be of equal relative importance as object j, a_{ij} is equal to 1; in particular, a_{ij} =1 for all i.

Now, suppose that there are n objects, I_1, $\cdots$, I_n and primary priorities are w_1, $\cdots$, w_n. Now relative importance of object Ii to object Ij a_{ij} is considered to $a_{ij} = w_i/w_j$. So an ideal pair wise comparison matrix can be expressed as following matrix (2).

$$\begin{pmatrix} 1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & 1 & \cdots & w_2/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & 1 \end{pmatrix} \tag{2}$$

However, Matrix (2) would be unrealistic to require these relations to hold in the general case.

Then, we can get the following relational expression (3).

$$\begin{pmatrix} 1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & 1 & \cdots & w_2/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_2/w_2 & \cdots & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = n \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \tag{3}$$

To get weight vector w, we solve the following equation (4).

$$(A - I)w = 0 \tag{4}$$

Then we define n as eigenvalues of this matrix A. n is the solution of equation (4) and there have n-number solutions including duplication. From the theory of linear algebra, the rank of matrix A is 1, so one eigenvalues is non-zero and the other is zero. And the sum of eigenvalues is n. So w corresponds to largest eigenvalues, and

$\sum_{i=1}^{n} w_i = 1.$

2.2 RANK ORDERING METHOD

Let us discuss the method proposed by Green *et al.* to rank candidates. Suppose that each candidate $m = 1, 2, \cdots, M$ has obtained some number y_{m1} of vote as first place, y_{m2} as second place, $\cdots$, y_{mk} of K-th place.

Further let w_{mk} be the weight of k-th place ($k = 1, 2, \cdots, K$). Every candidate wishes to assign the each weight w_{mk} so as to maximize the weighted sum to his votes, which is referred to "preference" in what follows. Then, the preference of candidate m is defined as follows:

$$\theta_{mm} = \sum_{k=1}^{K} w_{mk} y_{mk}. \qquad (5)$$

In order to rank candidates properly, there should be some constraint to each weight w_{mk}. Otherwise the total ranking of candidates cannot be determined. Thus, they build in the following inequalities to LP as constraints so as to determine each preference:

$$\theta_{mq} = \sum_{k=1}^{K} w_{mk} y_{qk} \le 1, \qquad (q = 1,2,\cdots,M) \qquad (6)$$

$$\begin{aligned} & w_{mk-1} - w_{mk} \ge d(k-1,\varepsilon) = \varepsilon \ge 0, \\ & or \\ & w_{m1} \ge w_{m2} \ge \ldots \ge w_{mK} \ge 0. \end{aligned} \qquad (7)$$

Then they proposed to composite an $M \times M$ matrix by taking θ_{mq} as the (m, q) element. Using cross evaluation of this matrix, they obtain the total ranking of candidates.

3. RANKING BY MULTI PURPOSES VOTING

To evaluate from multi objects, it is not suitable to use only DEA. DEA is applied to the problem of evaluating the relative efficiency of a set of decision making units, but DEA is very sensitive to data. AHP is useful method to find priorities of candidate s through the stratification of decision making problem and pair wise comparison from each criterion. But if data set is complex, it is difficult to evaluate candidates. In DEA model especially ranking method, the constraints about multipliers are very important and the result depends on these constraints. It may not be suitable to apply DEA model only to decision making problem needed to fairness. On the other hand, in AHP, the numbers of pair wise matrix or objects/candidates are large, it is not easy for one decision maker to consider all structure of this problem. In modern decision making problem, it occurs these difficulties, the number of candidate is especially large not suitable to analysis by AHP, the number of DMU compared to that of candidate is not large enough to calculate by DEA.

In this section, we propose a new procedure to evaluate candidate s from multi purposes when each candidate has plural elements complementing AHP with DEA.

3.1 RANKING PROCESSES

Now we present our method to choose candidates from multi objects. Let us assume that there are more than one, say p (number) purposes or objects for ranking. Next, let m be the number of candidates and n be the number of voters. They vote candidates as first place, second place, $\cdots$, and K-th place.

3.2 GETTING PRIORITIES

We make pair wise comparisons between p purposes or objects for ranking and set each element a_{ut} of $p \times p$ matrix, the evaluation value indicating how important decision maker regard object u, comparing to object t, when the importance of object t is regarded to 1. We allocate a_{ut} as element of pair-wise comparison matrix as (8).

$$\begin{pmatrix} 1 & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{p1} & \cdots & 1 \end{pmatrix} \qquad (8)$$

From pair-wise matrix (8), we can get priorities as inherent vector x of this matrix. Priorities mean how important decision maker regards each object.

3.3 GETTING PREFERENCES

From some object s ($s = 1, 2, \cdots, p$), voters vote candidates, as first place, second place, $\cdots$, and K-th place. Following voting model formulated as (9), we get preference rates θ_{jq}^{s} of candidate q maximizing preference rate of candidate j in object s.

$$\begin{aligned} & \max \ \theta_{jj}^{\ s} = \sum_{i=1}^{K} w_{ji}^{\ s} y_{ji}^{\ s} \\ & subject \ to \ \theta_{jq}^{\ s} = \sum_{i=1}^{K} w_{ji}^{\ s} y_{qi}^{\ s} \le 1 \\ & w_{j1}^{\ s} \ge 2w_{j2}^{\ s} \ge \cdots \ge kw_{jK}^{\ s} \\ & w_{jK}^{\ s} \ge \frac{2}{nK(K+1)} \end{aligned} \qquad (9)$$

$$j,q=1,\cdots,m$$

We make preference rate $m\times m$ matrix B^s, whose element is θ_{jq}^{s}. By using cross-evaluation of this matrix, they obtain the preference rate vector Θ_s in criterion s.

The preference matrix Θ can be expressed as follows:

$$\Theta=(\Theta_1\cdots\Theta_p)$$
$$=\begin{pmatrix} \sqrt[m]{\theta_{11}{}^{1}\cdots\theta_{m1}{}^{1}} & \cdots & \sqrt[m]{\theta_{11}{}^{p}\cdots\theta_{m1}{}^{p}} \\ \vdots & \cdots & \vdots \\ \sqrt[m]{\theta_{1m}{}^{1}\cdots\theta_{mm}{}^{1}} & \cdots & \sqrt[m]{\theta_{1m}{}^{p}\cdots\theta_{mm}{}^{p}} \end{pmatrix} \quad (10)$$

3.4 TOTAL INDICES

Finally, from inherent vector x in 3.2 and preference rate matrix Θ in 3.3, we can get total indices $r=\Theta x$, according to r, we determine their total ranking.

4. APPLICATIONS

In this section, we will demonstrate the effectiveness of our method shown in Section 3, by an example. Let us suppose that there are twenty merchandisers in a department store, and each of them votes two best alternatives out of four brand makers where he thinks the first and second best for one tie up maker.

For this purpose, let us assume that skills, which the tie up maker requires, are tentatively (a) sewing, (b) assortment, and (c) design. Then, the maker, which is evaluated best in weighted average of these three categories, will be selected for tie up maker. Figure 1 shows the hierarchy of this decision making problem to select one maker. Table 1 shows the rank vote result of twenty merchandisers evaluating four alternatives A, B, C, and D.

Table 2 shows the result of preference rate and its geometric mean from amount of each criterion voting data with respect to of each criterion.

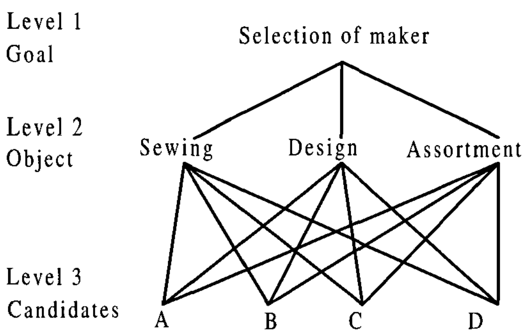

Figure 1. The hierarchy of selecting maker

Table 1. Voting data

	Sewing		Design		Assortment		Amount	
	1st	2nd	1st	2nd	1st	2nd	1st	2nd
Maker A	8	9	7	7	1	0	16	16
Maker B	6	7	7	8	2	2	15	17
Maker C	5	3	3	3	4	8	12	14
Maker D	1	1	3	2	13	10	17	13

Table 2. Result of preference rate and geometric mean from amount of voting data

	Maker A	Maker B	Maker C	Maker D
Maximize A	1	0.9783	0.7908	0.9808
Maximize B	1	0.9791	0.7916	0.9791
Maximize C	1	0.9791	0.7916	0.9791
Maximize D	0.9618	0.9127	0.7324	1
Mean	0.9903	0.9619	0.7762	0.9847

Table 3 shows the result of efficiency regarded all voting data as outputs of DMUs, with following DEA model (11).

$$\begin{aligned} \max \quad & w_{o1}y_{o1}+w_{o2}y_{o2} \\ subject\ to \quad & w_{o1}y_{j1}+w_{o2}y_{j2}\le 1 \\ & w_{1o}\ge 2w_{o2}\ge \tfrac{1}{20}\times 2\times 3 \end{aligned} \quad (11)$$

Table 3. Results of efficiency and cross evaluation from (11)

	D-efficiency	Cross Evaluation
Maker A	1	0.974
Maker B	1	0.909
Maker C	0.9364	0.7686
Maker D	1	0.9216

From the result showed in Table 2, it is not easy to select one maker because the difference of total scores is not large enough to decide which candidate is superior to others. And amounting voting data, this implies that we ignored objects to select candidates. When we calculate all categories voting data as output data, we only know which object is considered to be important. From the result shown in Table 3, it occurs that D-efficiency of three candidates is calculated 1. This is why the number of

DMU is smaller than that of indices. Also the result by cross evaluation in Table 3, there is no contribution of decision maker. So, in this case, it should be separate data into some categories to analyze. Next subsection we use our procedure considering priorities of each object.

4.1 Preference rate by voting method of each criterion

By using our procedure proposed in Section 2, results of calculation from three objects, sewing, design, and assortment are shown in Table 4, Table 5, and Table 6.

Table 4. Preference ratio from object of sewing

	Maker A	Maker B	Maker C	Maker D
Maximize A	1	0.759	0.5297	0.1204
Maximize B	1	0.76	0.52	0.12
Maximize C	1	0.7541	0.5814	0.1229
Maximize D	1	0.7541	0.5814	0.1229
Mean	1	0.7568	0.5524	0.1215

Table 5. Preference ratio from object of design

	Maker A	Maker B	Maker C	Maker D
Maximize A	0.9834	1	0.4214	0.4048
Maximize B	0.9545	1	0.409	0.3636
Maximize C	0.9834	1	0.4214	0.4048
Maximize D	0.9834	1	0.4214	0.4048
Mean	0.9761	1	0.4183	0.3941

Table 6. Preference ratio from object of assortment

	Maker A	Maker B	Maker C	Maker D
Maximize A	0.0641	0.1615	0.3894	1
Maximize B	0.0277	0.1666	0.4444	1
Maximize C	0.0277	0.1666	0.4444	1
Maximize D	0.0641	0.1615	0.3894	1
Mean	0.0597	0.164	0.416	1

4.2 Priority calculated by AHP

By pair wise comparison, priorities of three objects are shown in Table 7.

From priorities and preference rate, the total score of maker A, B, C, and D are calculated as follows:

Table 7. Pair wise comparison and priorities of substitutions

	Sewing	Design	Assortment	Priority
Sewing	1	5	3	0.6495
Design	1/5	1	3	0.2295
Assortment	1/3	1/3	1	0.121

A : 0.6495*1.0000+0.2294*0.9811+0.1209*0.0553=0.8814,
B : 0.6495*0.7272+0.2294*1.0000+0.1209*0.1656=0.7219,
C : 0.6495*0.5455+0.2294*0.4205+0.1209*0.4265=0.5024,
D : 0.6495*0.1213+0.2294*0.4010+0.1209*1.0000=0.2918.

From this result, we judge that maker A should be selected. It commonly happens that, when one has to selects among many alternatives, a particular alternative is rated as the best in one evaluation, while others are selected by other evaluation methods. However, our method has a definite advantage, since the weight of each ranking is determined automatically by the total votes each alternative captures. Thus, this ranking obtained with these weights will satisfy each alternative. Various applications of this method can be possible for cases in which multi-purposes ranking are necessary.

5. CONCLUSIONS

In this paper, we discussed the decision making method based on DEA and AHP to analyze structural decision making problem which is too large not to easy to analyze AHP or too large or small number of features to analyze DEA.

Nevertheless, Data Envelopment Analysis (DEA) is the useful method to determine the weight to maximize the ratio scale of each alternative. Like voting data, DEA is very useful method. The total ranking from voting data should be evaluated fairly, but cannot evaluate alternative from multi-purposes. Analytic Hierarchy Process (AHP) is the method to find the priority of alternatives through the stratification of the relation of each criterion, and can apply various decision-making situations.

But when each alternative has plural elements from more than one criterion, it is impossible to pair-wise comparison. To settle those weak points, we proposed a new procedure complementing AHP by DEA. Applying voting method from categorized data in conventional decision-making is very rare, and much less multi-purposes though this type evaluation of alternatives is useful and very fare.

There still have some problems. In our proposed method and application, the structure of decision making problem is applicable based on AHP complementing with DEA. Pair wise comparison about objects of voter is calculated by AHP, and analysis of voting data we applied DEA. But in other case, another methods should be more applicable ANP and PCA. The question which we must consider next is what methods should we apply depends on the structure of problem.

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