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Statistical Inference for an Arithmetic Process

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Abstract. A stochastic process $\{A_n, n = 1, 2, ...\}$ is an arithmetic process (AP) if there exists some real number, d, so that $\{A_n + (n-1)d, n = 1, 2, ...\}$ is a renewal process (RP). AP is a stochastically monotonic process and can be used for modelling a point process, i.e. point events occurring in a haphazard way in time (or space), especially with a trend. For example, the events may be failures arising from a deteriorating machine; and such a series of failures is distributed haphazardly along a time continuum. In this paper, we discuss estimation procedures for an AP, similar to those for a geometric process (GP) proposed by Lam (1992). Two statistics are suggested for testing whether a given process is an AP. If this is so, we can estimate the parameters d, μ_{A_1} and $\sigma^2_{A_1}$ of the AP based on the techniques of simple linear regression, where μ_{A_1} and $\sigma^2_{A_1}$ are the mean and variance of the first random variable A_1 respectively. In this paper, the procedures are, for the most part, discussed in reliability terminology. Of course, the methods are valid in any area of application, in which case they should be interpreted accordingly.

Keywords: arithmetic processes, common difference, deteriorating systems, Laplace's trend test, renewal processes, simple linear regression techniques

1. INTRODUCTION

In the statistical analysis of a series of events, a common method is to model the series using a point process. To start with, it is essential to test whether the data of successive inter-event times, denoted by X_i ($i = 1, 2, ...$), demonstrate a trend. If there is no trend, we may model the data using a stationary point process (e.g. by a counting process which has stationary, but not necessarily independent, increments), or using a sequence of independent and identically distributed (i.i.d.) random variables $X \equiv X_i$ for all i. For the latter, we may model the corresponding counts of events in time using a renewal process (RP). In particular, if X is exponentially distributed with a rate parameter λ, we may use a homogeneous Poisson process (HPP) with a constant rate λ to model the data. The HPP is one of the most common stochastic processes for modelling counts of events in time (or area / volume). This process is a standard for "randomness", for the assumptions involved state that events must occur independently and any two non-overlapping or overlapping intervals of the same size have the same probability of capturing one of the events of interest. However, in practice the data of successive inter-event times usually exhibit a trend. We may model them using a non-stationary model, or using a non-homogeneous Poisson process (NHPP) in which the rate at time t is a function of t. The NHPP is a popular approach to model data having a trend. For more details of these methods, see Cox and Lewis (1966).

An arithmetic process (AP), which is a non-stationary model, can be used as an alternative to the NHPP in analysing data of inter-event times that exhibit a trend. This appears to be a useful model for failure or repair data arising from a single system. Consider the maintenance problems of a repairable system and bear in mind that most repairable systems, like engines, are deteriorative. Two basic characteristics of a deteriorating system are that because of ageing or irreversible wear, the system's successive operating times decrease and so the system's life is finite; while because it is more difficult and hence takes more time to rectify accumulated wear, the corresponding consecutive repair times increase until finally the system is beyond repair. Most of the research on the maintenance of a deteriorating system has made the minimal repair assumption. Minimal repair means that a

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failed system will function, after repair, with the same rate of failure and the same effective age as at the epoch of the last failure. For a minimal repair model where repair time is assumed negligible, an NHPP in which the rate of occurrence of failures (ROCOF) over time is monotone can provide at least a good first-order model for a deteriorating system; see Ascher and Feingold (1984). If repair time has to be taken into account, the NHPP approach cannot be used. Based on this understanding, an AP approach proposed by Leung (2001) is considered more relevant, realistic and direct for the modelling of the maintenance problems in a deteriorating system.

2. ARITHMETIC PROCESSES

A definition of an arithmetic process (AP) is given below.

Definition: Given a sequence of non-negative random variables $A_1, A_2, \ldots$ if for some real number d, $\{A_n + (n-1)d, n = 1, 2, \ldots\}$ forms a renewal process (RP), then $\{A_n, n = 1, 2, \ldots\}$ is an AP. d is called the common difference of the AP.

Three specialisations of an AP are given below.

If $d \in \left(0, \frac{\mu_{A_1}}{n-1}\right]$, where $n = 2, 3, \ldots$ and μ_{A_1} is the mean of the first random variable A_1, then the AP is called a decreasing AP. If $d < 0$, then the AP is called an increasing AP. If $d = 0$, then the AP reduces to an RP.

The upper bound of d in the first specialisation can be obtained as follows. By the definition, the expression for the general term of an AP is given by $A_n \stackrel{d}{=} A_1 - (n-1)d$. Taking expectations on both sides of this expression, and remembering that A_n is a non-negative random variable and hence $\mathrm{E}(A_n) \equiv \mu_{A_n} \geq 0$ for $n = 1, 2, \ldots$; we obtain, after transposition, the upper bound of d given by $\frac{\mu_{A_1}}{n-1}$ for $n = 2, 3, \ldots$. Clearly, the positive integer n is limited for a decreasing AP. Moreover, if the value of d is close to its upper bound, we will obtain a short sequence of non-negative random variables. However, such a subtractive process is likely to be useful in a deteriorating system (e.g. an engine), which fails rarely (e.g. two or three times) over its usual span of life (e.g. ten years). This implicitly means that the system wears out, between two successive failures, to such an extraordinary extent that the corresponding system's successive operating time decreases dramatically.

Therefore, for a deteriorating system, it is reasonable to assume that the successive operating times of the system form a decreasing AP; whereas the corresponding consecutive repair times constitute an increasing AP. However, the replacement times for the system are usually stochastically the same no matter how old the used system is; hence, these will form an RP. This is the motivation behind the introduction of the AP approach.

Given an AP $\{A_n, n = 1, 2, \ldots\}$, we have $A_n \stackrel{d}{=} A_1 - (n-1)d$ by the definition. Therefore, the means and variances of A_n can respectively be written as

$$\begin{aligned}\mu_{A_n} &\equiv \mathrm{E}(A_n) \\ &= \mathrm{E}(A_1) - (n-1)d \equiv \mu_{A_1} - (n-1)d \end{aligned} \qquad (1)$$

and

$$\begin{aligned}\sigma^2_{A_n} &\equiv \mathrm{V}(A_n) \\ &= \mathrm{V}[A_1 - (n-1)d] = \mathrm{V}(A_1) \equiv \sigma^2_{A_1} \end{aligned} \qquad (2)$$

Thus, d, μ_{A_1} and $\sigma^2_{A_1}$ are the most important para meters in an AP because the means and variances of the A_ns are completely determined by these three parameters. In view of this fact, in this paper the author will define the procedure for applying the AP approach in a reliability context and derive estimators for the three fundamental parameters. Now, there are two questions. The first is, given a set of data of successive inter-event times of a point process, how do we test whether this is consistent with an AP? The second question is, if the data do come from an AP, how can we estimate the parameters d, μ_{A_1} and $\sigma^2_{A_1}$?

In this paper, the statistical inference for an AP is investigated and these two questions are answered using well-known statistical methods. In Section 3, Laplace's statistic is recommended for testing whether a process has a trend, and a graphical technique is suggested for testing whether a process is an AP as well as having a trend. In Section 4, the parameters d, α and σ^2_ε are estimated using simple linear regression techniques. In Section 5, a statistic is introduced for testing whether a process is an AP. In Section 6, first μ_{A_1} and $\sigma^2_{A_1}$ are estimated based on the results derived in Section 4, and then μ_{A_n} and $\sigma^2_{A_n}$ for $n = 2, 3, \ldots$ are correspondingly estimated using equations (1) and (2) respectively. In Section 7, two concluding remarks are given. Finally, the derivations of some key results are outlined in the Appendix.

3. TESTING FOR TRENDS

Given the data $\{A_n, n = 1, 2, \ldots\}$ of successive inter-event times of a point process, first of all we need to test whether the A_ns are identically distributed by checking for the existence of a trend. To do this, many techniques discussed in Ascher and Feingold (1984) can be used. Laplace's trend test for a time-truncated data set (where the time of the conclusion of observation is fixed and the number of events is random) or for an event-truncated data set (where the number of events is fixed before observation begins and the time of the conclusion of the observation is random) is used for ease of manipulation and interpretation.

Rigdon and Basu (1989), page 259 reach the conclusion that "using any model for event times, one should indicate the time that data collection started and the time that it ceased. This is necessary so that the appropriate analysis, that is, an analysis based on time-truncated or event-truncated data, can be applied and maximum information can be obtained from the data. For time-truncated data, the time between the last event and the termination of the test contains some information that should not be wasted."

Another possible approach is to use the simple linear regression techniques. To start with, let

$$W_n = A_n + (n-1)d \tag{3}$$

From the definition, W_ns are i.i.d. and can be written as

$$W_n = \alpha + \varepsilon_n \tag{4}$$

where

$$\mathrm{E}(W_n) = \alpha \tag{5}$$

and ε_ns are also i.i.d. (not necessarily normally distributed if our objective is estimation only, e.g. see Gujarati (1988), page 281) with

$$\mathrm{E}(\varepsilon_n) = 0 \text{ and } \mathrm{V}(\varepsilon_n) \equiv \sigma^2_{\varepsilon_n} \equiv \sigma^2_\varepsilon \tag{6}$$

Combining equations (3) and (4) yields

$$A_n = -d(n-1) + \alpha + \varepsilon_n \text{ for } n = 1, 2, \cdots, N \tag{7}$$

which is a simple linear regression equation. Therefore, we can plot A_n against $(n-1)$ for $n = 1, 2, ..., N$ to see whether there is a linear relationship between them. Clearly, this is also useful for testing if the observations $\{A_n, n = 1, 2, ...\}$ come from an AP as well as having a trend.

4. ESTIMATING THE PARAMETERS d, α AND σ^2_ε

We can estimate the parameters d, α and σ^2_ε using the simple linear regression method. The least squares point estimates $\hat{d}, \hat{\alpha}$ and $\hat{\sigma}^2_\varepsilon$ of the parameters d, α and σ^2_ε are calculated respectively using the following formulae:

$$\hat{d} = \frac{6(N-1)\sum_{n=1}^{N} A_n - 12\sum_{n=1}^{N}(n-1)A_n}{(N-1)N(N+1)} \tag{8}$$

$$\hat{\alpha} = \frac{\sum_{n=1}^{N} A_n}{N} + \frac{\hat{d}(N-1)}{2} = \frac{2(2N-1)\sum_{n=1}^{N} A_n - 6\sum_{n=1}^{N}(n-1)A_n}{N(N+1)} \tag{9}$$

and

$$\hat{\sigma}^2_\varepsilon = \frac{\sum_{n=1}^{N} A_n^2 - \frac{1}{N}\left(\sum_{n=1}^{N} A_n\right)^2}{N-2} - \frac{-\hat{d}\left[\frac{(N-1)}{2}\sum_{n=1}^{N} A_n - \sum_{n=1}^{N}(n-1)A_n\right]}{N-2} \tag{10}$$

The derivations of equations (8) to (10) are outlined in the Appendix.

5. DISTINGUISHING A RENEWAL PROCESS FROM AN ARITHMETIC PROCESS

We test whether the data comes from an arithmetic process or a renewal process.
Null hypothesis $H_0 : d = 0$
Alternative hypothesis $H_1 : d \neq 0$
The t-test statistic is given by

$$t = \frac{-\hat{d}\sqrt{(N-1)N(N+1)}}{\sqrt{12}\hat{\sigma}_\varepsilon} \tag{11}$$

where t is distributed as a Student's t with $(N-2)$ degrees of freedom if the null hypothesis of $d = 0$ applies. One point worth noting is that for testing purposes, each ε_n is approximately normally distributed, e.g. see Gujarati (1988), page 282. It is difficult to evaluate the normality assumption for a sample of only 20 observations, and formal test procedures are presented in Ramsey and Ramsey (1990).

If $|t| >$ critical value $t_{N-2, 0.025}$, then H_0 is rejected at the 5% level of significance, i.e. the data set $\{A_1, A_2, ..., A_N\}$ comes from an AP.

6. ESTIMATING THE MEANS AND VARIANCES OF A_nS

First, the mean and variance of A_1 are estimated respectively using the relevant estimators with formulae given below.
From the definition, W_ns are i.i.d., we have

$$\mathrm{E}(W_n) = \mu_{A_1}$$

and

$$V(W_n) \equiv \sigma_{W_n}^2 = \sigma_{A_1}^2$$

From equations (5), (4) and (6), we obtain

$$E(W_n) = \alpha$$

and

$$V(W_n) = V(\alpha + \varepsilon_n) = V(\varepsilon_n) \equiv \sigma_\varepsilon^2$$

Therefore, the first estimators for μ_{A_1} and $\sigma_{A_1}^2$ are denoted and given by

$$\hat{\mu}_{A_1,1} = \hat{\alpha} \quad (12)$$

and

$$\hat{\sigma}_{A_1,1}^2 = \hat{\sigma}_\varepsilon^2 \quad (13)$$

Alternatively, since W_ns are i.i.d. with mean $\mu_{W_n} = \mu_{A_1}$ and variance $\sigma_{W_n}^2 = \sigma_{A_1}^2$, it is plausible to estimate μ_{A_1} and $\sigma_{A_1}^2$ by the sample mean and sample variance of $\hat{W}_n$s, where $\hat{W}_n = A_n + (n-1)\hat{d}$. The second possible estimators for μ_{A_1} and $\sigma_{A_1}^2$ are given by

$$\hat{\mu}_{A_1,1} = \frac{\sum_{n=1}^{N} \hat{W}_n}{N} = \frac{\sum_{n=1}^{N} [A_n + (n-1)\hat{d}]}{N} = \frac{\sum_{n=1}^{N} A_n}{N} + \frac{\hat{d}(N-1)}{2} = \hat{\alpha}$$

and

$$\hat{\sigma}_{A_1}^2 = \frac{\sum_{n=1}^{N} [A_n + (n-1)\hat{d}]^2}{N-1} - \frac{\frac{\left\{\sum_{n=1}^{N} [A_n + (n-1)\hat{d}]\right\}^2}{N}}{N-1} \equiv \hat{\sigma}_{A_1,2}^2 \quad (14)$$

Hence, we only have the second estimator for $\sigma_{A_1}^2$ denoted by $\hat{\sigma}_{A_1,2}^2$ and given by equation (14).

We can also deduce $\hat{\mu}_{A_1,1}$ as follows. Let $S_N = \sum_{n=1}^{N} A_n = \sum_{n=1}^{N} [A_1 - (n-1)d]$. Then

$$E(S_N) = \sum_{n=1}^{N} E(A_1) - d \sum_{n=1}^{N} (n-1) = N\mu_{A_1} - \frac{dN(N-1)}{2}$$

After transposition, we have

$$\mu_{A_1} = \frac{E(S_N)}{N} + \frac{d(N-1)}{2}$$

and the possible estimator for μ_{A_1} is

$$\hat{\mu}_{A_1} = \frac{S_N}{N} + \frac{\hat{d}(N-1)}{2} = \frac{\sum_{n=1}^{N} A_n}{N} + \frac{\hat{d}(N-1)}{2} = \hat{\alpha} = \hat{\mu}_{A_1,1}$$

It is also possible to obtain second and third estimators for μ_{A_1}. In view of the fact that $E(W_n) = \mu_{A_1}$, we can write

$$W_n = \mu_{A_1}(1 + \delta_n) \quad (15)$$

(a) We have

$$E\left(\frac{W_n}{\mu_{A_1}}\right) = 1 + E(\delta_n)$$

this follows that

$$E(\delta_n) = 0 \quad (16)$$

(b) We obtain

$$V\left(\frac{W_n}{\mu_{A_1}}\right) = V(1 + \delta_n)$$

$$\frac{V(W_n)}{\mu_{A_1}^2} = V(\delta_n)$$

this follows that

$$V(\delta_n) = \frac{\sigma_{A_1}^2}{\mu_{A_1}^2} \quad (17)$$

(c) Taking logarithms for equations (4) and (15) obtain respectively

$$\ln W_n = \ln\left[\alpha\left(1 + \frac{\varepsilon_n}{\alpha}\right)\right] = \ln\alpha + \ln\left(1 + \frac{\varepsilon_n}{\alpha}\right) \quad (18)$$

and

$$\ln W_n = \ln \mu_{A_1} + \ln(1 + \delta_n) \quad (19)$$

Taking expectation on equations (18) and (19), equating them, and expanding the logarithm series, we have

$$\ln\alpha + E\left(\frac{\varepsilon_n}{\alpha} - \frac{\varepsilon_n^2}{2\alpha^2} + \frac{\varepsilon_n^3}{3\alpha^3} - \cdots\right)$$

$$= \ln \mu_{A_1} + \mathrm{E}\left(\delta_n - \frac{\delta_n^2}{2} + \frac{\delta_n^3}{3} - \cdots \right)$$

$$\ln \alpha + \frac{1}{\alpha}\mathrm{E}(\varepsilon_n) - \frac{1}{2\alpha^2}\mathrm{E}(\varepsilon_n^2)$$

$$\approx \ln \mu_{A_1} + \mathrm{E}(\delta_n) - \frac{1}{2}\mathrm{E}(\delta_n^2)$$

$$\ln \alpha - \frac{1}{2\alpha^2}\mathrm{V}(\varepsilon_n)$$

$$= \ln \mu_{A_1} - \frac{1}{2}\mathrm{V}(\delta_n) \quad \text{by equations (6) and (16)}$$

$$\ln \alpha - \frac{\sigma_\varepsilon^2}{2\alpha^2}$$

$$= \ln \mu_{A_1} - \frac{\sigma_{A_1}^2}{2\mu_{A_1}^2} \quad \text{by equations (6) and (17)}$$

(d) μ_{A_1} must satisfy the equation

$$\ln\left(\frac{\mu_{A_1}}{\alpha}\right) - \frac{1}{2}\left(\frac{\sigma_{A_1}^2}{\mu_{A_1}^2} - \frac{\sigma_\varepsilon^2}{\alpha^2}\right) = 0$$

Finally, we can estimate μ_{A_1} by $\hat{\mu}_{A_1,2}$ which satisfies the equation

$$\ln\left(\frac{\mu_{A_1}}{\hat{\alpha}}\right) - \frac{1}{2}\left(\frac{\hat{\sigma}_{A_1,1}^2}{\mu_{A_1}^2} - \frac{\hat{\sigma}_\varepsilon^2}{\hat{\alpha}^2}\right) = 0 \qquad (20)$$

or by $\hat{\mu}_{A_1,3}$ which satisfies the equation

$$\ln\left(\frac{\mu_{A_1}}{\hat{\alpha}}\right) - \frac{1}{2}\left(\frac{\hat{\sigma}_{A_1,2}^2}{\mu_{A_1}^2} - \frac{\hat{\sigma}_\varepsilon^2}{\hat{\alpha}^2}\right) = 0 \qquad (21)$$

where $\hat{\alpha}, \hat{\sigma}_\varepsilon^2, \hat{\sigma}_{A_1,1}^2$ and $\hat{\sigma}_{A_1,2}^2$ are given by equations (9), (10), (13) and (14) respectively.

Clearly, if $d = 0$, the parameters μ_{A_1} and $\sigma_{A_1}^2$ can be estimated using the sample mean and sample variance which are given by

$$\hat{\mu}_{A_1} = \frac{\sum_{n=1}^{N} A_n}{N} \quad \text{and} \quad \hat{\sigma}_{A_1}^2 = \frac{\sum_{n=1}^{N} (A_n - \hat{\mu}_{A_1})^2}{N-1} \qquad (22)$$

Secondly, using equations (1) and (2), the means and variances of A_n for $n = 2, 3, ..., N$ are estimated using the following formulae.

$$\hat{\mu}_{A_n} = \hat{\mu}_{A_1} - (n-1)\hat{d} \quad \text{and}$$

$$\hat{\sigma}_{A_n}^2 = \hat{\sigma}_{A_1}^2 \quad \text{for } n = 2,3,\ldots,N \qquad (23)$$

7. CONCLUDING REMARKS

Below are two notes concerning the statistical inference for an AP. Note 2 reveals how the inference method is relevant and applicable to solve reliability (comprising inspection, maintenance, replacement, etc.) problems.

7.1 Note (1)

Leung *et al.*(2002), performed some simulation studies to evaluate various estimators of the parameters of an AP, given by equations (12) to (14) and (20) to (22). The authors make some suggestions, based on the results of the simulation studies, for selecting the best estimators when $d \in \left(0, \frac{\mu_{A_1}}{n-1}\right]$, $d<0$ or $d=0$, with respect to three different criteria. For easy reference, the recommended estimators are summarized in Table 1, where $\phi = |\hat{\mu}_{A_1,i} - \mu_{A_1}|$ is the deviation of $\hat{\mu}_{A_1}$ from μ_{A_1}, $i = 1, 2, 3, 4$; $\mathrm{MSE} = \frac{\sum_{n=1}^{N}(\hat{A}_{n,i} - A_n)^2}{N}$ is the mean square error between the fitted values and observations, and $\Phi = \phi + \sqrt{\mathrm{MSE}}$. Moreover, the estimates $\hat{\sigma}_{A_1,1}^2$ and $\hat{\sigma}_{A_1,2}^2$ can be compared by their standard deviations from $\sigma_{A_1}^2$.

7.2 Note (2)

Fitting a model to failure and/or repair data is preliminary to utilizing an optimisation model, from which optimal maintenance policy based on minimizing loss, cost or downtime may be found. A system fails more frequently as it grows older; the age T or the Nth failure at which it should be replaced to minimize the long-run expected loss, cost or downtime per unit time may be calculated from the model. Leung (2001) formulates an AP replacement model; for replacement policy T or N by which we replace the system at period T or at the time of the Nth failure, he individually derived explicit expressions for the long-run average performance measure (e.g. loss or its negation profit, cost, and downtime or its

Table 1. Recommended estimators for μ_{A_1} and $\sigma_{A_1}^2$

d	ϕ μ_{A_1}	MSE A_n	Φ μ_{A_1} & A_n	Standard Deviation $\sigma_{A_1}^2$
$=0$	$\hat{\mu}_{A_1,4}$	$\hat{\mu}_{A_1,2}$ or $\hat{\mu}_{A_1,4}$	$\hat{\mu}_{A_1,3}$ or $\hat{\mu}_{A_1,4}$	$\hat{\sigma}_{A_1,3}^2$
<0	$\hat{\mu}_{A_1,1}$ or $\hat{\mu}_{A_1,2}$	$\hat{\mu}_{A_1,2}$	$\hat{\mu}_{A_1,2}$	$\hat{\sigma}_{A_1,2}^2$
$\left(0, \frac{\mu_{A_1}}{n-1}\right]$	$\hat{\mu}_{A_1,2}$	$\hat{\mu}_{A_1,2}$	$\hat{\mu}_{A_1,2}$	$\hat{\sigma}_{A_1,2}^2$

complement availability) per unit of total time and those for the long-run average performance measure per unit of operation time, and the optimal replacement T^* or N^* is analytically determined. Two models, given by equation (4) with $r_b = 1$ and equation (5) with $r_a = r_b = 1$ in Leung (2001), used in resolving replacement problems are extracted, namely the long-run expected loss per unit time $l(T)$ under policy T which is given by

$$l(T) = \frac{(c_o - w)T + c_f\left\{\sum_{j=1}^{\infty}[\mu_{Y_1} - (j-1)d_b]F_j(T)\right\} + c_{RT}u_{RT}}{T + \sum_{j=1}^{\infty}[\mu_{Y_1} - (j-1)d_b]F_j(T) + u_{RT}} \tag{24}$$

and the long-run expected loss per unit time $l(N)$ under policy N which is given by

$$l(N) = \frac{(c_o - w)\left\{\frac{N}{2}[2\mu_{X_1} - (N-1)d_a]\right\} + c_f\left\{\frac{N-1}{2}[2\mu_{Y_1} - (N-2)d_b]\right\} + c_{RN}u_{RN}}{\frac{N}{2}[2\mu_{X_1} - (N-1)d_a] + \frac{N-1}{2}[2\mu_{Y_1} - (N-2)d_b] + u_{RN}} \tag{25}$$

where F_j is the cumulative distribution function of $\sum_{i=1}^{j} X_i$, μ_{X_1} is the mean operating time after installation, μ_{Y_1} is the mean repair time after the first failure, d_a and d_b are the common differences corresponding to the failure and repair processes of a system respectively, c_o is the average operating cost rate, c_f is the average repair cost rate, c_{RT} (or c_{RN}) is the average replacement cost rate under policy T (or N), u_{RT} (or u_{RN}) is the fixed replacement downtime under policy T (or N), and w is the average revenue rate of a working system.

We notice that Models (24) and (25) depend on the AP through the parameters d_a, d_b and μ_{X_1}, μ_{Y_1} only.

A case study using the results given in this paper, Leung *et al.* (2002) and Leung (2001) applied to the same set of real maintenance data of engines in Leung and Lee (1998) was completed and the findings were compared with those obtained in that paper. The case study was presented in a paper entitled "An engine-maintenance case study using arithmetic-process approach" which was submitted to the *International Journal of Industrial Engineering*. To illustrate the applicability of a methodology, the author considers a case study to be more interesting and convincing than a numerical example.

APPENDIX

To determine the "best" fit line to the N paired-observations, we minimise the sum of squared errors $S(d, \alpha) = \sum_{n=1}^{N}[A_n + d(n-1) - \alpha]^2$. Partially differentiating $S(d, \alpha)$ with respect to d and α, setting them equal to zero and solving the associated equations simultaneously, we obtain equations (8) and (9). It is well known that $\hat{\sigma}_\varepsilon^2 = \frac{S(\hat{d}, \hat{\alpha})}{N-2}$, which is equation (10).

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