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# Model Predictive Control for Productions Systems Based on Max-plus Algebra

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**Abstract.** Among the state-space description of discrete event systems, the max-plus algebra is known as one of the effective approach. This paper proposes a model predictive control (MPC) design method based on the max-plus algebra. Several studies related to these topics have been done so far under the constraints that system parameters are constant. However, in practical systems such as production systems, it is common and sometimes inevitable that system parameters vary by each event. Therefore, it is of worth to design a new MPC controller taking account of adjustable system parameters. In this paper, we formulate system parameters as adjustable ones, and they are solved by a linear programming method. Since MPC determines optimal control input considering future reference signals, the controller can be more robust and the operation cost can be reduced. Finally, the proposed method is applied to a production system with three machines, and the effectiveness of the proposed method is verified through a numerical simulation.

**Keywords:** model predictive control, production system, max-plus algebra, linear programming

## 1. INTRODUCTION

The max-plus linear (MPL) system is one group of the studying method for system representation and controller design of discrete event systems (Cohen *et al.*, 1989; Baccelli *et al.*, 1992; Boimond and Ferrier, 1996; Schutter and Boom, 2001; Boom and Schutter, 2001; Suzuki and Masuda, 1998).

Max-plus algebra is an algebra in which max operation is addition and plus operation is multiplication. It has some same operation rules as satisfies in the usual algebra. Using an independent variable called an event counter in linear state-space representation model, transitions of events in the linear state-space are described. Concerning with MPL systems, Internal Model Control (IMC) (Boimond and Ferrier, 1996; Suzuki and Masuda, 1998) and Model Predictive Control are studied in recent years.

In the past papers related to these topics, studies are done under the constraints that system parameters are constants. However, in practical systems, to change or adjust system parameters is common and sometimes

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inevitable. For example, production system of assembly line or batch processing line produces multiple kinds of products on the same equipments, and the processing times differ by each kind. This means that the system parameters (processing times) are dependent upon the event counter (production No.). Furthermore, there is a case to adjust the number of working members according to the volume of order. In such case, the processing times differ by the number of workers. This means that the system parameters are adjustable.

Therefore, the approach proposed by Masuda *et al.* (2003) is useful for expanding the application field of MPL systems. However, the inverse system of Masuda *et al.* (2003) has cases that the system parameters vary abruptly. This is because the control inputs are determined under restrictions that the corresponding outputs should be equal or less than to the desired reference signal. Moreover, the system parameters are considered to be a kind of constant and not explicitly to be adjustable.

In this paper, we propose a MPC design method in MPL systems with adjustable parameters by expanding the method of (Masuda *et al.*, 2003). Using MPC, moderate changes of the system parameters can be accomplished by foreseeing the former event, and the outputs are within the desired reference signal. Moreover, a control method can be obtained at the lower cost. As the systems parameters are defined as to depend upon the event counter, the adjustment of system parameters can be done freely.

Generally, the design approach of MPC consists of the following steps.

1) Introducing output prediction equation using a model of controlled system
2) Determining optimal control law based on prediction equation
3) Applying Receding Horizon method

As a whole, the same procedures described above can be adopted in MPL systems. However, as the optimal control law 2) cannot be formulated like the conventional(+, ×) operation, control methods peculiar to the max-plus algebra should be constructed. Schutter and Boom (2001), Boom and Schutter (2001) propose some ideas by utilizing the solution of Extended Linear Complementary Problem (ELCP), but it is difficult to extend them to the subjects of adjustable parameters we are going to handle. Hence, in this paper, we utilize the calculation method of the greatest subsolution derived in (Masuda *et al.*, 2003) after we show that the prediction equation can be expressed as linear summation function of adjustable parameters.

The outline of this paper is as follows:

Section 2 gives the mathematical preliminaries. Section 3 represents a MPL system that depends on the event counter. Introducing a modeling example of production system, we show that it is suitable to assume that systems parameters are dependent upon the event counter. Section 4 introduces a MPC control law for MPL system, which depends on the event counter. We show that the prediction equation can be expressed as a linear summation function of the adjustable parameters, and then we utilize the calculation method of the greatest subsolution derived in (Masuda *et al.*, 2003). Section 5 proposes an adjusting method of the systems parameters after verifying that the constraints on system matrices are reduced to a linear programming problem. Section 6 shows a result of numerical simulation. Finally, section 7 gives concluding remarks.

## 2. MATHEMATICAL PRELIMINARIES

The basic operations of max-plus algebra are denoted by $\oplus$ and $\otimes$ for addition and multiplication, which are defined as follows.

$$x \oplus y = \max(x, y), \quad x \otimes y = x + y \tag{1}$$

where $\boldsymbol{R}_\varepsilon = \boldsymbol{R} \cup \{-\infty\}$, and $\boldsymbol{R}$ stands for the real field. Let $\varepsilon$ be defined as $-\infty$, which is a neutral element of the addition $\oplus$, and let $e$ defined as 0, which is a neutral element of the multiplication $\otimes$.

The following two operators are also defined.

$$x \wedge y = \min(x, y), \quad x \backslash y = -x + y$$

Operations to multiple numbers are defined as follows. When $m \leq n$, then

$$\bigoplus_{k=m}^{n} a_k = a_m \oplus a_{m+1} \oplus \cdots \oplus a_n = \max(a_m, a_{m+1}, \cdots, a_n) \tag{2}$$

$$\bigotimes_{k=m}^{n} a_k = a_m \otimes a_{m+1} \otimes \cdots \otimes a_n = a_m + a_{m+1} + \cdots + a_n \tag{3}$$

$$\bigwedge_{k=m}^{n} a_k = a_m \wedge a_{m+1} \wedge \cdots \wedge a_n = \min(a_m, a_{m+1}, \cdots, a_n) \tag{4}$$

The above operations are extended to matrix calculations whose elements belong to $\boldsymbol{R}_\varepsilon$.

For instance, in $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{R}_\varepsilon^{m \times n}$,

$$[\boldsymbol{A} \oplus \boldsymbol{B}]_{ij} = [\boldsymbol{A}]_{ij} \oplus [\boldsymbol{B}]_{ij} = \max([\boldsymbol{A}]_{ij}, [\boldsymbol{B}]_{ij}) \tag{5}$$

$$[\boldsymbol{A} \wedge \boldsymbol{B}]_{ij} = [\boldsymbol{A}]_{ij} \wedge [\boldsymbol{B}]_{ij} = \min([\boldsymbol{A}]_{ij}, [\boldsymbol{B}]_{ij}) \tag{6}$$

$$1 \leq i \leq m, \; 1 \leq j \leq n$$

where $[\cdot]_{ij}$ stands for the element of the $i$-th row, $j$-th column of the matrix. $[\cdot]_{<IJ>(mn)}$ stands for the element of the $I$-th row, $J$-th column block of the block matrix and its size is $m \times n$. For simplicity, $(mn)$ might be suppressed.

When, $\boldsymbol{A} \in \boldsymbol{R}_\varepsilon^{m \times l}$, $\boldsymbol{B} \in \boldsymbol{R}_\varepsilon^{l \times p}$, then

$$[\boldsymbol{A} \otimes \boldsymbol{B}]_{ij} = \bigoplus_{k=1}^{l}([\boldsymbol{A}]_{ik} \otimes [\boldsymbol{B}]_{kj}) = \max_{k=1,\ l}([\boldsymbol{A}]_{ik} + [\boldsymbol{B}]_{kj}) \tag{7}$$

$$[A \ominus B]_{ij} = \bigwedge_{k=1}^{l}\left([A]_{ij} \setminus [B]_{kj}\right) = \min_{k=1,\ldots,l}\left(-[A]_{ij} + [B]_{kj}\right) \quad (8)$$
$$1 \le i \le m,\ 1 \le j \le p$$

Eqs. (2)-(4) are analogically extended to the operations of matrices using Eqs. (5) and (7). Neutral elements of addition and multiplication of matrices are represented as follows.

$\varepsilon_{mn}$ : All elements are in $\varepsilon_{mn} \in R_\varepsilon^{m\times n}$

$e_m$ : Only diagonal elements are $e$ and all other elements are $\varepsilon$ in $e_m \in R_\varepsilon^{m\times m}$

If $d \in R_\varepsilon$, $A \in R_\varepsilon^{m\times n}$, then

$$[d \otimes A]_{ij} = d \otimes [A]_{ij} \quad (9)$$

If $a, b \in R_\varepsilon^{\,n}$, $a \le b$ implies $[a]_i \le [b]_i$ for all $(1 \le i \le n)$

## 3. PRODUCTION SYSTEMS BASED ON MAX-PLUS ALGEBRA

The state-space representation for discrete event systems based on max-plus algebra, which is similar to the traditional one for linear continuous-time systems, is described in the following way (Baccelli *et al.*, 1992):

$$x(k+1) = A\,x(k) \oplus B\,u(k+1) \quad (10)$$

$$y(k) = C\,x(k) \quad (11)$$

where $k$ is the event counter, which means the number of event occurrence from initial state. $x(k) \in R_\varepsilon^{\,n}$, $u(k) \in R_\varepsilon^{\,p}$ and $y(k) \in R_\varepsilon^{\,q}$ are state variables, control inputs and con trolled outputs, respectively. $A \in R_\varepsilon^{n\times n}$, $B \in R_\varepsilon^{n\times p}$ and $C \in R_\varepsilon^{q\times n}$ are matrices which depend upon the structure of systems.

Past researches handled systems parameters as constants. In practical systems, as there are cases that the processing times vary by each event, it would be better to treat systems parameters as variables dependent upon the event counter.

In Masuda *et al.* (2003), application scope is expanded to the case that the systems parameters are adjustable, but it cannot be applied to the case that they change by each event. Hence, this paper regards them as variables dependent upon the event counter $k$, and let system matrices be denoted by $A_k$, $B_k$ and $C_k$. Then, Eqs. (10) and (11) will be expressed as

$$x(k+1) = A_{\underline{k}}\,x(k) \oplus B_{\underline{k}}\,u(k+1) \quad (12)$$

$$y(k) = C_{\underline{k}}\,x(k) \quad (13)$$

As an example, let us consider a two-input one-output production system depicted in Figure 1. This production system processes two pieces of parts, produces one processed part, and then sends to the next production line.

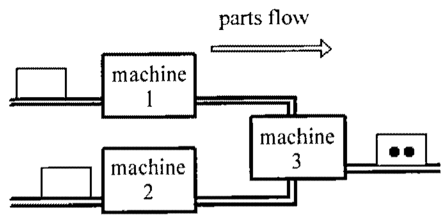


**Figure 1.** Two-inputs and one-output production system

Machine 1, machine 2 process parts in $d_1$, $d_2$ time for each, and send them to machine 3. Machine 3 processes parts in $d_3$ time and send it to the next production line.

Set,

$u_1(k)$ : time for $k$-th parts to put to machine 1
$u_2(k)$ : time for $k$-th parts to put to machine 2
$y(k)$ : time for machine 3 to output processed parts
$x_1(k)$ : time to start processing $k$-th parts in machine 1
$x_2(k)$ : time to start processing $k$-th parts in machine 2
$x_3(k)$ : time to start processing $k$-th parts in machine 3

$u_1(k)$ and $u_2(k)$ are inputs to the system, $y(k)$ is output of the system, $x_1(k)$, $x_2(k)$ and $x_3(k)$ are internal state of the system.

There are the following constraints on these variables.

- Machine 1 and 2 can not start processing until former parts complete processing and next parts being inputted
- Machine 3 can not start processing until former parts complete processing and receive next parts from machine 1 and 2

which are described as

$$x_1(k+1) = \max\{x_1(k)+d_1(k),\ u_1(k+1)\} \quad (14)$$

$$x_2(k+1) = \max\{x_2(k)+d_2(k),\ u_2(k+1)\} \quad (15)$$

$$x_3(k+1) = \max\{y(k),\ x_1(k)+d_1(k),\ x_2(k)+d_2(k)\} \quad (16)$$

$$y(k) = x_3(k)+d_3(k) \quad (17)$$

As these equations are expressed by max operation and addition operation, they are represented by the max-plus algebra. Set,

$$A_{\underline{k}} = \begin{bmatrix} d_1(k) & \varepsilon & \varepsilon \\ \varepsilon & d_2(k) & \varepsilon \\ d_1(k)d_1(k+1) & d_2(k)d_2(k+1) & d_3(k) \end{bmatrix} \quad (18)$$

$$B_{\underline{k}} = \begin{bmatrix} e & \varepsilon \\ \varepsilon & e \\ d_1(k+1) & d_2(k+1) \end{bmatrix} \quad (19)$$

$$C_{\underline{k}} = [\varepsilon \quad \varepsilon \quad d_3(k)] \quad (20)$$

$$x(k) = [x_1(k) \quad x_2(k) \quad x_3(k)]^T \tag{21}$$

$$y(k) = [y(k)]^T \tag{22}$$

$$u(k) = [u_1(k) \quad u_2(k)]^T \tag{23}$$

Eqs. (14)-(17) can be described in the forms of Eqs. (12) and (13). It is easily verified that $A_{\underline{k}}$, $B_{\underline{k}}$ and $C_{\underline{k}}$ include the systems parameters and are dependent upon the event counter.

## 4. MPC FOR PRODUCTION SYSTEMS

This section introduces a MPC framework for production systems based on max-plus algebra. First, a prediction equation is derived, and then a method for determining the optimal control inputs is developed.

### 4.1 Derivation of output prediction equation

The prediction equations on $k+1, \cdots, k+N$ step can be obtained by using Eq. (12) iteratively.

$$\begin{cases} x(k+1) &= A_{\underline{k}}\, x(k) \oplus B_{\underline{k}}\, u(k+1) \\ x(k+2) &= A_{\underline{k+1}} A_{\underline{k}}\, x(k) \\ & \quad \oplus A_{\underline{k+1}} B_{\underline{k}}\, u(k+1) \oplus B_{\underline{k+1}}\, u(k+2) \\ \vdots & \\ x(k+N) &= A_{\underline{k+N-1}} \cdots A_{\underline{k}}\, x(k) \\ & \quad \oplus A_{\underline{k+N-1}} \cdots B_{\underline{k}}\, u(k+1) \oplus \cdots \\ & \quad \cdots \oplus B_{\underline{k+N-1}}\, u(k+N) \end{cases} \tag{24}$$

Multiplying $C_{\underline{k+1}}, \cdots, C_{\underline{k+N}}$ on both sides of Eq. (24), and utilizing Eq. (13), Eq. (24) can be expressed as

$$Y(k+1) = \Gamma_{\underline{k}}\, x(k) \oplus \Delta_{\underline{k}}\, U(k+1) \tag{25}$$

where

$$Y(k+1) = \begin{bmatrix} y(k+1) \\ y(k+2) \\ \vdots \\ y(k+N) \end{bmatrix}, \quad U(k+1) = \begin{bmatrix} u(k+1) \\ u(k+2) \\ \vdots \\ u(k+N) \end{bmatrix} \tag{26}$$

$$\Gamma_{\underline{k}} = \begin{bmatrix} C_{\underline{k+1}} A_{\underline{k}} \\ C_{\underline{k+2}} A_{\underline{k+1}} A_{\underline{k}} \\ \vdots \\ C_{\underline{k+N}} A_{\underline{k+N-1}} \cdots A_{\underline{k}} \end{bmatrix} \tag{27}$$

$$\Delta_{\underline{k}} = \begin{bmatrix} C_{\underline{k+1}} B_{\underline{k}} & \cdots & \varepsilon_{qp} \\ C_{\underline{k+2}} A_{\underline{k+1}} B_{\underline{k}} & & \mathrm{M} \\ \vdots & \ddots & \varepsilon_{qp} \\ C_{\underline{k+N}} A_{\underline{k+N-1}} \Lambda\, A_{\underline{k+1}} B_{\underline{k}} & \cdots & C_{\underline{k+N}} B_{\underline{k+N-1}} \end{bmatrix} \tag{28}$$

When the desired reference signals are set as

$$R(k+1) = \begin{bmatrix} r(k+1) \\ r(k+2) \\ \vdots \\ r(k+N) \end{bmatrix} \quad r(k+i) = \begin{bmatrix} r_1(k+i) \\ r_2(k+i) \\ \vdots \\ r_q(k+i) \end{bmatrix} \tag{29}$$

Optimal control inputs can be obtained by solving $U(k+1)$ in the following equation.

$$R(k+1) = \Gamma_{\underline{k}}\, x(k) \oplus \Delta_{\underline{k}}\, U(k+1) \tag{30}$$

This paper gives a solution to this problem by utilizing the calculation method of the greatest subsolution derived in (Cohen *et al.*, 1989).

For preparation, the following section shows that the system matrices can be represented as linear summation function of the system parameters.

### 4.2 Linear summation representation of system matrix

Although the system matrices Eqs. (18)-(20) depend upon the event counter $k$, which means the parts number, the positions of $\varepsilon$ and $d_i(k)$ do not depend upon $k$. Therefore, Eqs. (18)-(20) can be represented as linear summation function of $d_i(k)$ dependent upon $k$ and matrices inherent in the system independent of $k$.

They are represented as

$$A_{\underline{k}} = \bigoplus_{l=1}^{L_A} f_{A_l}(d_k) A_l, \quad B_{\underline{k}} = \bigoplus_{l=1}^{L_B} f_{B_l}(d_k) B_l$$
$$C_{\underline{k}} = \bigoplus_{l=1}^{L_C} f_{C_l}(d_k) C_l \tag{31}$$

where $d_k$ denotes a vector whose elements consist of $e$ and the systems parameters included in $A_{\underline{k}}$, $B_{\underline{k}}$ and $C_{\underline{k}}$. $f_{A_l}(d_k)$, $f_{B_l}(d_k)$ and $f_{C_l}(d_k)$ represent linear functions of elements of $d_k$. $A_l$, $B_l$ and $C_l$ have only $\varepsilon$ and $e$ in their elements, and their size are equal to $A$, $B$ and $C$, respectively.

For example, $B_{\underline{k}}$ in Eq. (19) can be expressed as

$$B_{\underline{k}} = e \begin{bmatrix} e & \varepsilon \\ \varepsilon & e \\ \varepsilon & \varepsilon \end{bmatrix} \oplus d_1(k+1) \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ e & \varepsilon \end{bmatrix} \oplus d_2(k+1) \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & e \end{bmatrix} \tag{32}$$

where

$$L_B = 3$$
$$d_k = [e, d_1(k), d_2(k), d_3(k), d_1(k+1), d_2(k+1)]^T$$
$$f_{B_1}(d_k) = e,\ f_{B_2}(d_k) = d_1(k+1),\ f_{B_3}(d_k) = d_2(k+1)$$

Let the next Theorem be derived in order to verify that $\Gamma_{\underline{k}}$ and $\Delta_{\underline{k}}$ in Eqs. (27) and (28) can be expressed by the linear summation function of the parameter $d_i(k)$.

**Theorem 1.** *When $X \in R_\varepsilon^{m\times h}$ and $Y \in R_\varepsilon^{h\times n}$ can be expressed as linear summation function of the parameter $d_i(k)$, $XY$ can also be expressed as linear summation function of the parameter $d_i(k)$.*

**Proof.** $X$ and $Y$ are represented as

$$X = \bigoplus_{i=1}^{L_X} f_{X_i}(d_{k_X})X_i \quad Y = \bigoplus_{j=1}^{L_Y} f_{Y_j}(d_{k_Y})Y_j$$

Here, $X_i$, $Y_i$ is the same size matrix as $X$, $Y$, and $d_{k_X}$, $d_{k_Y}$ is the vector with the parameter element included in $X$ and $Y$ for each. $f_{X_i}(d_{k_X})$ and $f_{Y_i}(d_{k_Y})$ are the linear functions which have $d_{k_X}$ and $d_{k_Y}$ in their elements respectively.

As distributive law holds in the max-plus algebra, $XY$ can be represented as

$$\begin{aligned} XY &= \bigoplus_{i=1}^{L_X}\bigoplus_{j=1}^{L_Y} f_{X_i}(d_{k_X})f_{Y_j}(d_{k_Y})X_iY_j \\ &= \bigoplus_{l=1}^{L_Z} f_{Z_l}(d_{k_X},d_{k_Y})Z_l \qquad L_z = L_X \cdot L_Y \end{aligned}$$

Here $Z_l$ is a matrix $m\times n$ in size, and has only $\varepsilon$ and $e$ in its elements.

In the max-plus algebra, as multiplication is equivalent to addition in $(+, \times)$ operation,

$$f_{Z_l}(d_{k_X},d_{k_Y}) = f_{X_i}(d_{k_X})f_{Y_j}(d_{k_Y})$$

is a linear function in $(+, \times)$ operation which has $d_{k_X}$ and $d_{k_Y}$ in the elements. □

Using Theorem 1 iteratively, also using linear function $f_{\Gamma_l}(\cdot)$, $f_{\Delta_l}(\cdot)$ in $(+, \times)$ operation, the block matrices of Eqs. (27) and (28) can be represented as

$$\begin{aligned} [\Gamma_{\underline{k}}]_{<I1>(qm)} &= C_{\underline{K+I}}\left(\bigoplus_{i=1}^{I} A_{\underline{k+I-i}}\right) \\ &= \bigoplus_{l=1}^{L_\Gamma} f_{\Gamma_l}(d_k, d_{k+1}, \cdots, d_{k+N})\,\Gamma_{Il} \\ &\qquad L_\Gamma = L_C \cdot L_A{}^N, \quad I = 1,\cdots,N \end{aligned} \tag{33}$$

$$\begin{aligned} [\Delta_{\underline{k}}]_{<I1>(qp)} &= C_{\underline{K+I}}\left(\bigoplus_{i=1}^{I-J} A_{\underline{k+I-i}}\right)B_{\underline{k+J-1}} \\ &= \bigoplus_{l=1}^{L_\Delta} f_{\Delta_l}(d_k, d_{k+1}, \cdots, d_{k+N})\,\Delta_{IJl} \\ L_\Delta &= L_C \cdot L_A{}^{N-1} \cdot L_B, \quad I,J = 1,\cdots,N, \quad I \ge J \end{aligned} \tag{34}$$

They are also linear summations represented by $d_i(k)$. Note that $\Gamma_{Il}$ and $\Delta_{IJl}$ have the elements consisted of only $\varepsilon$ and $e$ and their sizes are $q\times n$, $q\times p$, respectively.

**Theorem 2.** *In, $X \in R_\varepsilon^{n\,N\times m\,M}$, when $X_{<IJ>(nm)} \in R_\varepsilon^{n\times n}$, $1 \le I \le N$, $1 \le J \le M$, $X$ is represented as a linear summation function of $X_{IJ}$, $X$ can also be expressed as a linear summation function of $d_i(k)$.*

**Proof.** Using $X$'s $(I, J)$ block element $X_{<IJ>}$, $X$ can be represented as

$$X = \bigoplus_{I=1}^{N}\bigoplus_{J=1}^{M} E_I X_{<IJ>} E_J^T \tag{35}$$

where

| | | |
|---|---|---|
| $E_{I(n\,N\times n)}$ | : element of $I$-th column | : $e_n$ |
| | else | : $\varepsilon_{nn}$ |
| $E_{J(m\,M\times m)}$ | : element of $J$-th column | : $e_m$ |
| | else | : $\varepsilon_{mm}$ |

As $X_{<IJ>}$ can be expressed as the linear summation function of $d_i(k)$, using linear function $f_{<IJ>l}(\cdot)$ under $(+, \times)$ operation, $X_{<IJ>}$ can be expressed as

$$X_{<IJ>} = \bigoplus_{l=1}^{L} f_{<IJ>l}(d_k)X_{IJl} \tag{36}$$

$X_{IJl}$ has only $\varepsilon$ and $e$ in the elements, and is $m\times n$ in size. Therefore, utilizing the distributive law, $X$ can be represented as

$$\begin{aligned} X &= \bigoplus_{I=1}^{N}\bigoplus_{J=1}^{M}\bigoplus_{l=1}^{L} f_{<IJ>l}(d_k)E_I X_{<IJ>l} E_J{}^T \\ &= \bigoplus_{h=1}^{H} f_h(d_k)Z_h \qquad H = N\cdot M\cdot L \end{aligned} \tag{37}$$

where $Z_h$ is the same size as $X$ and has $\varepsilon$ and $e$ in its elements, $f_h(d_k)$ is a linear function under $(+, \times)$ operation which is composed of the element of $d_k$. Consequently, $X$ is expressed as a linear summation function of $d_i(k)$. □

Using Eqs. (33), (34) and Theorem 2, $\Gamma_{\underline{k}}$ and $\Delta_{\underline{k}}$ can be expressed as a linear summation function of $d_i(k)$ and are represented in the following way.

$$\begin{aligned} \Gamma_{\underline{k}} &= \bigoplus_{l=1}^{L_\Gamma} f_l(d_k, d_{k+1}, \cdots, d_{k+N})\,\Gamma_l \\ &\qquad L_\Gamma = L_C \cdot L_A{}^N \cdot N \end{aligned} \tag{38}$$

$$\begin{aligned} \Delta_{\underline{k}} &= \bigoplus_{l=1}^{L_\Delta} g_l(d_k, d_{k+1}, \cdots, d_{k+N})\,\Delta_l \\ &\qquad L_\Delta = \frac{L_C \cdot L_A{}^{N-1} \cdot L_B \cdot N \cdot (N+1)}{2} \end{aligned} \tag{39}$$

$\Gamma_l$ and $\Delta_l$ have only $\varepsilon$ and $e$ in the elements, and their sizes are the same with $\Gamma$ and $\Delta$, respectively, where $f_l(\cdot)$ and $g_l(\cdot)$ are linear functions under $(+, \times)$ operation.

### 4.3 Definition of control inputs using the greatest subsolution

Optimal control inputs for the system can be obtained by solving $U(k+1)$ which satisfies Eq. (30). Here, we utilize the greatest subsolution introduced in Cohen *et al.* (1989) after transforming Eq. (30).

Firstly, we transform the output prediction equation Eq. (30) into

$$\Delta_{\underline{k}} U(k+1) = \boldsymbol{R}(k+1) \oplus \boldsymbol{\Gamma}_{\underline{k}}\, \boldsymbol{x}(k) \tag{40}$$

This transformation can be justified by the following reason (Cohen *et al.*, 1989).

1) Case of $\boldsymbol{R}(k+1) \geq \boldsymbol{\Gamma}_{\underline{k}}\boldsymbol{x}(k)$
Eq. (30) is equivalent to $\boldsymbol{R}(k+1) = \Delta_{\underline{k}}\, U(k+1)$.

2) Case of $\boldsymbol{R}(k+1) < \boldsymbol{\Gamma}_{\underline{k}}\boldsymbol{x}(k)$
Since exact solution of Eq. (30) does not exist, $U(k+1)$ will be determined by getting the maximum solution by which the values of the both sides of Eq. (30) do not change.
Consequently, $\boldsymbol{\Gamma}_{\underline{k}}\boldsymbol{x}(k) = \Delta_{\underline{k}}\, U(k+1)$ should be solved.

Concerning with 1) and 2), it follows that getting the desired input of Eq. (30) can be reduced to Eq. (40).

Eq. (40) is a linear equation of $U(k+1)$ in the max-plus algebra. As $\Delta_{\underline{k}}$ can be expressed as a linear summation function of $d_i(k)$ in Eq. (39), the following formula of the greatest subsolution can be utilized, which is derived in (Masuda *et al.*, 2003).

When, $\boldsymbol{M}_l \in \boldsymbol{R}_\varepsilon^{m\times n}, z \in \boldsymbol{R}_\varepsilon^{n}, v \in \boldsymbol{R}_\varepsilon^{m}$, the greatest subsolution $\underline{z}$ of linear equation

$$\left(\bigoplus_{l=1}^{L} \boldsymbol{M}_l\right) z = v \tag{41}$$

is given by

$$\underline{z} = \bigwedge_{l=1}^{L}\left(\boldsymbol{M}_l^{\,T} \ominus v\right) \tag{42}$$

Using this formula, the greatest subsolution of Eq. (40) can be expressed as

$$\begin{aligned} U(k+1) = \bigwedge_{l=1}^{L_\Delta}\Big\{ & g_l(\boldsymbol{d}_k, \boldsymbol{d}_{k+1}, \cdots, \boldsymbol{d}_{k+N})\Delta_l^T \\ & \ominus (\boldsymbol{R}(k+1) \oplus \boldsymbol{\Gamma}_{\underline{k}}\, \boldsymbol{x}(k)\Big\} \end{aligned} \tag{43}$$

The greatest subsolution has the property that when the correct solution of Eq. (40) is available, the exact solution can be obtained, and when the correct solution of Eq. (40) is not available, the maximum solution, which does not exceed the value of the right hand side, can be obtained.

The example of the production system depicted in Figure 1 is the problem to determine parts input times $\boldsymbol{u}(k+i)$ so as to be just in time for the predetermined due date $\boldsymbol{r}(k+i)$. When the processing completes on the due date, $U(k+i)$ indicates the times of input. Otherwise, it follows the latest inputting time within the due date.

In the inverse system, each input is determined only by considering the corresponding due date. In MPC, the inputs are determined by considering the further finishing time of parts, and then they can accomplish the output times with less delay.

Using the Receding Horizon method in MPC, the control input will be given only by the element of the first block (1-P column) of Eq. (43) such as

$$\boldsymbol{u}(k+1) = [\boldsymbol{e}_p, \varepsilon_{pp}, \cdots, \varepsilon_{pp}]\, U(k+1) \tag{44}$$

Thus, feedback control can be realized using the current outputs result.

## 5. ADJUSTMENT OF PARAMETERS

This section gives an adjustment method of parameters when the systems parameter $\boldsymbol{d}_k$ is adjustable as for the control inputs given in Eqs. (43) and (44).

The objective of the control is to determine the outputs, which do not exceed the desired reference signal, and simultaneously to reduce the operating cost of the system. Masuda *et al.* (2003) gives a method by adjusting the system parameters in accordance with

$$[\boldsymbol{\Gamma}_{\underline{k}}\, \boldsymbol{x}(k)]_i \leq [\boldsymbol{R}(k+1)]_i, \quad i = 1, \dots, q\cdot N \tag{45}$$

Hereafter, an adjusting method of $\boldsymbol{d}_k$ will be formulated considering the constraints of Eq. (45).

### 5.1 Constraints on parameter

This subsection considers constraints on the system parameter $d_i(k)$.

As processing times take non-negative values,

$$d_i(k+j) \geq 0 \quad i = 1, \cdots, h, \quad j = 1, \cdots, N \tag{46}$$

should be satisfied.

Next, we consider Eq. (45). Multiplying $\boldsymbol{x}(k)$ on both sides of Eq. (38), the following equation holds.

$$\begin{aligned} f_l(\boldsymbol{d}_k, \boldsymbol{d}_{k+1}, \cdots, \boldsymbol{d}_{k+N})\, [\boldsymbol{\Gamma}_l\, \boldsymbol{x}(k)]_i \leq [\boldsymbol{R}(k+1)]_i \\ l = 1, \cdots, L_\Gamma, \; i = 1, \cdots, q\cdot N \end{aligned} \tag{47}$$

where $f_l(\cdot)$ is a linear function under $(+, \times)$ operation. As $x(k)$ is known, $[\Gamma_l x(k)]_l$ becomes constant, and consequently Eq. (47) can be expressed as

$$\begin{aligned}&\phi_{ijl}(\boldsymbol{d}_k, \boldsymbol{d}_{k+1}, \cdots, \boldsymbol{d}_{k+N}) \leq r_i(k+j)\\ &\quad i=1,\cdots,q, \quad j=1,\cdots,N, \quad l=1,\cdots,L_\Gamma\end{aligned} \tag{48}$$

As $\phi_{ijl}(\cdot)$ includes the parameter $\boldsymbol{d}_k$ as multiplication in the max-plus algebra, it is reduced to linear constraints under $(+, \times)$ operation.

When the state variables are larger than $\boldsymbol{R}(k+1)$, parameters $d_l(k+j)$ which satisfy Eq. (48) cannot be obtained. Hence, the constraints should be relaxed. Using slack variables $\mu_{ij}$, the inequalities are transformed into

$$\begin{aligned}&\phi_{ijl}(\boldsymbol{d}_k, \boldsymbol{d}_{k+1}, \cdots, \boldsymbol{d}_{k+N}) \leq r_i(k+j)+\mu_{ij}\\ &\quad i=1,\cdots,q, \quad j=1,\cdots,N, \quad l=1,\cdots,L_\Gamma\end{aligned} \tag{49}$$

Adding slack variables, a solution can be found even when the outputs exceed the desired reference signal.

## 5.2 Decreasing calculation load using Control Horizon

Eq. (48) indicates that the number of unknown variables (system parameter included in $\phi_{ijl}(\cdot)$) grows large in proportion to the prediction step $N$, and the calculation volume becomes huge. Therefore, we consider decreasing the calculation load by introducing the Control Horizon.

The Control Horizon method is a technique to shorten the calculation time by reducing unknown variables. Supposing that the systems parameters are the same at the further step than a certain $N_c(<N)$. The number of unknown variables can be decreased accordingly, and $d_l(k+j)$ can be set as

$$d_l(k+j) = \begin{cases} d_l(k+j) & j=1,\cdots,N_c \\ d_l(k+N_c) & j=N_c+1,\cdots,N \end{cases} \tag{50}$$

## 5.3 Criteria function

A criteria function in the linear programming is defined considering both the operation cost of the system and the penalty of errors. The error is defined as a difference between the actual output and the desired reference signal. In the previous production system, they correspond to the processing cost and the delay from due date.

Firstly, we consider the processing cost. The constraints given in Eq. (45) can be achieved by decreasing the value of $\boldsymbol{d}_k$, which means a shortening of the processing time. However, as the smaller system parameters imposes higher operation cost, we set an objective function as

$$P_1 = -\sum_{j=1}^{N_c}\sum_{l=1}^{h} \alpha_{lj} d_l(k+j), \quad \alpha_{lj} > 0 \tag{51}$$

when $\boldsymbol{d}_k$ becomes smaller, the penalty grows large.

Next, we consider the output error. In Eq. (49), the slack variables are appended to get a solution even when output dates exceed the due dates. As for this delay, the penalty is burdened as

$$P_2 = -\sum_{j=1}^{N_c}\sum_{i=1}^{q} \beta_{ij}\mu_{ij}, \quad \beta_{ij} > 0 \tag{52}$$

Using Eqs. (51) and (52), a general criteria function for the system is given by

$$P = P_1 + P_2 \tag{53}$$

From above equations, solving $\boldsymbol{d}_k$ is reduced to linear programming, which minimizes Eq. (53) under the constraints of Eqs. (45) and (46).

In this way, the optimization problem is formulated, which considers both the system operating cost and capability to follow the desired reference signal.

# 6. NUMERICAL SIMULATION

In order to confirm the effectiveness of this proposed control law, this section shows a numerical simulation of two-inputs one-output production system (Figure 1).

We set the desired reference signal as

$$\begin{aligned}&r(k) = r(k-1)+1.5 \quad k=2,\cdots,15, \quad r(1)=0\\ &r(k) = r(k-1)+0.6 \quad k=16,\cdots,20\\ &r(k) = r(k-1)+1.2 \quad k=21,\cdots,35\end{aligned} \tag{54}$$

Initial values and the lowest values of the system parameters, control horizon, and coefficients of the objective function are set as follows.

$$\begin{aligned}&d_1(1) = d_2(1) = d_3(1) = 1.5\\ &d_1(1), d_2(1), d_3(1) \geq 0.8, \quad k=2,\cdots,35\\ &N_c = 1, \alpha_{11} = 1.0, \alpha_{21} = 3.0, \alpha_{31} = 5.0\end{aligned} \tag{55}$$

Concerning Eqs. (54) and (55), the reference signals for $16 \leq k \leq 20$ exceeds the maximum ability (minimum value of the system parameters) of the machines.

Figure 2 shows the processing time of each part at machine 1 for the prediction step $N$=1, 5, 10, 15. $N$=1 is equivalent to the inverse system. $N$=1 makes the processing ability to the maximum 0.8 at step $k$=17. On the other hand, the profiles for $N>1$ are different from that of $N$=1. For instance, $N$=5 considers the future schedules, and then shortens the processing time at the earlier step.

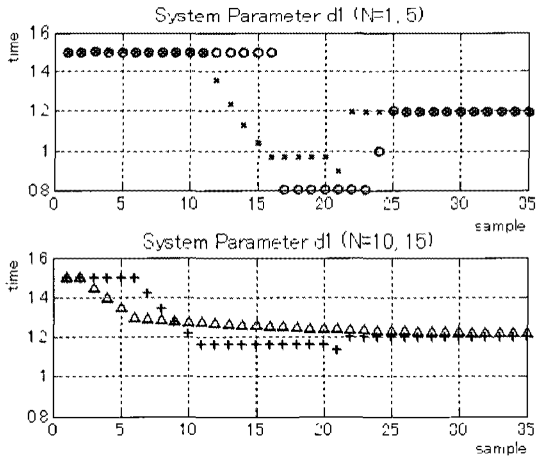


**Figure 2.** System parameter $d_1$ for $N$=1 (o), $N$=5 (x), $N$=10 (+) and $N$=15 (△)

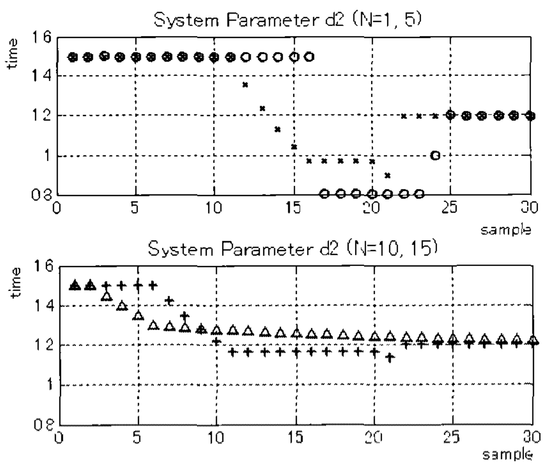


**Figure 3.** System parameter $d_2$ for $N$=1 (o), $N$=5 (x), $N$=10 (+) and $N$=15 (△)

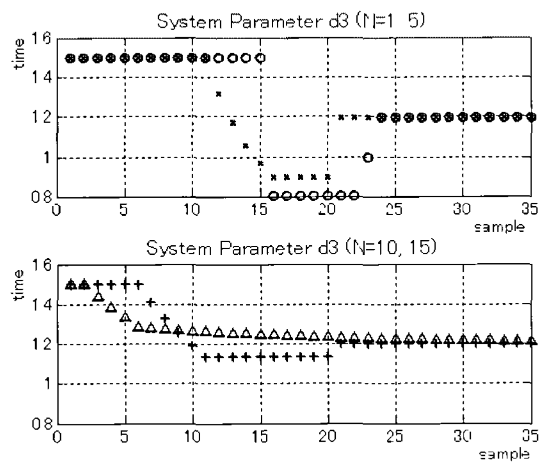


**Figure 4.** System parameter $d_3$ for $N$=1 (o), $N$=5 (x), $N$=10 (+) and $N$=15 (△)

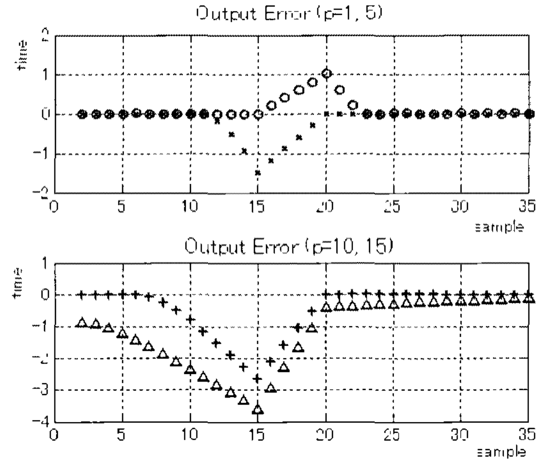


**Figure 5.** Output error (delay) for $N$=1 (o), $N$=5 (x), $N$=10 (+) and $N$=15 (△)

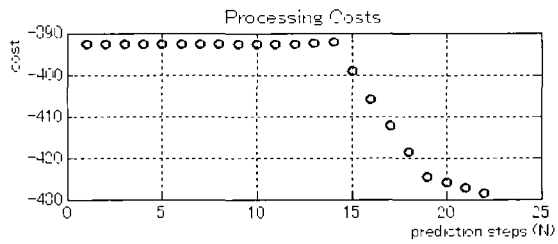


**Figure 6.** Processing costs versus prediction step $N$

Figure 3 shows the processing time of each part at machine 2 for the prediction step $N$=1, 5, 10, 15. Changes of the processing times have the same behavior to machine 1.

Figure 4 shows the processing time of each part at machine 3 for the prediction step $N$=1, 5, 10, 15. Although the behavior is similar to machine 1 and 2, a response to the reduction of desired signal can be seen in the one step earlier. This may be caused by the location of machine 3, which is placed in downstream compared with machine 1 and 2.

Figure 5 shows the deviation of the actual finishing time from the due date. Recall that the interval of the desired signals is decreased from step $k$=16. While $N$=1 causes output delays from step $k$=16, $N$=5 reduces the systems parameters from step $k$=12, and complete processing earlier than the reference signal. As a result, the delays can be avoided. $N$=10 and $N$=15 also make speedups, yet they are performed in the earlier steps.

Figure 6 shows how the prediction step number gives an influence to the processing cost. Here, the processing cost is set as a summation for all $1 \le k \le 35$ in Eq. (51), which is defined as follows.

$$C = -\sum_{k=1}^{35} \sum_{j=1}^{N_i} \sum_{i=1}^{3} \alpha_{ij} d_i (k+j)$$

While the processing cost keeps nearly constant in $1 \leq N \leq 14$, it can be reduced for $N \geq 15$ . When $N$ is small, delays of completion can be reduced by increasing $N$. Furthermore, when $N$ is sufficiently large, the cost reduction can be accomplished. From these matters, it follows that the prediction step $N$ contributes in getting outputs within the reference signals and in minimizing the value of criteria function.

From above results, our proposed method determines the optimal control inputs considering both the due date and the cost. Moreover, it realizes desirable control properties that the conventional inverse system or the other ideas using fixed parameters cannot achieve.

## 7. CONCLUDING REMARKS

In this paper, we introduce a MPC framework for the production systems in which the system parameters depend upon the event counter, and propose a method to give optimal input parameters.

In the past studies of MPC using the max-plus algebra, only the cases that system parameters are constant were handled.

After considering how to represent the system matrices as linear summation function of the parameters, we give a method to obtain the optimal inputs using the greatest subsolution. In addition, we make a consideration of the case that the input parameters are adjustable and show that the adjustment method is reduced to a linear programming problem.

Furthermore, we demonstrate that MPC contributes the cost reduction through a numerical simulation.

As an application of this proposed method in this paper, a scheduling algorithm of a production system is examined, where the system parameters can be determined continuously. Therefore, if we would like to apply to business applications such as workforce assignment, considering discrete system parameters would be better. The expansion for this viewpoint is an issue for the future.

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