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Convergence and Measurement of Inter-Departure Processes in a Pull Serial Line:

Entropy and Augmented Lagrange Multiplier Approach

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Abstract. In this study, we consider infinite supply of raw materials and backlogged demands as given two boundary conditions. And we need not make any specific assumptions about the inter-arrival of external demand and service time distributions. We propose a numeric model and an algorithm in order to compute the first two moments of inter-departure process. Entropy enables us to examine the convergence of this process and to derive measurable relations of this process. Also, lower bound on the variance of inter-departure process plays an important role in proving the existence and uniqueness of an optimal solution for a numeric model and deriving the convergence order of augmented Lagrange multipliers method applied to a numeric model. Through these works, we confirm some structural properties and numeric examples show the validity and applicability of our study.

Keywords: entropy, augmented lagrange multiplier, boundary conditions, general service and demand scheme

1. INTRODUCTION

1.1 Motivation

Material planning and control schemes can be classified as push, pull or hybrid systems. Push systems are typically associated with material requirements planning (MRP) systems. Pull systems are also called kanban (card) control systems. The distinction between push and pull systems is made on the basis of how production orders are released to stages. In a push system, the amount and time of material flow at each stages are forecasted in advance. Based on this forecast value, materials are pushed from a downstream stage to an upstream stage. In a pull system, the succeeding stage orders and withdrawls materials from the preceding stage, only at the rate and at the time it has consumed the items.

In recent years, several variants of push and pull schemes have been proposed, which combine features of either different push systems, or different pull systems, or both. So they are referred to as hybrid systems such as CONWIP and POLCA systems. The CONWIP control system was initially presented as an alternative pull control strategy (Spearman *et al.*, 1992). The basic idea behind CONWIP is to maintain a constant amount of WIP inventory in the entire manufacturing line by releasing new jobs to the front of the line only when WIP at the end of the line is used to satisfy customer demand. POLCA (paired-cell overlapping loops of cards with authorization) strategy is described in (Suri, 1998). In POLCA, the card loops encompass pairs of cells(stages), a compromise between the small loops of the kanban strategy and long loops of the CONWIP strategy.

In the last two decades, there has been considerable interest in the study and analysis of the pull systems. The majority of pull researches treats the analysis problem. Researches dealing with both performance analysis problems and derivation of structural properties belong to this direction. They investigate important steady state performance measures such as throughput, average WIP and average flow time under the ergodicity.

The models used include analytical approaches as well as simulation approaches. Analytical solutions exist almost exclusively for the pull serial lines with deterministic or exponentially distributed times(see e.g. Bardinelli, 1992; Bitran *et al.*, 1987; Buzacott, 1989; Deleersnyder *et al.*, 1989; Kim, 1985; Mitra *et al.*, 1990, 1991; So *et al.*, 1988; Spearman, 1992 and Tayur, 1993). On the other hand, more complex systems are investigated by simulation (see e.g. Aytug *et al.*, 1998; Hum *et al.*, 1988; Blair *et al.*, 1991; Huang *et al.*, 1983; Sarker *et al.*, 1988, 1989, and Philipoom *et al.*, 1987). However, simulation by itself can not solve any optimization problem.

The successful implementation of pull systems as well as analytical studies done on serial lines have led to the belief that the performance of pull systems and its variations are generally superior (Spearman *et al.*, 1992; Muckstadt *et al.*, 1995a, 1995b). And the superior performance of pull systems extends to more general environments. Clearly there is still a need for the development of quantitative models to gain insight in the mechanics of a card controlled pull system. Useful models for serial pull systems are provided by the finite-buffer literature for tandem queues (see e.g. Gershwin, 1987; Suresh *et al.*, 1990 and Whitt, 1984).

Although the results of many theories and applications are very promising, this author have strong reasons to believe that this might not be so. They have adopted the strong assumption such as the infinite supply of raw material and the infinite external demand process. An ideal pull system should have no backlogged demand. However, due to different variations and uncertainties in processing times, demand and availability of machines (or mans) in the production process, it is only possible when there is a sufficiently large amount of inventory in each station. From these motives, we have been interested in the implications and effects of boundary conditions.

Thus, the validity and applicability of these findings have been fully re-examined in terms of the implication and the effect of these strong assumptions. To date little work has been done on the analytical approach to these problems.

1.2 Decision Problems

Form these motives, three major problems in the pull serial lines can be identified :

(1) What is the effect of infinite supply of raw material on the first two moments of inter-departure process in the steady state? With respect to this problem, the promising findings described in Tayur(1993, corollary 1, corollary 2 and remark) may be re-examined : True or not?
(2) What is the implication of the assumption combined infinite supply of raw material with infinite or backlogged external demand process?
(3) If distribution free, what notions should be required?
(4) What theories should be followed to solve the above two questions? With respect to this problem, it may be proved to be true that the optimal solution for the arrival rates to each cell is unique, which is the conjecture described in (Mitra *et al.*, 1990).

These issues are addressed in this paper. In particular, convergence of the inter-departure process and measurement of its the first two moments are required to answer to the problem (3). The theories and algorithm proposed in this study may be ultimately applied to the following topics :

(1) It may be proved that there exists the VPP(variability propagation principle) described in Suresh *et al.* (1990) in any pull serial line under general environments.
(2) The Bull-whip effect in SCM (supply chain management) may be quantified.
(3) Any steady state performances including distributions may be easily computed, since they functionally relate to the first two moments of the inter-departure process.
(4) Optimal arrangement of cells(stages) in the pull serial line and
(5) An equivalent push type serial line in view of the inter-departure process may be obtained.

2. CONVERGENCE OF INTER-DEPARTURE PROCESS

2.1 Model Formulation Of The Pull Serial Line

Henceforth, the "pull" in this paper is meant for the pure (traditional) card (kanban) control. Although there are several ways of achieving a pull type control systems, actual physical implementation of a pull control is most often achieved by means of a card (kanban) system. As a result, the terms "kanban (card)" and "pull" are used without distinction.

Pull systems may be either constant order quantity, non-constant withdrawl cycle or constant withdrawl cycle, non-constant order quantity. In particular, the latter is also called a periodic pull system. In this paper, the former is adopted. Although the order quantity is fixed, the period between "pulls" varies due to the randomness of manual or machine processing time and external demand process.

In detail a series of cell may be represented as Figure 1, which is the same as shown in (Kimura *et al.*, 1981 and Schonberger, 1982). In Figure 1, w-card post is unnecessary since the existence of information sensing and material handling capabilities are taken for granted (Mitra *et al.*, 1990). As a result, descriptions of implementations is identical to Mitra *et al.* (1990), Tayur (1993).

Cell (i) owns a finite number of cards which are collected in the p-card post. If there is at lease one card in that post, and if there is at least one full container in the outbound of cell (i-1), then one of these containers is moved to cell (i). Here, that container is ties with one card from the p-card post of cell (i). Now the pair(container, card of cell (i)) enters the buffer(inbound). In the case the server is busy, the pair has to wait : otherwise processing is started. After processing the pair proceeds to the outbound where it the p-card post of the succeeding cell (i+1). Then the pair(container, card of cell (i)) is separated,

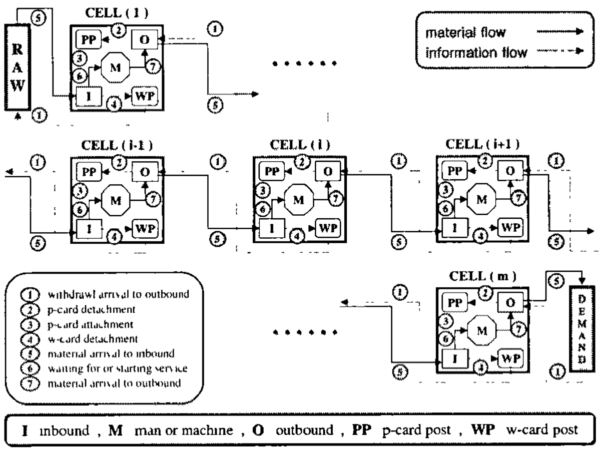

Figure 1. Flow of cards and materials(containers) in a pull serial line

the card of cell (i) is retuned to its p-card post and the container is moved to cell (i+1).

In this paper, the pull production line consists of a series of cell, which is composed of a manufacturing node (inbound + man or machine), a bulletin board (p-card post) and outbound. A manufacturing node and a bulletin board can be described as the queueing model. Thus each cell in the pull serial line consists of queue & outbound.

Items flow through the cells in sequence and one operation is performed in each cell which consists of one machine or one server. The lot size and the batch size is 1 and there is only 1 type of item produced in each cell. External demand arrives in a single unit. The service time in each cell and the inter-arrival time of external demand are assumed to be i.i.d and their means and variances are known (distribution free, general distribution).

In addition to, there is no transit time for the movement of items between cells; no scrap or defectives are produced, and there is no down time.

There is an infinite supply of raw materials to the cell (1) and the external demand is permitted to be infinite or backlogged. Finally, there is a finite buffer size in each cell, which is said to be a maximum inventory level or a fixed number of cards.

2.2 Nomenclature

$(N1)\ i \in [0, m] \cdots$

cell index, and note that 0 denotes raw material pool.

$(N2)\ T_i,\ i \neq 0 \cdots$

maximum of inventory levels or number of cards in cell (i).

$$(N3)\ K_i = \left\{ k \mid k \geq 1 + \sum_{a=i+1}^{m} T_a \right\},\ K_{(i)} = \left\{ k \mid k \geq 1 + \sum_{a=i}^{m} T_a \right\}$$

$(N4)\ C_i^k,\ k \in K_i \cdots$

time at which the withdrawl order for the k_th material arrives at the outbound of cell (i).

$(N5)\ D_i^k,\ k \in K_i \cdots$

time at which the k_th material arrives at the outbound of cell (i).

$(N6)\ Z_i^k,\ k \in K_i \cdots$

time at which the k_th material departs from the outbound of cell (i), in other words, the k_th material arrives at the the queue of cell (i+1).

$(N7)\ S_i^k,\ i \neq 0, k \in K_{(i)} \cdots$

processing(service) time of the k_th material at the queue of cell (i).

$(N8)\ A(k),\ k \in K_m \cdots$

time at which the k_th external demand arrives at the outbund of cell (m).

$(N9)\ \{IC_i^k = C_i^{k+1} - C_i^k,\ k \in K_i\} \cdots$

increments process of $C_i^k, k \in K_i$, and note that m indicates the inter-arrivals (increments) process of external demands denoted by $IA(k), k \in K_m$.

$(N10)\ \{ID_i^k = D_i^{k+1} - D_i^k,\ k \in K_i\} \cdots$

increments process of $D_i^k, k \in K_i$.

$(N11)\ \{IZ_i^k = Z_i^{k+1} - Z_i^k,\ k \in K_i\} \cdots$

increments process of $Z_i^k, k \in K_i$, which denotes the inter-departutre process of cell (i).

$(N12)\ U_i^k = S_i^k - IZ_{i-1}^k,\ i \neq 0, k \in K_{(i)}$

$(N13)\ W_{Q,i}^k = Max\left(D_i^{k-1} - Z_{i-1}^k, 0\right),\ i \neq 0, k \in K_{(i)} \cdots$

waiting time of the k_th material in the queue of cell (i).

$(N14)\ I_i^k = Max\left(Z_{i-1}^{k+1} - D_i^k, 0\right), i \neq 0, k \in K_{(i)} \cdots$

virtual idle time of man or machine in cell (i).

$$(N15) \begin{cases} 0,\ if\ C_i^k \geq D_i^k \Leftrightarrow Z_i^k = C_i^k \\ 1,\ if\ C_i^k < D_i^k \Leftrightarrow Z_i^k = D_i^k \\ k \in K_i \end{cases}$$

$(N16)\ f_i^k = P(J_i^k = 0) = P(Z_i^k = C_i^k), k \in K_i$

$(N17)\ N_i = M_i + B_i$

$M_i, i \neq 0 \cdots$

number of materials being or being served in the queue of cell (i) in the steady state.

$B_i, i \neq 0 \cdots$
number of materials not immediately satisfied when a withdrawal order from cell (i+1) arrives at the outbound of cell (i) in the steady state.

$O_i, i \neq 0 \cdots$
number of materials in the outbound of cell (i) in the steady state.

$(N18)\ \lambda_d\ (\neq 0) \cdots$ external demand rate.
$\mu_i\ (\neq 0), i \neq 0 \cdots$ service rate at cell (i).

$$(N19)\begin{cases} C_{A,i+1}^2 = V(IZ_i) \big/ [E(IZ_i)]^2,\ i \in [0, m+1] \\ C_{DM}^2 = V(IA)\lambda_d^2 \\ C_{S,i}^2 = V(S_i)\mu_i^2,\ i \neq 0 \\ IZ_i^k,\ IA(k),\ S_i^k \rightarrow IZ_i,\ IA,\ S_i \end{cases}$$

$(N20)\ T(i, i+1) = T_i + T_{i+1},\ i \neq 0$

$(N21)\ \lambda_i^- = \lambda_d \cdot \sum_{n=0}^{T(i,i+1)-1} P(N_i = n),\ i \neq 0 \cdots$
effective arrival rate at cell (i).

$\mu_i^- = \mu_i \cdot \sum_{n=1}^{T(i,i+1)} P(N_i = n),\ i \neq 0 \cdots$
effective service rate at cell (i).

$\rho_i = \lambda_d \big/ \mu_i,\ i \neq 0 \cdots$
traffic intensity of cell (i).

2.3 Preliminaries

Let R_+ denote the set of nonnegative real numbers and let B_O, the subsets of R_+, denote the class of bounded Borel sets. The stochastic processes $\{C_i^k,\ k \in K_i\}$ and $\{D_i^k,\ k \in K_i\}$ defined on the probability space (R_+, B_O, P) have independent increments and are generated by (1) and (2) respectively.

$$C_i^k = \begin{cases} Z_{i+1}^{k-T_{i+1}},\ i \neq m \\ A(k),\ i = m \end{cases} \tag{1}$$

$$D_i^k = Max\left(Z_{i-1}^k, D_i^{k-1}\right) + S_i^k,\ i \neq 0, k \in K_{(i)} \tag{2}$$

It follows from (N4) and (N5) that the stochastic processes $\{C_i^k,\ k \in K_i\}$ and $\{D_i^k,\ k \in K_i\}$ are mutually independent. Also, by equation (1), a stochastic process $\{Z_i^k,\ k \in K_i\}$ defined on the probability space (R_+, B_O, P) has independent increments and is generated by

$$Z_i^k = Max\left(C_i^k, D_i^k\right) = \begin{cases} Max\left(Z_{i+1}^{k-T_{i+1}}, D_i^k\right),\ i \neq m \\ Max\left(A(k), D_m^k\right),\ i = m \end{cases} \tag{3}$$

In particular, the boundary condition of an infinite supply of raw materials requires $\{C_0^k, k \in K_0\} \equiv \{Z_0^k, k \in K_0\}$. Thus, we need not consider $\{D_0^k, k \in K_0\}$.

Similarly, $\{Z_i^k, k \in K_i, i \neq 0\}$ may be generated by

$$Z_i^k = C_{i-1}^{k+T_i},\ i \neq 0 \tag{4}$$

, where $\{C_0^{k+T_1}, k \in K_1\}$ implies that a withdrawl order for the $(k+T_1)_{th}$ material arrives at the raw materials pool.

In general, the stochastic processes $\{Z_i^k, k \in K_i\}$, $\{C_i^k, k \in K_i\}$ and $\{D_i^k, k \in K_i\}$ have nonstationary increments since they can not be represented as a sum of i.i.d.(independent and identically distributed) random variables. In a pull serial line, if either the p-card post of cell (i) or the outbound of cell (i-1) is empty, then the server remains idle until the earliest point in time until both material and card are available. In other words, a pull serial line is subject to blocking (back order, backlogging) from time to time. This is the reason why the stochastic processes $\{Z_i^k, k \in K_i\}$, $\{C_i^k, k \in K_i\}$ and $\{D_i^k, k \in K_i\}$ can not be renewal processes.

Lemma 1. *If there exists a pull serial line such that there is no down time, then we get the following two results :*

(i)

Three sequences of integrable random variables $\{W_{Q,i}^k, k \in K_{(i)}\}$, $\{ID_i^k, k \in K_{(i)}\}$ *and* $\{IZ_{i-1}^k, k \in K_{(i-1)}\}$ *defined on the probability space* (R_+, B_O, P) *converge in* L^2 *to some random variables* $W_{Q,i}$, ID_i *and* IZ_{i-1} *respectively for each i. That is,*

$$\lim_{k \to \infty} E\left[\left(W_{Q,i}^k - W_{Q,i}\right)^2\right] = 0$$

$$\lim_{k \to \infty} E\left[\left(ID_i^k - ID_i\right)^2\right] = 0$$

$$\lim_{k \to \infty} E\left[\left(IZ_{i-1}^k - IZ_{i-1}\right)^2\right] = 0$$

(ii)

$$\begin{cases} E(ID_i) = E(IZ_{i-1}) \\ V(ID_i) = 2 \cdot V(S_i) + 2 \cdot E(W_{Q,i})[E(S_i) - E(IZ_{i-1})] \\ \qquad + V(IZ_{i-1}) \end{cases} \tag{5}$$

Proof. From the assumption that there is no down time, any given pull serial line must be always stable. This implies that

$$\left\{ \left(W_{Q,i}^k\right)^2,\ k \in K_{(i)} \right\}$$

is uniformly integrable and $W_{Q,i}^k$ converges in probability to some random variable $W_{Q,i}$. That is, for every

$$\varepsilon > 0,\ Sup_k\ E\left[\left(W_{Q,i}^k\right)^2\right] < \infty \text{ and}$$

$$\lim_{k \to \infty} P\left(\left| W_{Q,i}^k - W_{Q,i} \right| > \varepsilon\right) = 0.$$

Thus, $W_{Q,i}^k$ converges in L_2 to some random variable $W_{Q,i}$. Using (N13) and (N14), we obtain

$$\begin{cases} W_{Q,i}^{k+1} - I_i^k = D_i^k - Z_{i-1}^{k+1} = U_i^k - W_{Q,i}^k \\ W_{Q,i}^{k+1} \cdot I_i^k = 0 \end{cases} \quad (6)$$

It follows from equation (6) and (N12) that

$$ID_i^k = U_i^{k+1} + I_i^k + IZ_{i-1}^{k+1} = S_i^{k+1} + I_i^k \quad (7)$$

Since S_i^{k+1} are i.i.d. random variables, it is clear that I_i^k and ID_i^k converges in L^2 to some random variables I_i and ID_i respectively. Thus U_i^k in equation (6) converges in L^2 to some random variable U_i. Similarly, IZ_{i-1}^k in (N12) converges L^2 in to some random variable IZ_{i-1}. Since all of $W_{Q,i}^k$, I_i^k, ID_i^k, U_i^k and IZ_{i-1}^k converge in L^2, as a matter of course, they converge in L^1. Hence, based on (6), (7) and **Lebesque's Dominated Convergence Theorem**, we have

$$E(I_i) = -E(U_i) \quad (8)$$

$$E(ID_i) = E(S_i) - E(U_i) = E(IZ_{i-1}) \quad (9)$$

$$E(W_{Q,i}) = \frac{E(I_i^2) - E(U_i^2)}{2 \cdot E(U_i)} \quad (10)$$

Note that S_i^{k+1} and I_i^k are mutually independent.
By equation (8),

$$V(ID_i) = V(S_i) + V(I_i) \quad (11)$$

which may be written as

$$\begin{aligned} V(ID_i) &= V(S_i) + 2 \cdot E(W_{Q,i}) \cdot E(U_i) + V(U_i) \\ &= 2 \cdot V(S_i) + 2 \cdot E(W_{Q,i}) \cdot \left[E(S_i) - E(IZ_{i-1})\right] \quad (12) \\ &\quad + V(IZ_{i-1}) \end{aligned}$$

since S_i^k and IZ_{i-1}^k in (N12) are mutually independent too. This completes the proof. □

2.4 Entropy of a Finite Scheme

The inter-departures $\{IZ_i^k,\ k \in K_{(i)}\}$process may be represented as

$$IZ_i^k = J_i^{k+1} \cdot \left(D_i^{k+1} - C_i^{k+1}\right) - J_i^k \cdot \left(D_i^k - C_i^k\right) + IC_i^k \quad (13)$$

In probability theory a complete system of events means a set of events such that one and only one of them must occur at each trial. If we are given the events of a complete system, together with their probabilities, then we say that we have a finite scheme. In this viewpoint, we consider a finite scheme of the inter-departures process that is given by

$$\Omega(IZ_i^k) = \begin{cases} IC_i^k, & w.p.\ f_i^k \cdot f_i^{k+1} \\ C_i^{k+1} - D_i^k, & w.p.\ (1 - f_i^k) \cdot f_i^{k+1} \\ D_i^{k+1} - C_i^k, & w.p.\ f_i^k \cdot (1 - f_i^{k+1}) \\ ID_i^k, & w.p.\ (1 - f_i^k) \cdot (1 - f_i^{k+1}) \end{cases} \quad (14)$$

Every finite scheme describes a state of uncertainty. It seems obvious that the amount of uncertainty is different in different schemes. For many applications it seems desirable to introduce a quantity that in a reasonable way measures the amount of uncertainty associated with a given finite scheme. We shall call the quantity $H_\Omega(IZ_i^k)$ the entropy of the finite scheme $\Omega(IZ_i^k)$, which represents the degree of uncertainty and difficulty in estimation.

Proposition 1. *J_i^k, $k \in K_{(i)}$ are i.i.d. random variables and $H_\Omega(IZ_i^k)$ is time-invariant.*

Proof. By the definition, it is obvious that J_i^k and J_i^{k+1} are mutually independent for each k. Let $C_{0,r}^k$ and $C_{1,r}^k$ denote the number of 0's and 1's in the pair $\left(J_i^k, J_i^{k+1}\right)$ of an event defined in equation (13). Then we obtain

$$P(J_i^k = 0) = \frac{\sum_{r=1}^{4} C_{0,r}^k \cdot P_r}{\sum_{r=1}^{4} 2 \cdot P_r} = \frac{f_i^k + f_i^{k+1}}{2}$$

$$P(J_i^k = 1) = \frac{\sum_{r=1}^{4} C_{1,r}^k \cdot P_r}{\sum_{r=1}^{4} 2 \cdot P_r} = \frac{2 - f_i^k - f_i^{k+1}}{2}$$

Since $f_i^k = \left(f_i^k + f_i^{k+1}\right)/2$ and $f_i^k = f_i^{k+1}$ for each k, we have $E(J_i^k) = E(J_i^{k+1}), V(J_i^k) = V(J_i^{k+1})$, which means $J_i^k, k \in K_{(i)}$ are i.i.d.

Now we set $f_i^k = f_i, \forall k$. f_i is a given constant such that $0 \le f_i \le 1$. Then the entropy may be calculated as

$$\begin{aligned} H_\Omega(IZ_i^k) &= -\sum_{r=1}^{4} P_r \ln P_r \\ &= -2 \cdot [f_i \ln f_i + (1-f_i)\ln(1-f_i)] \end{aligned}$$

And ln denotes the natural logarithm. Thus $H_\Omega(IZ_i^k)$ is time-invariant. This completes the proof. □

Under a finite scheme $\Omega(IZ_i^k)$, we may calculate $E_\Omega(IZ_i^k)$ and $V_\Omega(IZ_i^k)$ with the conditional expectation and variance, which are given by

$$\begin{cases} E_\Omega(IZ_i^k) = f_i \cdot E(IC_i^k) + (1-f_i) \cdot E(ID_i^k) \\ V_\Omega(IZ_i^k) = f_i^2 \cdot V(IC_i^k) + (1-f_i)^2 \cdot V(ID_i^k) \\ \quad + f_i \cdot (1-f_i) \cdot E\left[\left(C_i^k - D_i^k\right)^2 + \left(C_i^{k+1} - D_i^{k+1}\right)^2\right] \end{cases} \tag{15}$$

If there exists a new finite scheme $\Omega_+(IZ_i^k)$ such that

$$E_{\Omega+}(IZ_i^k) = E_\Omega(IZ_i^k), V_{\Omega+}(IZ_i^k) \le V_\Omega(IZ_i^k)$$

and

$$H_{\Omega+}(IZ_i^k) \le H_\Omega(IZ_i^k),$$

it is worth while for us to consider and see what will come out of it the more because $\Omega_+(IZ_i^k)$ is superior to $\Omega(IZ_i^k)$.

To begin with, (13) and (14) are transformed into (13-1) and (14-1) respectively.

$$IZ_i^k = \alpha_i^k \cdot IC_i^k + (1-\alpha_i^k) \cdot ID_i^k, \ 0 \le \alpha_i^k \le 1 \tag{13-1}$$

$$\Omega_+(IZ_i^k) = \begin{cases} IC_i^k, \ s.t.\ \alpha_i^k = 1 \\ ID_i^k, \ s.t.\ \alpha_i^k = 0 \\ \alpha_i^k \cdot IC_i^k + (1-\alpha_i^k) \cdot ID_i^k, \ s.t.\ 0 < \alpha_i^k < 1 \end{cases} \tag{14-1}$$

α_i^k is a mixed random variable with the distribution is given by

$$\alpha_i^k = \begin{cases} 0, \ w.p.\ (1-f_i)^2 \\ 1, \ w.p.\ f_i^2 \\ \left(0, x_i^k\right], \ w.p.\ -4 \cdot f_i \cdot (1-f_i) \cdot \\ \quad \int_0^{x_i^k} \left[\alpha_i^k \cdot \ln \alpha_i^k + (1-\alpha_i^k) \cdot \ln(1-\alpha_i^k)\right] d\alpha_i^k \end{cases} \tag{16}$$

When α_i^k lies in an interval (0, 1), a distribution function must satisfies the following conditions.

(1) $\int_0^1 dF(\alpha_i^k) = 2f_i(1-f_i)$, $F(\alpha_i^k)$ denotes the distribution function.

(2) $E_{\Omega+}(IZ_i^k) = E_\Omega(IZ_i^k), V_{\Omega+}(IZ_i^k) \le V_\Omega(IZ_i^k)$ and $H_{\Omega+}(IZ_i^k) \le H_\Omega(IZ_i^k)$.

(3) Since equation (13-1) represents the stochastic convex combination, $\frac{dF(\alpha_i^k)}{d\alpha_i^k} = \frac{dF(1-\alpha_i^k)}{d\alpha_i^k}$.

Proposition 2. *α_i^k, $k \in K_{(i)}$ are i.i.d. random variables and $H_{\Omega+}(IZ_i^k)$ is time-invariant.*

Proof. It is obvious that α_i^k and α_i^{k+1} are mutually independent for each k. It follows from equation (16) that

$$E(\alpha_i^k) = f_i, V(\alpha_i^k) = 11/18 \cdot f_i(1-f_i)$$

This implies that α_i^k, $k \in K_{(i)}$ are i.i.d.

And $H_{\Omega+}(IZ_i^k)$ is given by

$$H_{\Omega+}(IZ_i^k) = -2 \cdot \left[\begin{array}{l} f_i \cdot \ln f_i + (1-f_i) \cdot \ln(1-f_i) + \\ f_i(1-f_i)(\ln 2 + 0.087213) \end{array}\right]$$

This completes the proof. □

Similarly, under a finite scheme $\Omega_+(IZ_i^k)$, we may calculate $E_{\Omega+}(IZ_i^k)$ and $V_{\Omega+}(IZ_i^k)$ with the conditional expectation and variance, which are given by

$$\begin{cases} E_{\Omega+}(IZ_i^k) = f_i \cdot E(IC_i^k) + (1-f_i) \cdot E(ID_i^k) \\ V_{\Omega+}(IZ_i^k) = f_i^2 \cdot V(IC_i^k) + (1-f_i)^2 \cdot V(ID_i^k) \\ \quad + \frac{11}{18} \cdot f_i \cdot (1-f_i) \cdot \left(V(IC_i^k) + V(ID_i^k)\right) \\ \quad + \frac{11}{18} \cdot f_i \cdot (1-f_i) \cdot \left(E(IC_i^k) - E(ID_i^k)\right)^2 \end{cases} \tag{17}$$

Hence, by (Proposition 1, 2), (15) and (17), we have

$$E_{\Omega+}(IZ_i^k) = E_\Omega(IZ_i^k), V_{\Omega+}(IZ_i^k) \le V_\Omega(IZ_i^k)$$

and

$$H_{\Omega+}(IZ_i^k) \le H_\Omega(IZ_i^k).$$

As a result, it is recommended to do use a finite scheme $\Omega_+(IZ_i^k)$ the more because the uncertainty of $\Omega_+(IZ_i^k)$ is less than that of $\Omega(IZ_i^k)$.

A finite scheme $\Omega_{+}(IZ_i^k)$ enables us to get a very useful lemma.

Lemma 2. *If there exists a pull serial line such that the external demand is permitted to be backlogged, then we get the following two main results :*

(i)
For each i, $\{IC_i^k,\ k \in K_{(i)}\}$ *and* $\{IZ_i^k,\ k \in K_{(i)}\}$ *defined on the probability space* (R_+, B_O, P) *converge in* L^2 *to some random variables*

$$IC_i \ \text{ and } \ IZ_i = \alpha_i IC_i + (1-\alpha_i) ID_i,\ 0 < \alpha_i \le 1.$$

(ii)

$$\begin{cases} E(IZ_i) = f_i \cdot E(IC_i) + (1-f_i) \cdot E(ID_i) \\ V(IZ_i) = f_i^2 \cdot V(IC_i) + (1-f_i)^2 \cdot V(ID_i) \\ \qquad + \frac{11}{18} \cdot f_i \cdot (1-f_i) \cdot \left(V(IC_i) + V(ID_i)\right) \\ \qquad + \frac{11}{18} \cdot f_i \cdot (1-f_i) \cdot \left(E(IC_i) - E(ID_i)\right)^2 \\ 0 < f_i \le 1 \end{cases} \tag{18}$$

Proof. In (Lemma 1), there is no knowing whether IZ_m^k converges in L^2 to some random variable IZ_m or not. In (Lemma 2), however, by the assumption of permitting backlogged demand, it follows from equation (1) and (N8) that $\{IC_m^k,\ k \in K_{(m)}\} = \{IA(k),\ k \in K_{(m)}\}$.
Since $IA(k)$ are i.i.d. random variables, it is clear that IC_m^k converges in L^2 to some random variable IC_m. Then a finite scheme $\Omega_{+}(IZ_i^k)$ described in equation (13-1) tells us that if we fix $i = m$ in equation (13-1), IZ_m^k converges in L^2 to some random variable IZ_m. In other words, note that (Lemma 1) and (Proposition 2). Now, applying this fact, (Lemma 1), (Proposition 2) and Continuous Mapping Principle to a finite scheme $\Omega_{+}(IZ_i^k)$, then we obtain IC_i^k converges in L^2 to some random variable IC_i for each $i \in [0, m-1]$. Therefore,

$$\lim_{k \to \infty} E\left[\left(IC_i^k - IC_i\right)^2\right] = 0 \quad \text{and}$$

$$\lim_{k \to \infty} E\left[\left(IZ_i^k - \alpha_i \cdot IC_i - (1-\alpha_i) \cdot ID_i\right)^2\right] = 0.$$

Now that IZ_i^k converges in L^2 to some random variable IZ_i, it is clear that a conditional expectation also converges in L^2 to some random variable and its some subsequence converges in w.p. 1(i.e. almost everywhere, almost surely) to some random variable. Thus equation (18) can be directly derived from equation (17). Finally,

$$E(\alpha_i^k) = f_i,\ V(\alpha_i^k) = {}^{11}/_{18} \cdot f_i (1-f_i) \ \text{ implies } f_i \ne 0,$$

which means also $\alpha_i \ne 0$. This completes the proof. □

2.5 True Lower Bound of $V(IZ_i)$

By (N16) and (N17), f_i must be equal to $P(B_i = 0)$. However, it is possible for us to derive the distributions of B_i, M_i and O_i only when the distribution of N_i should be given in advance.

In this paper, we need not make any specific assumptions about the inter-arrivals of external demands and service time distributions. Consequently, only approximate distributions of the steady state random variables such as B_i, M_i, O_i and N_i may be available. This implies that there are many possibilities of approximating their distributions. Therefore, it is necessary that we should derive a true or desirable lower bound on $V(IZ_i)$ applicable to any approximate to f_i or $P(B_i = 0)$ on the basis of N_i.

Also this necessity forces us to modify equation (13-1). Fortunately, there is at least one mathematical technique to solve this problem, which is the Taylor's Series. Maintaining an identical expectation of IZ_i, we will utilize the Taylor's Series.

Proposition 3. *The lower bound on* $V(IZ_i)$ *applicable to any given approximate to* f_i *or* $P(B_i = 0)$ *is represented as*

$$\begin{cases} IZ_i = f_i \cdot IC_i + (1-f_i) \cdot ID_i + \\ \qquad \left(E(IC_i) - E(ID_i)\right) \cdot (\alpha_i - f_i) \\ E(IZ_i) = f_i \cdot E(IC_i) + (1-f_i) \cdot E(ID_i) \\ V(IZ_i) = f_i^2 \cdot V(IC_i) + (1-f_i)^2 \cdot V(ID_i) \\ \qquad + \frac{11}{18} \cdot f_i \cdot (1-f_i) \cdot \left(E(IC_i) - E(ID_i)\right)^2 \end{cases} \tag{19}$$

Proof. From equation (13-1), the lower bound may be obtained by the first order Taylor's Series at the neighborhood of

$$\left(\alpha_i^k, IC_i^k, ID_i^k\right) = \left(E(\alpha_i^k), E(IC_i^k), E(ID_i^k)\right)$$

without remainder. Then we have

$$\begin{cases} IZ_i^k = f_i \cdot IC_i^k + (1-f_i) \cdot ID_i^k + \\ \qquad \left(E(IC_i^k) - E(ID_i^k)\right) \cdot (\alpha_i^k - f_i) \\ E(IZ_i^k) = f_i \cdot E(IC_i^k) + (1-f_i) \cdot E(ID_i^k) \\ V(IZ_i^k) = f_i^2 \cdot V(IC_i^k) + (1-f_i)^2 \cdot V(ID_i^k) \\ \qquad + \frac{11}{18} \cdot f_i \cdot (1-f_i) \cdot \left(E(IC_i^k) - E(ID_i^k)\right)^2 \end{cases} \tag{13-2}$$

By (Lemma 1), (Lemma 2) and (Proposition 2), equation (19) can be derived from equation (13-2) in the steady state. This completes the proof. □

Actually, it is impossible to overestimate the importance of the lower bound on $V(IZ_i)$. A concise functional form compared to the true value of $V(IZ_i)$ enables us to easily manipulate problems associated with proofs and structural properties. It is not too much to say that we cannot pay too much attention to this fact.

2.6 First Two Moments of under Given Two Boundary Conditions

In this section, we explicitly consider two boundary conditions that infinite supply of raw materials and backlogged demands are permitted to cell (1) and cell (m) respectively. Therefore, what is the implication of these conditions? We propose the equivalent statement to these assumptions. Then, relied upon these assumptions, we are with intention of investigating into measurable relations among ID_i, IZ_i and IC_i representative of cell (i-1), cell (i) and cell (i+1) respectively.

Theorem 1. *Suppose that there exists a pull serial line such that infinite supply of raw materials and backlogged demands are permitted, besides $E(U_i) < 0$. Then we have the following results :*

(i)

$$E(IZ_{i+1}) = E(IZ_0) = 1/\lambda_d \ , i \in [0, m] \quad (20\text{-}1)$$

(ii)

$$V(IZ_i)_{TRUE} = f_i^2 \cdot V(IC_i) + (1-f_i)^2 \cdot V(ID_i) + \frac{11}{18} \cdot f_i \cdot (1-f_i) \cdot \left(V(IC_i) + V(ID_i)\right) \quad (20\text{-}2)$$

(iii)

$$V(IZ_i)_{LOWER} = f_i^2 \cdot V(IC_i) + (1-f_i)^2 \cdot V(ID_i) \quad (20\text{-}3)$$

Proof. In this proof, (Lemma 1) and (Lemma 2) are implicitly used.

It follows from equation (1) or (4) that $IZ_1 = IC_0 = IZ_0$. Hence,

$$E(IZ_1) = E(IZ_0)$$

And by (4), (5) and (18), we obtain

$$f_i \cdot E(IZ_{i+1}) - E(IZ_i) + (1-f_i) \cdot E(IZ_{i-1}) = 0$$

which may be rewritten as

$$E(IZ_{i+1}) = E(IZ_0) + [E(IZ_0) - E(IZ_{-1})].\ S(i), \qquad S(i) = \sum_{j=0}^{i} \prod_{k=0}^{j} \left(\frac{1}{f_k} - 1 \right),\ i \in [0,\ m] \quad (21)$$

Infinite supply of raw materials implies $f_0 = 1$. Thus, from equation [21], we have

$$S(i) = \sum_{j=0}^{i} \prod_{k=0}^{j} \left(\frac{1}{f_k} - 1 \right) = 0$$

In addition,

$$E(IZ_{m+1}) = E(IC_m) = E(IA) = 1/\lambda_d$$

This completes the proof of (20-1).

It follows from (4), (5) and (20-1) that $E(IC_i) = E(ID_i)$. Applying this relation to (18) and (19), then we complete the proof of (20-2) and (20-3). □

Proposition 3. *Suppose $E(U_i) < 0$. Then the following two statements are equivalent.*

(i)

The infinite supply of raw materials and the backlogged demands are permitted.

(ii)

Either backlogged demands or infinite demands are permitted, and

$$P(N_i = T(i,i+1)) = 0,\ P(N_i = 0) = 1 - \rho_i,\ \rho_i < 1, \forall i \in [1, m]$$

Proof. (1) (i) implies (ii).

By equation (20-1) and (N21), we have

$$\begin{aligned} E(IZ_i) &= (\lambda_d \cdot [1 - P(N_i = T(i,i+1))])^{-1} \\ &= (\mu_i \cdot [1 - P(N_i = 0)])^{-1} \\ &= 1/\lambda_d \ , i \in [0, m+1] \end{aligned}$$

(2) (ii) implies (i).

By (N21) and equation (9), we have

$$\lambda_i^- = \mu_i^- = 1/E(ID_i) = 1/E(IZ_{i-1}).$$

Since $\rho_i < 1$,

$$1/E(IZ_{m+1}) = \lambda_d = \lambda_i^- = 1/E(IZ_{i-1}),\ i \in [1, m]$$

Thus, we obtain either

$$1/E(IZ_{m+1}) = 1/E(IZ_m)$$

or

$$1/E(IZ_m) = 1/E(ID_m) = 1/E(IZ_{m-1})$$

This completes the proof. □

3. MEASUREMENT OF INTER-DEPARTURE PROCESS

3.1 Numeric model

Henceforth, we would like to focus on the squared coefficient of variation of the inter-departure process. To begin with, we present the (Corollary 1).

Corollary 1. *Suppose that there exists a pull serial line such that infinite supply of raw materials and backlogged demands are permitted, besides $E(U_i) < 0$. Then we have the following results :*

$$\begin{cases} C^2_{A,m+2} = C^2_{DM} \\ C^2_{A,i+2} = M(i)\cdot C^2_{A,i+1} - N(i)\cdot C^2_{A,i} - 2\cdot\lambda_d^2\cdot N(i)\cdot SS(i) \\ SS(i) = \dfrac{C^2_{S,i}}{\mu_i^2} + E(W_{Q,i})\cdot\left(\dfrac{1}{\mu_i} - \dfrac{1}{\lambda_d}\right) \\ C^2_{A,2} = C^2_{A,1} \\ i \in [1, m] \;\; for \;\; any \;\; t \in [0 \;\; m+1] \end{cases} \tag{22}$$

$$M(i) = \begin{cases} \left(\dfrac{1}{f_i}\right)^2 & , \; under \; lower \; bound \\ \dfrac{18}{7\cdot f_i^2 + 11\cdot f_i} & , \; under \; true \; value \end{cases} \tag{23}$$

$$N(i) = \begin{cases} \left(\dfrac{1-f_i}{f_i}\right)^2 & , \; under \; lower \; bound \\ \dfrac{7\cdot f_i^2 - 25\cdot f_i + 18}{7\cdot f_i^2 + 11\cdot f_i} & , \; under \; true \; value \end{cases} \tag{24}$$

Proof. We replace $V(ID_i)$ in (20-2) and (20-3) with (12). And then, by (N19) and (20-1), the proof is completed. □

Indeed, there is no computing f_i, $E(W_{Q,i})$ in (22), (23) and (24) if not given $C^2_{A,i}$ beforehand. In this sense, $C^2_{A,i}$ is the key factor that enables us to analyze the interaction between every three adjacent cells.

A way of solving a set of simultaneous algebraic equations such as the Table 1 is to form an objective function and minimize it numerically.

There never exists a unique optimal solution in case of a lower bound without existing it in case of a true value, since structural properties hold due to the similarities of functional form such as (23) and (24).

In particular, functional form of lower bound is more concise than that of true value. Thus, we can use the case of lower bound in order to prove the existence, the uniqueness and the necessary and sufficient conditions of an optimal solution.

Table 1. Nonlinear simultaneous equations

$FIND : C^2_{A,h} \geq 0 \;,\; h \in [1, m+2] \;,\; i \in [1, m]$	(25)
$\begin{cases} C^2_{A,m+2} = C^2_{DM} \\ C^2_{A,i+2} = \left(\frac{1}{f_i}\right)^2 C^2_{A,i+1} - \left(\frac{1-f_i}{f_i}\right)^2 C^2_{A,i} - \\ 2\cdot\lambda_d^2\left(\frac{1-f_i}{f_i}\right)^2\left\{\frac{C^2_{S,i}}{\mu_i^2} + E(W_{Q,i})\left(\frac{1}{\mu_i} - \frac{1}{\lambda_d}\right)\right\} \\ C^2_{A,2} = C^2_{A,1} \\ E(W_{Q,i}) = \frac{E(M_i)}{\lambda_d} - \frac{1}{\mu_i} \\ \rho_i = \frac{\lambda_d}{\mu_i} \end{cases}$	(26)
$\begin{cases} E(M_i) = \sum_{x=0}^{T_i} x\cdot P(M_i = x) \\ f_i = 1 - \rho_i + C_{B,i}\cdot\sum_{n=1}^{T_i} P(N_i = n) \\ C_{B,i} = \left(\rho_i\cdot\sum_{n=1}^{T(i,i+1)-1} P(N_i = n)\right)^{-1} \\ T_{m+1} = big\ M \\ P(M_i = x) = \begin{cases} 1-\rho_i \;,\; x=0 \\ C_{B,i}\cdot P(N_i = x) \;,\; x \in [1, T_i - 1] \\ C_{B,i}\cdot\sum_{n=T_i}^{T(i,i+1)-1} P(N_i = n) \;,\; x = T_i \end{cases} \end{cases}$	(27)
$\begin{cases} P(N_i = n) = \begin{cases} (1-\rho_i)\cdot\delta_i \;,\; n=0 \\ \rho_i(1-r_i)\cdot r_i^{n-1}\cdot\delta_i \;,\; n \in [1, T(i,i+1)-1] \\ \rho_i(1-\rho_i) r_i^{T(i,i+1)-1}\cdot\delta_i \;,\; n = T(i,i+1) \end{cases} \\ \delta_i = \left(1 - \rho_i^2\cdot r_i^{T(i,i+1)-1}\right)^{-1} \\ r_i = 1 - \frac{\rho_i}{E(L_i)} \\ E(L_i) = \frac{0.5\cdot\rho_i^2(C^2_{S,i}+1)(\rho_i^2 C^2_{S,i} + C^2_{A,i})}{(1-\rho_i)(\rho_i^2 C^2_{S,i}+1)} + \rho_i \end{cases}$	(28)

3.2 Existence and Uniqueness of Optimal Solution

We define Λ and $C^2_{A,i}$ as

$$\begin{cases} C^2_{A,i} = k_i(C^2_{A,1}), \\ C^2_{A,1} = k_1(C^2_{A,1}) = k_2(C^2_{A,1}), \\ \Lambda = \left\{ C^2_{A,1} \middle| C^2_{A,i} = k_i(C^2_{A,1}) \geq 0, \forall i \in [1, m+2] \right\} \\ \quad \subset H^1 = [0, \infty) \end{cases} \tag{29}$$

Table 1 may be transformed into (30) and (30) is the same as (31).

$$\begin{aligned} &FIND: C^2_{A,1} \in \Lambda \subset H^1 = [0, \infty) \\ &\quad k_{m+2}(C^2_{A,1}) - C^2_{DM} = 0 \\ &where\ C^2_{A,m+2} = k_{m+2}(C^2_{A,1}) \end{aligned} \tag{30}$$

$$\begin{aligned} &Minimize: \Phi(C^2_{A,1}) = 0.5 \cdot \left[k_{m+2}(C^2_{A,1}) - C^2_{DM} \right]^2 \\ &Subject\ to: C^2_{A,1} \in \Lambda \subset H^1 = [0, \infty) \\ &where: C^2_{A,m+2} = k_{m+2}(C^2_{A,1}) \end{aligned} \tag{31}$$

Theorem 2. *$\Phi(C^2_{A,1})$ are Λ given by equation (31).*
Suppose
(1) $\Phi: H^1 \to H^1$ is a continuous function.
(2) Λ is a compact set.
And we define

$$(a)\quad \nu = \inf\left\{ \Phi(C^2_{A,1}), C^2_{A,1} \in \Lambda \right\} \qquad (b)\quad \Phi\left(\left(C^2_{A,1}\right)^* \right) = 0$$

Then there is an optimal solution $(C^2_{A,1})^*$ in Λ that minimizes $\Phi(.)$ over Λ. And the optimal solution $(C^2_{A,1})^*$ is unique.

Proof. (1) Existence :

By the definition of ν (infimum), there exists a sequence $\{C^2_{A,1,k}\}$ such that

$$\lim_{k\to\infty} \Phi(C^2_{A,1,k}) = \nu \text{ in } \Lambda.$$

Thus there exists a set K such that

$$C^2_{A,1,k} \to \left(C^2_{A,1}\right)^* \in \Lambda, k \in K \subset \{positive\ integer\}$$

due to the compactness of Λ.
Continuity $\Phi(.)$ of implies

$$\lim_{k\in K} \Phi(C^2_{A,1,k}) = \Phi\left(\left(C^2_{A,1}\right)^* \right).$$

And as $K \subset \{positive\ integer\}$,

$$\lim_{k\in K} \Phi(C^2_{A,1,k}) = \lim_{k\to\infty} \Phi(C^2_{A,1,k}).$$

Hence,

$$\Phi\left(\left(C^2_{A,1}\right)^* \right) = \lim_{k\in K} \Phi(C^2_{A,1,k}) = \lim_{k\to\infty} \Phi(C^2_{A,1,k}) = \nu.$$

From the definition of ν (infimum),

$$\Phi(C^2_{A,1}) \geq \nu = \Phi\left(\left(C^2_{A,1}\right)^* \right), \forall C^2_{A,1} \in \Lambda.$$

Consequently, there exists an optimal solution $(C^2_{A,1})^*$ that minimizes $\Phi(.)$ over Λ.
(2) Uniqueness :
If we assume that equation (31) is multimodal,
Ω, Ω_1 and Ω_2 may be defined as

$$\begin{cases} \Omega = \left\{ \omega \middle| \Phi(\omega) \leq \Phi(C^2_{A,1}), \forall C^2_{A,1} \in \Lambda \right\} \\ |\Omega| \geq 2 \\ \Omega_1 = \{\omega_1\}, \Omega_2 = \Omega_1^C, |\Omega_2| \geq 1 \\ \Omega_1, \Omega_2 \subset \Omega \subset \Lambda \end{cases} \tag{32}$$

By this definition,

$$\Phi(\omega_1) \leq \Phi(C^2_{A,1}) \wedge \Phi(\omega_2) \leq \Phi(C^2_{A,1}), \\ \forall \left(C^2_{A,1} \in \Lambda, \omega_2 \in \Omega_2 \right)$$

Thus,

$$\Phi(\omega_1) = \Phi(\omega_2), \omega_1 \neq \omega_2, \forall \omega_2 \in \Omega_2 \tag{33}$$

(31) and (33) imply,

$$k_{m+2}(\omega_1) = k_{m+2}(\omega_2), \omega_1 \neq \omega_2, \forall \omega_2 \in \Omega_2 \tag{34}$$

It follows from Table 1 that for any ε_1, ε_2, equation (36) must be satisfied in order for equation (35) to have a meaning.

$$k_{i+2}(\varepsilon_1) = k_{i+2}(\varepsilon_2), \forall \varepsilon_1, \varepsilon_2 \in \Lambda \tag{35}$$

$$\begin{aligned} &\left(k_{i+1}(\varepsilon_1) = k_{i+1}(\varepsilon_2)\right) \wedge \left(k_i(\varepsilon_1) = k_i(\varepsilon_2)\right) \\ &\Leftrightarrow \varepsilon_1 = \varepsilon_2, \forall (\varepsilon_1, \varepsilon_2 \in \Lambda, i \in [1\ m]) \end{aligned} \tag{36}$$

Hence, by equation (36), equation (38) must be satisfied in order that equation (37) may be significant.

$$k_{m+2}(\omega_1) = k_{m+2}(\omega_2), \forall \omega_2 \in \Omega_2 \tag{37}$$

$$\omega_1 = \omega_2, \forall \omega_2 \in \Omega_2 \tag{38}$$

This contradicts equation (34). Therefore, the assumption that equation (31) is multimodal is rejected. This completes the proof. □

A set Λ is said to be compact if any sequence(or subsequence) contains a convergent subsequence whose limit is in Λ. In Euclidean spaces it can be shown that compact sets correspond to closed and bounded sets. Thus a compact set must contain all of its edges and cannot be extended to infinity in any direction. The points generated by most algorithms can be contained in such sets. If there exists Λ, it is required to establish an algorithm to be capable of finding a compact set Λ and an optimal solution to equation (31).

3.3 Augmented Lagrange Multiplier Approach

The constrained minimization problem such as equation (31) can be solved by solving a sequence of unconstrained problems, that in effect, by providing some penalty to limit constraint violations. Because the way in which this penalty is imposed often leads to a numerically ill-conditioned problem, a method is devised whereby only a moderate penalty is provided in the initial stages, and this penalty is increased as the optimization progresses.

This requires the solution of several unconstrained minimization problems in obtaining the optimum constrained design ; thus the term Sequential Unconstrained Minimization Techniques or SUMT to identify these methods.

The numerical ill-conditioning often encountered in the SUMT can be substantially reduced by incorporating the Lagrange multipliers into the optimization strategy. The most common method is known as the augmented Lagrange multiplier(ALM) method.

Indeed, Powell (1979) notes that the use of SUMT which do not include the Lagrange multipliers is obsolete as a practical optimization tool. The ALM method has been studied exhaustively by numerous authors. (see e.g. Imai *et al.*, 1981; Gross *et al.*, 1985; Pierre *et al.*, 1975; Rockafellar, 1973).

The first step is to convert equation (31) to an equivalent equality-constrained problem by adding slack variables to the constraints and then, these slack variables can be eliminated in the augmented Lagrange function.

Equation (31) is equivalent to

$$\begin{aligned}&Minimize.: \Phi(C_{A,1}^2) = \frac{1}{2}\cdot\left[k_{m+2}(C_{A,1}^2) - C_{DM}^2\right]^2\\&subject\ to: k_i(C_{A,1}^2) \geq 0\ ,\ i \in [1\ ,\ m+2]\end{aligned} \tag{39}$$

AL function to equation (39) is given by

$$\begin{aligned}AL(C_{A,1}^2,\ \lambda,\ r_p) =\ & \Phi(C_{A,1}^2) - \sum_{i=1}^{m+2}\left(\lambda_i \cdot Min\left[k_i(C_{A,1}^2), \frac{\lambda_i}{2\cdot r_p}\right]\right)\\&+ r_p \cdot \sum_{i=1}^{m+2}\left(Min\left[k_i(C_{A,1}^2), \frac{\lambda_i}{2\cdot r_p}\right]\right)^2\\&= L(C_{A,1}^2, \lambda, r_p) + r_p \cdot P(C_{A,1}^2, \lambda, r_p)\end{aligned} \tag{40}$$

In equation (40),

$$\lambda = (\lambda_1, \lambda_2, \ldots\ldots, \lambda_{m+2})^T$$

is a Lagrange multiplier vector. If an optimal Lagrange multiplier vector

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots\ldots, \lambda_{m+2}^*)^T$$

is known, then

$$\begin{aligned}\underset{C_{A,1}^2 \in \Lambda}{Min}.\Phi(C_{A,1}^2) &\equiv \underset{C_{A,1}^2 \in \Lambda}{Min}.L(C_{A,1}^2, \lambda^*, r_p)\\&\equiv \underset{C_{A,1}^2 \in H^1}{Min}.AL(C_{A,1}^2, \lambda^*, r_p)\end{aligned} \tag{41}$$

The foregoing observations suggest that, by updating the Lagrange multipliers so that they approach the optimal values, convergence may occur without the need for penalty parameter to be very large. Thus the ill-conditioning associated with penalty parameter can be avoided.

The update formula for the Lagrange multiplier vector may be suggested by

$$\begin{cases}\lambda_{k+1} = \lambda_k - 2\cdot r_{p,k}\cdot Min\left[K(C_{A,1,k}^2), \dfrac{\lambda_k}{2\cdot r_{p,k}}\right]\\ \lambda_0 = 0\\ K(C_{A,1,k}^2) = \left[k_1(C_{A,1,k}^2), k_2(C_{A,1,k}^2), \ldots\ldots, k_{m+2}(C_{A,1,k}^2)\right]^T\\ \Leftrightarrow\quad \lambda_{i,k+1} = \lambda_{i,k} - 2\cdot r_{p,k}\cdot Min\left[k_i(C_{A,1,k}^2), \dfrac{\lambda_{i,k}}{2\cdot r_{p,k}}\right]\end{cases} \tag{42}$$

And given constants c_1 and c_2 a penalty parameter is generated by

$$r_{p,k+1} = \begin{cases}c_1 \cdot r_{p,k}\ ,\ if\ r_{p,k} \leq \dfrac{c_2}{c_1}\\ c_2\ ,\ if\ r_{p,k} > \dfrac{c_2}{c_1}\end{cases} \tag{43}$$

And

$$c_1 > 1\ ,\ c_2 = Max\{r_{p,k}\}$$

Theorem 3. *The Lagrange multipliers are updated by equation (42) and equation (43) generates the penalty* *parameter. An algorithm of minimizing the unconstrained subproblem is given. Suppose a set Λ is compact. then there exists a convergent subsequence $\{C^2_{A,1,k}, k\in K\}$ whose limit $(C^2_{A,1})^+$ is in Λ and is an optimal solution $(C^2_{A,1})^*$. In other words, the limit point $(C^2_{A,1})^+$ is a Kuhn-Tucker point.*

Proof. First, Kronecker delta function is defined as

$$\delta_{i,k} = \begin{cases} 1 \ , \ if \ \ k_i(C^2_{A,1,k}) - \dfrac{\lambda_{i,k}}{2\cdot r_{p,k}} \le 0 \\ 0 \ , \ otherwise \end{cases}$$

And we use the following notations.

$$\lim_{k\in K} \lambda_k = \lambda^+ = \left(\lambda_1^+, \lambda_2^+, \ldots\ldots, \lambda_{m+2}^+\right)^T,$$
$$\lim_{k\in K} C^2_{A,1,k} = \left(C^2_{A,1}\right)^+, \ \lim_{k\in K} r_{p,k} = r_p^+ \ \ and$$
$$\lim_{k\in K} \delta_{i,k} = \delta_i^+$$

(1) $(C^2_{A,1})^+ \in \Lambda$

Continuity of $\Phi(.)$, $AL(.)$ *and* $L(.)$ implies

$$\begin{cases} \lim_{k\in K} \Phi(C^2_{A,1,k}) = \Phi(\left(C^2_{A,1}\right)^+) \\ \lim_{k\in K} L(C^2_{A,1,k}, \lambda_k, r_{p,k}) = L(\left(C^2_{A,1}\right)^+, \lambda^+, r_p^+) \\ \lim_{k\in K} AL(C^2_{A,1,k}, \lambda_k, r_{p,k}) = AL(\left(C^2_{A,1}\right)^+, \lambda^+, r_p^+) \end{cases}$$

And

$$\Phi(\left(C^2_{A,1}\right)^+) = L(\left(C^2_{A,1}\right)^+, \lambda^+, r_p^+) = AL(\left(C^2_{A,1}\right)^+, \lambda^+, r_p^+) \tag{44}$$

By (41) and (44), we have

$$\lim_{k\in K} r_{p,k}\cdot P(C^2_{A,1,k}, \lambda_k, r_{p,k}) \tag{45}$$
$$= \lim_{k\in K} AL(C^2_{A,1,k}, \lambda_k, r_{p,k}) - \lim_{k\in K} L(C^2_{A,1,k}, \lambda_k, r_{p,k})$$
$$= AL\left(\left(C^2_{A,1}\right)^+, \lambda^+, r_p^+\right) - L\left(\left(C^2_{A,1}\right)^+, \lambda^+, r_p^+\right) \equiv 0$$

Since $r_{p,k+1} \ge r_{p,k} > 0$, $P(C^2_{A,1,k}, \lambda_k, r_{p,k}) \ge 0, \forall k\in K$, equation (46) must be satisfied in order for equation (45) to have a meaning.

$$\lim_{k\in K} P(C^2_{A,1,k}, \lambda_k, r_{p,k}) \equiv 0 \tag{46}$$

Since $P(.)$ is continuous,

$$\lim_{k\in K} P(C^2_{A,1,k}, \lambda_k, r_{p,k}) = P((C^2_{A,1})^+, \lambda^+, r_p^+) \equiv 0$$

Thus, $(C^2_{A,1})^+ \in \Lambda$

(2) $(C^2_{A,1})^+ = (C^2_{A,1})^*$

$C^2_{A,1,k}$ is a value such that

$$AL(C^2_{A,1,k}, \lambda_k, r_{p,k}) \equiv \underset{C^2_{1,i}\in H^1}{Min.} AL(C^2_{A,1}, \lambda_k, r_{p,k}) \tag{47}$$

Thus,

$$\begin{aligned} AL(C^2_{A,1,k}, \lambda_k, r_{p,k}) &= L(C^2_{A,1,k}, \lambda_k, r_{p,k}) \\ &\quad + r_{p,k}\cdot P(C^2_{A,1,k}, \lambda_k, r_{p,k}) \\ &\le AL(\left(C^2_{A,1}\right)^*, \lambda_k, r_{p,k}) = L(\left(C^2_{A,1}\right)^*, \lambda_k, r_{p,k}) \end{aligned} \tag{48}$$

And $r_{p,k+1} \ge r_{p,k} > 0$, $P(C^2_{A,1,k}, \lambda_k, r_{p,k}) \ge 0, \forall k\in K$. Hence, it follows that

$$L(C^2_{A,1,k}, \lambda_k, r_{p,k}) \le L(\left(C^2_{A,1}\right)^*, \lambda_k, r_{p,k}) \tag{49}$$

Continuity of $L(.)$ and equation (49) imply

$$\begin{aligned} \lim_{k\in K} L(C^2_{A,1,k}, \lambda_k, r_{p,k}) &= L\left(\left(C^2_{A,1}\right)^+, \lambda^+, r_p^+\right) \\ &\le L\left(\left(C^2_{A,1}\right)^*, \lambda^+, r_p^+\right) \end{aligned} \tag{50}$$

After all, $(C^2_{A,1})^+ \in \Lambda$ means $(C^2_{A,1})^+ = (C^2_{A,1})^*$.

(3) $(C^2_{A,1})^+$ *is a Kuhn-Tucker point.*

Since $\lambda_{i,k} \ge 0$, $\lambda_i^+ \ge 0 \ \ \forall i,k$,
there is two possibilities :

ⓐ $\lambda_i^+ \ge 0 \wedge k_i\left(\left(C^2_{A,1}\right)^+\right) = 0$

ⓑ $\lambda_i^+ = 0 \wedge k_i\left(\left(C^2_{A,1}\right)^+\right) > 0$

Hence,

$$\lambda_i^+ \cdot k_i\left(\left(C^2_{A,1}\right)^+\right) = 0, \ \ k_i\left(\left(C^2_{A,1}\right)^+\right) \ge 0, \ \forall i$$

Using Kronecker delta function, it follows from (42) that

$$\begin{aligned} &\lim_{k\in K} Min.\left(k_i(C^2_{A,1,k}), \frac{\lambda_{i,k}}{2\cdot r_{p,k}}\right) \\ &\equiv \lim_{k\in K}\left(\delta_{i,k}\cdot k_i(C^2_{A,1,k}) + (1-\delta_{i,k})\cdot\frac{\lambda_{i,k}}{2\cdot r_{p,k}}\right) \\ &\equiv Min.\left(k_i\left(\left(C^2_{A,1}\right)^+\right), \frac{\lambda_i^+}{2\cdot r_p^+}\right) \\ &\equiv \delta_i^+\cdot k_i\left(\left(C^2_{A,1}\right)^+\right) + (1-\delta_i^+)\cdot\frac{\lambda_i^+}{2\cdot r_p^+} \end{aligned} \tag{51}$$

and

$$\lim_{k\in K}\left(\lambda_{i,k+1}=\lambda_{i,k}-2\cdot r_{p,k}\cdot Min.\left[k_i(C^2_{A,1,k}),\frac{\lambda_{i,k}}{2\cdot r_{p,k}}\right]\right)$$
$$\equiv\left(\lambda_i^+=\delta_i^+\cdot\left[\lambda_i^+-2\cdot r_p^+\cdot k_i\left(\left(C^2_{A,1}\right)^+\right)\right]\right) \tag{52}$$
$$\equiv\left(\delta_i^+=\frac{\lambda_i^+}{\lambda_i^+-2\cdot r_p^+\cdot k_i\left(\left(C^2_{A,1}\right)^+\right)}\right),\ k_i\left(\left(C^2_{A,1}\right)^+\right)\neq\frac{\lambda_i^+}{2\cdot r_p^+}$$

Equations (51) and (40) imply that

$$\begin{aligned}&\nabla AL\left(\left(C^2_{A,1}\right)^+,\lambda^+,r_p^+\right)=\nabla\Phi\left(\left(C^2_{A,1}\right)^+\right)\\&-\sum_{i=1}^{m+2}\left[\lambda_i^+\cdot\delta_i^+\cdot\nabla k_i\left(\left(C^2_{A,1}\right)^+\right)\right]\\&+2\cdot r_p^+\cdot\sum_{i=1}^{m+2}\left[\begin{aligned}&\delta_i^+\cdot k_i\left(\left(C^2_{A,1}\right)^+\right)+\\&(1-\delta_i^+)\cdot\delta_i^+\cdot\frac{\lambda_i^+}{2\cdot r_p^+}\cdot\nabla k_i\left(\left(C^2_{A,1}\right)^+\right)\end{aligned}\right]\equiv 0\end{aligned} \tag{53}$$

By (52) and (53),

$$\begin{aligned}&\nabla AL\left(\left(C^2_{A,1}\right)^+,\lambda^+,r_p^+\right)=\nabla\Phi\left(\left(C^2_{A,1}\right)^+\right)\\&-\sum_{i=1}^{m+2}\left\{\left[\lambda_i^+-2\cdot r_p^+\cdot k_i\left(\left(C^2_{A,1}\right)^+\right)\right]\cdot\delta_i^+\cdot\nabla k_i\left(\left(C^2_{A,1}\right)^+\right)\right\}\\&=\nabla\Phi\left(\left(C^2_{A,1}\right)^+\right)-\sum_{i=1}^{m+2}\left[\lambda_i^+\cdot\nabla k_i\left(\left(C^2_{A,1}\right)^+\right)\right]\equiv 0\end{aligned} \tag{54}$$

Since $\forall i$,

$$\lambda_i^+\geq 0,\ \lambda_i^+\cdot k_i\left(\left(C^2_{A,1}\right)^+\right)=0\ \text{and}\ k_i\left(\left(C^2_{A,1}\right)^+\right)\geq 0,$$

equation (54) implies

$$\forall i,$$
$$\nabla\Phi\left(\left(C^2_{A,1}\right)^+\right)-\sum_{i=1}^{m+2}\left[\lambda_i^+\cdot\nabla k_i\left(\left(C^2_{A,1}\right)^+\right)\right]\equiv 0,$$

$$\lambda_i^+\geq 0,\ \lambda_i^+\cdot k_i\left(\left(C^2_{A,1}\right)^+\right)=0\ \text{and}\ k_i\left(\left(C^2_{A,1}\right)^+\right)\geq 0.$$

Thus, these are Kuhn-Tucker conditions for (31) or (39). Consequently, $(C^2_{A,1})^+$ is a Kuhn-Tucker point. This completes the proof. □

Through this proof, we have

$$\lim_{k\in K}\lambda_k=\lambda^+=\lambda^*,\ \lim_{k\in K}C^2_{A,1,k}=\left(C^2_{A,1}\right)^+=\left(C^2_{A,1}\right)^*$$

Convergence condition of subsequence $\{C^2_{A,1,k},\ k\in K\}$ is

$$\lim_{k\in K}AL(C^2_{A,1,k},\lambda_k,r_{p,k})=AL\left(\left(C^2_{A,1}\right)^+,\lambda^+,r_p^+\right)\equiv 0$$

$\{C^2_{A,1,k},\ k\in K\}$ is determined by the process to solve the unconstrained minimization subproblem. Equation (40) has a discontinuous second derivative at $k_i(C^2_{A,1})=\lambda_i/2r_p$. Thus second order techniques should be used with caution for solving the unconstrained minimization subproblem. In this paper, we will use only the augmented Lagrange function value and its gradient in order to solve subproblems. The gradient may be computed by the numeric method that is said to be the second order midpoint difference formula given by

$$\nabla AL(C^2_{A,1},\lambda,r_p)\cong\frac{AL(C^2_{A,1}+\Delta,\lambda,r_p)-AL(C^2_{A,1}-\Delta,\lambda,r_p)}{2\Delta},\Delta>0$$

3.4 Algorithm to Minimize the Unconstrained Subproblem

Let $C^2_{A,1,k,n}$ be the n_{th} of $C^2_{A,1,k}$ value in the process of minimizing equation (55).

$$C^2_{A,1,k,n}:AL(C^2_{A,1,k},\lambda_k,r_{p,k})\equiv\underset{C^2_{A,1}\in H^1}{Min}\ AL(C^2_{A,1},\lambda_k,r_{p,k}),\ k,n\geq 0 \tag{55}$$

In the subsequence $\{C^2_{A,1,k,n},\ n\in N_K\}$, suppose that the point $C^2_{A,1,k,n+1}$ lies in a given direction $d_{k,n}$ from the point $C^2_{A,1,k,n}$. Then we get

$$\nabla AL(C^2_{A,1,k,n},\lambda_k,r_{p,k})\cdot d_{k,n-1}\cong 0,\ n\geq 1,\ \forall k$$

If we define

$$\tau_n\equiv\frac{AL(C^2_{A,1,k,n+1},\lambda_k,r_{p,k})}{AL(C^2_{A,1,k,n},\lambda_k,r_{p,k})},\ n\geq 0,\ \forall k$$

then τ_n means the ratio of the augmented Lagrange function value in the process of solving the k_{th} unconstrained minimization subproblem.

If compact set Λ exists, the augmented Lagrange function and its first derivative can be expanded into a Taylor series by the (Theorem 2).

Theorem 4. *For some constants* $0<r_a,\ c_3<1$, τ_n *is given by equation (56). Then the algorithm to generate the subsequence* $\{C^2_{A,1,k,n},\ n\in N_K\}$ *by the equation (57) has* *the order of convergence 1.*

$$\tau_n = r_a^{n+1} + c_3 \ , n \ge 0 \ , \forall \ k \tag{56}$$

$$C^2_{A,1,k,n+1} = C^2_{A,1,k,n} - \frac{2(1-\tau_n)AL(C^2_{A,1,k,n}, \lambda_k, r_{p,k})}{\nabla AL(C^2_{A,1,k,n}, \lambda_k, r_{p,k})} \tag{57}$$

Proof. It is obvious that

$$AL(C^2_{A,1,k,n+1}, \lambda_k, r_{p,k}) - AL(C^2_{A,1,k,n}, \lambda_k, r_{p,k}) \cong \frac{\nabla AL(C^2_{A,1,k,n}, \lambda_k, r_{p,k}) \cdot d_{k,n} \cdot s_{k,n}}{2} \tag{58}$$

And we set

$$s_{k,n} \cong \frac{2 \cdot \left[AL(C^2_{A,1,k,n+1}, \lambda_k, r_{p,k}) - AL(C^2_{A,1,k,n}, \lambda_k, r_{p,k})\right]}{\nabla AL(C^2_{A,1,k,n}, \lambda_k, r_{p,k}) \cdot d_{k,n}} \tag{59}$$

By (58) and (59), we obtain

$$C^2_{A,1,k,n+1} = C^2_{A,1,k,n} + \frac{2\left[AL(C^2_{A,1,k,n+1}, \lambda_k, r_{p,k}) - AL(C^2_{A,1,k,n}, \lambda_k, r_{p,k})\right]}{\nabla AL(C^2_{A,1,k,n}, \lambda_k, r_{p,k})}$$

From (56), $\lim_{n \in N_k} \tau_n = c_3$

For sufficiently large number, AL function value is decreased geometrically. That is,

$$AL(C^2_{A,1,k,n}, \lambda_k, r_{p,k}) = A \cdot c_3^n \ , A > 0 \ , \forall \ k$$

Order of convergence is described as

$$\lim_{n \in N_k} \frac{\log AL(C_{A,1,k,n+1}, \lambda_k, r_{p,k})}{\log AL(C_{A,1,k,n}, \lambda_k, r_{p,k})} = \lim_{n \in N_k} \frac{\log A + (n+1)\log c_3}{\log A + n \log c_3} \equiv 1$$

This completes the proof. □

In the subsequence $\{C^2_{A,1,k,n}, n \in N_K\}$, necessary and sufficient condition for convergence is

$$\lim_{n \in N_k} AL(C_{A,1,k,n}, \lambda_k, r_{p,k}) = AL(C_{A,1,k}, \lambda_k, r_{p,k}) \equiv 0$$

The remainder of work is two : one is to confirm the existence of a compact set and the other is to estimate an initial value $C^2_{A,1,0,0}$. The necessary and sufficient condition for the existence of a compact set Λ is that there exists $0 \le l < u < \infty$ such that

$$\nabla AL(l, \lambda_0, r_{p,0}) \le 0 \ , \nabla AL(u, \lambda_0, r_{p,0}) \ge 0 \tag{60}$$

Finally, the initial value $C^2_{A,1,0,0}$ can be estimated with the method of Cubic Interpolation that has the order of convergence 2.

4. NUMERIC EXAMPLES

The nonlinear simultaneous equations proposed as Table 1 may be converted into a constrained optimization problem. Also it can be proved that this constrained optimization problem is unimodal, which means that there exists a unique optimal solution. In addition, we can model this problem with the augmented Lagrange multipliers and prove that the algorithm used to solve this problem has the convergence order of 1. All proofs associated with these issues has been already presented in section 3 in detail. In this section, we consider five toy problems as examples in order to show the validity and applicability of the proposed theories and algorithm. And codes implementing a proposed algorithm are compiled with the **Borland C++** (***Version 3.0 or 3.1***).

To begin with, we set

$$\lambda_d = 1 \ , V(IA) = V(S_i) = 0.2^2$$
$$T_i = (i) \ mod \ 2 + 1 \ , big \ M = 20 \ , m = 5.$$

and consider

$$[P1] \ \mu = (1.9, 1.9, 1.5, 1.5, 1.1)$$
$$[P2] \ \mu = (1.1, 1.5, 1.5, 1.9, 1.9)$$
$$[P3] \ \mu = (1.9, 1.5, 1.1, 1.5, 1.9)$$
$$[P4] \ \mu = (1.5, 1.9, 1.9, 1.5, 1.1)$$
$$[P5] \ \mu = (1.1, 1.5, 1.9, 1.9, 1.5)$$

The results to be reported on the experiments performed with our study can be divided into the following two groups:

(1) computing the squared coefficients of variation of inter-departure process under the lower bound of $V(IZ_i)$.
(2) if (1) is successfully performed, quantifying some issues associated with distribution including $(C^2_{A,i})^*$ under the true value of $V(IZ_i)$.

4.1 $(C^2_{A,i})^*, i \in [1, m+2]$ under $V(IZ_i)_{LOWER}$

Results of experiments are given by Table 2.

Table 2. $(C_{A,i}^2)^*$ under lower bound

$(C_{A,i}^2)^*$	[P1]	[P2]	[P3]	[P4]	[P5]
1	0.0310	0.0349	0.0336	0.0310	0.0348
2	0.0310	0.0349	0.0336	0.0310	0.0348
3	0.0310	0.0349	0.0336	0.0310	0.0348
4	0.0323	0.0381	0.0365	0.0323	0.0380
5	0.0323	0.0381	0.0365	0.0323	0.0380
6	0.0350	0.0389	0.0399	0.0350	0.0397
7	0.0397	0.0399	0.0399	0.0397	0.0400

Table 2 shows that there exists an optimal solution for (22), (23) and (24) in all problems. Thus, further experiments are required.

4.2 $(C_{A,i}^2)^*, i \in [1, m+2]$ under $V(IZ_i)_{TRUE}$

Results of experiments are given by Table 3.

Table 3. $(C_{B,i})^*$

$(C_{B,i})^*$	[P1]	[P2]	[P3]	[P4]	[P5]
1	1.0016	1.0161	1.0017	1.0034	1.0161
2	1.0016	1.0035	1.0035	1.0016	1.0035
3	1.0034	1.0035	1.0162	1.0016	1.0016

$(C_{B,i})^*$ has been defined in equation (27).

Table 4. $(f_i)^*$

$(f_i)^*$	[P1]	[P2]	[P3]	[P4]	[P5]
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	0.9763	0.9514	0.9511	0.9763	0.9514
3	1.0000	1.0000	1.0000	1.0000	1.0000
4	0.9519	0.9764	0.9516	0.9519	0.9764
5	0.9243	0.9988	0.9988	0.9243	0.9960

$(f_i)^*$ implies the probability there is no back order in the steady state. In particular, $(f_5)^*$ in Table 4 is equal to the probability there occurs no backlogged demands.

Table 5. $(C_{A,i}^2)^*$ under true value

$(C_{A,i}^2)^*$	[P1]	[P2]	[P3]	[P4]	[P5]
1	0.0409	0.0415	0.0421	0.0409	0.0415
2	0.0409	0.0415	0.0421	0.0409	0.0415
3	0.0409	0.0415	0.0421	0.0409	0.0415
4	0.0405	0.0405	0.0410	0.0405	0.0404
5	0.0405	0.0405	0.0410	0.0405	0.0404
6	0.0394	0.0400	0.0400	0.0394	0.0400
7	0.0400	0.0400	0.0400	0.0400	0.0400

Table 6. Time in system

	[P1]	[P2]	[P3]	[P4]	[P5]
Mean	8.0000	8.0000	8.0000	8.0000	8.0000
variance	2.0558	1.9885	1.9360	2.0558	1.9940

Note that a sequence [P3] is better than any other sequences in terms of the variance of time in system.

Table 7. Interval in which compact set exists: Necessary and sufficient conditions

Interval	[P1]	[P2]	[P3]	[P4]	[P5]
l	0.0390	0.0391	0.0394	0.0390	0.0391
u	0.0450	0.0450	0.0450	0.0450	0.0450

l and u have been appeared in equation (60).

In Table 8, note that the value of $P(N_i = 1), \forall i$ is higher than any other value of n, which implies that the desirable number of cards in each cell is almost 1.

Table 8. Distribution of $(N_i)^*$ in case of [P3]

n	cell (1)	cell (2)	cell (3)	cell (4)	cell (5)
0	0.4737	0.3333	0.0909	0.3333	0.4737
1	0.5023	0.6177	0.7033	0.6183	0.5015
2	0.0240	0.0489	0.2058	0.0484	0.0237
3	0.0000	0.0000	0.0000	0.0000	0.0011
4					0.0001
5					0.0000
…					…
21					0.0000
22					0.0000

5. CONCLUDING REMARKS

Our model is described in terms of a 2-Card configuration scheme. One aspect of the Pull serial line is that of a system for extracting and converting information. The other aspect is the item handling. It has shown that the performances of the Pull serial line are able to be compared to those of the Push serial line with an associated buffer of finite capacity.

This viewpoint makes this author be interested in the inter-departure process of the pull serial line. We have proposed a numeric model and algorithm for the purpose of computing the first two moments of the inter-departure process subject to given service rate, demand rate and number of cards in each cell. And via some experiments, we have confirmed the validity and applicability of the proposed model and algorithm.

Through these works, some structural properties have been proved under the assumptions of an infinite supply of raw materials and a backlogged demands :

(1) The assumption of an infinite supply of raw material results in the same throughput in each cell, which indicates that material flow in the pull serial line must be conserved.

In addition, variance of inter-departure in each cell is considerably reduced compared to a finite supply of raw material in any case of either lower or true.

(2) Besides, if backlogged demands are permitted, then these assumptions are equivalent to the statement that the pull serial line is stable, that is, traffic intensities of each cell must be smaller than 1.

Also, the throughput in each cell is identical to the external demand rate. But there is no change in the variance of inter-departure process in each cell. It follows from this finding that management of raw material pool is more important than that of external demand pool in a pull serial line.

(3) Necessary and sufficient conditions for the existence of optimal solution which enables us to get insight to the interaction between cells, and algorithm implementing optimal solution have been theoretically provided.

In particular, lower bound of variance of inter-departure process has been derived. We can use this lower bound in any approximate distribution associated with performances under a general service scheme and demand scheme.

Now we have been ready to give answers to the two previous promising studies : Tayu r(1993) and Mitra *et al.* (1990).

To begin with, we consider the study of Tayur(1993). Corollary 1 and corollary 2 described in Tayur(1993) are all accepted but his remark may be rejected since throughput in each cell is exact to demand rate under a backlogged demand model. (see Theorem 1)

The conjecture mentioned in Mitra *et al.*(1990) that the optimal solution for arrival rates to each cell is unique has been proved through our works(see Theorem 2 and Theorem 3).

Finally, further researches with respect to this study are briefly listed as follows :

(1) It may be proved that there exists the VPP(variability propagation principle) in any pull serial line under general environment (Suresh *et al.* (1990)).
(2) The Bull-whip effect in SCM (supply chain management) may be quantified.
(3) Any steady state performances including distributions may be easily computed due to their relation to the first two moments of the inter-departure process.
(4) Optimal arrangement of cells(stages) in the pull serial lime and
(5) An equivalent push type serial line may be obtained in view of the inter-departure process.

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