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SYMMETRIC DUALITY FOR A CLASS OF NONDIFFERENTIABLE VARIATIONAL PROBLEMS WITH INVEXITY

WON JUNG LEE

ABSTRACT. We formulate a pair of nondifferentiable symmetric dual variational problems with a square root term. Under invexity assumptions, we establish weak, strong, converse and self duality theorems for our variational problems by using the generalized Schwarz inequality. Also, we give the static case of our nondifferentiable symmetric duality results.

1. INTRODUCTION

Symmetric duality in nonlinear programming was introduced in Dorn [4] by defining a symmetric dual program for quadratic proqrams. Subsequently Dantzig, Eisenberg and Cottle [3] first formulated a pair of symmetric dual nonlinear programs in which the dual of the dual equals the prime and established weak and strong duality for these problems concerning convex and concave functions. Later on, Mond and Hanson [9] extended the symmetric duality results to variational problems, giving continuous analogues of the results of [3]. Since the invexity conditions on functions were first defined by Hanson [5] as a generalization of convexity ones, many authors ([6], [10], [13]) have extended the concepts of invexity and generalized invexity to continuous functions. Smart and Mond [13] extended the symmetric duality results to variational problems by using the continuous version of invexity.

On the other hand, Mond [8] and Mehndiratta [7] gave symmetric dual theorems for certain nondifferentiable programs which involve square roots of quadratic forms in the objective functions ([1], [2], [11]).

Mond and Smart [11] extended the duality theorems for a class of static nondifferentiable problems with Wolfe type and Mond-Weir type duals under invexity, and further extended these for the continuous analogues.

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We are motivated to consider symmetric duality for nondifferentiable variational problems involving weaker convexity assumptions [1].

In this paper, we formulate a pair of nondifferentiable symmetric dual variational problems with a square root term. Under invexity assumptions, we establish weak, strong, converse and self duality theorems for our variational problems by using the generalized Schwarz inequality. Also, we give the static case of our nondifferentiable symmetric duality results.

2. Notations and Statement of the Problems

The following conventions for vectors $x = (x_1, x_2, \cdots, x_n)^T$ and $y = (y_1, y_2, \cdots, y_n)^T$ in R^n will be used:

$$x < y \Leftrightarrow x_i < y_i, \quad i = 1, 2, \cdots, n;$$
$$x \leqq y \Leftrightarrow x_i \leqq y_i, \quad i = 1, 2, \cdots, n;$$
$$x \leq y \Leftrightarrow x_i \leqq y_i, \quad i = 1, 2, \cdots, n \quad \text{but} \quad x \neq y;$$
$$x \nleq y \quad \text{is the negation of} \quad x \leq y.$$

Let $[a, b]$ be a real interval and $f : [a, b] \times R^n \times R^n \times R^m \times R^m \to R$. Consider the real scalar function $f(t, x, x', y, y')$, where $t \in [a, b]$, x and y are functions of t with $x(t) \in R^n$ and $y(t) \in R^m$, and x' and y' denote the derivatives of x and y, respectively, with respect to t. Assume that f has continuous fourth-order partial derivatives with respect to x, x', y and y'. f_x and $f_{x'}$ denote the gradient vectors of $f(t, x, x', y, y')$ at (t, x, x', y, y') with respect to x and x', respectively. Similarly f_y and $f_{y'}$ denote the gradient vectors of $f(t, x, x', y, y')$ at (t, x, x', y, y') with respect to y and y', respectively. Subsequently f_{yy}, $f_{y'y'}$, $f_{y'y}$, f_{yx}, $f_{yx'}$, $f_{y'x}$ and $f_{y'x'}$ will denote the $(m \times m)$, $(m \times m)$, $(m \times m)$, $(n \times m)$, $(n \times m)$, $(n \times m)$ and $(n \times m)$ matrices of second order partial derivatives, respectively. Similarly, f_{xx}, $f_{x'x'}$, $f_{x'x}$, f_{xy}, $f_{xy'}$, $f_{x'y}$ and $f_{x'y'}$ will denote the $(n \times n)$, $(n \times n)$, $(n \times n)$, $(m \times n)$, $(m \times n)$, $(m \times n)$ and $(m \times n)$ matrices of second order partial derivatives, respectively.

Let $C([a, b], R^n)$ be the space of piecewise smooth functions $x : [a, b] \to R^n$ with norm given by $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where D is the differentiation operator defined by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)ds,$$

where α is a given boundary value. So $D = \frac{d}{dt}$ except at discontinuities.

We consider the problem of finding functions $x : [a, b] \to R^n$ and $y : [a, b] \to R^m$, with $(x'(t), y'(t))$ piecewise smooth on $[a, b]$, to solve the following pair of symmetric dual variational problems with a sequare root term:

Primal (P)

$$\text{Minimize } F(x,y,w) = \int_a^b \{f(t,x,x',y,y') - y(t)^T \left[f_y(t,x,x',y,y') - \frac{d}{dt} f_{y'}(t,x,x',y,y')\right] + \left(x(t)^T B(t) x(t)\right)^{\frac{1}{2}}\Big\} dt$$

$$\text{subject to } x(a) = x_0,\ \ x(b) = x_1,\ \ y(a) = y_0,\ \ y(b) = y_1,$$

$$\left[f_y(t,x,x',y,y') - \frac{d}{dt} f_{y'}(t,x,x',y,y')\right] - C(t)w(t) \leqq 0, \tag{1}$$

$$w(t)^T C(t) w(t) \leqq 1, \tag{2}$$

Dual (D)

$$\text{Maximize } G(u,v,z) = \int_a^b \Big\{f(t,u,u',v,v') - u(t)^T \left[f_x(t,u,u',v,v') - \frac{d}{dt} f_{x'}(t,u,u',v,v')\right] - (v(t)^T C(t) v(t))^{\frac{1}{2}}\Big\} dt$$

$$\text{subject to } u(a) = x_0,\ \ u(b) = x_1,\ \ v(a) = y_0,\ \ v(b) = y_1,$$

$$\left[f_x(t,u,u',v,v') - \frac{d}{dt} f_{x'}(t,u,u',v,v')\right] + B(t)z(t) \geqq 0, \tag{3}$$

$$z(t)^T B(t) z(t) \leqq 1, \tag{4}$$

where (1) and (3) may fail to hold at corners of $(x'(t), y'(t))$ and $(u'(t), v'(t))$ respectively, but must be satisfied for unique right-hand and left-hand limits.

3. Symmetric Duality

We consider the following nondifferentiable variational problem:

$$(\mathrm{P}_0) \qquad \text{Minimize} \quad \int_a^b \Big\{f(t,x(t),x'(t)) + (x(t)^T B(t) x(t))^{\frac{1}{2}}\Big\} dt$$

$$\text{subject to} \quad x(a) = \alpha,\ \ x(b) = \beta,$$

$$g(t,x(t),x'(t)) \leqq 0,\ \ t \in [a,b],$$

where $f : [a,b] \times R^n \times R^n \to R$, $g : [a,b] \times R^n \times R^n \to R^m$ are assumed to be continuously differentiable functions,$B(t)$ is an $n \times n$ positive semidefinite (symmetric) matrix, with $B(\cdot)$ continuous on $[a,b]$.

Let K be the set of feasible solutions for (P_0) given by

$$K = \{x \in C\left([a,b], R^n\right) \mid x(a) = \alpha, x(b) = \beta,\ g\left(t, x(t), x'(t)\right) \leqq 0,\ \ t \in [a,b]\}.$$

Now we define the invexity as follows.

Definition 3.1. The functional $\int_a^b \left\{ f(t,\cdot,\cdot,v,v') + (\cdot)^T B(t)z(t) \right\} dt$ is invex in x and x' if for each $y : [a,b] \to R^m$, with y' piecewise smooth, there exists a function $\eta : [a,b] \times R^n \times R^n \times R^n \times R^n \to R^n$ such that

$$\begin{aligned} &\int_a^b \left\{ \left(f(t,x,x',v,v') + x(t)^T B(t)z(t) \right) - \left(f(t,u,u',v,v') + u(t)^T B(t)z(t) \right) \right\} dt \\ &\geqq \int_a^b \eta(t,x,x',u,u')^T \left[f_x(t,u,u',v,v') - \frac{d}{dt} f_{x'}(t,u,u',v,v') + B(t)z(t) \right] dt \end{aligned}$$

for all $x : [a,b] \to R^n, u : [a,b] \to R^n$ with $(x'(t), u'(t))$ piecewise smooth on $[a,b]$.

Definition 3.2. The functional $-\int_a^b \left\{ f(t,x,x',\cdot,\cdot) - (\cdot)^T C(t)w(t) \right\} dt$ is invex in y and y' if for each $x : [a,b] \to R^n$, with x' piecewise smooth, there exists a function $\xi : [a,b] \times R^m \times R^m \times R^m \times R^m \to R^m$ such that

$$\begin{aligned} &- \int_a^b \left\{ \left(f(t,x,x',v,v') - v(t)^T C(t)w(t) \right) - \left(f(t,x,x',y,y') - y(t)^T C(t)w(t) \right) \right\} dt \\ &\geqq - \int_a^b \xi(t,v,v',y,y')^T \left[f_y(t,x,x',y,y') - \frac{d}{dt} f_{y'}(t,x,x',y,y') - C(t)w(t) \right] dt \end{aligned}$$

for all $v : [a,b] \to R^m, y : [a,b] \to R^m$ with $(v'(t), y'(t))$ piecewise smooth on $[a,b]$.

In the sequel, we will write $\eta(x,u)$ for $\eta(t,x,x',u,u')$ and $\xi(v,y)$ for $\xi(t,v,v',y,y')$.

Theorem 3.1 (Weak Duality). *Let (x,y,w) be feasible for (P) and (u,v,z) be feasible for (D). Assume that for all $t \in [a,b]$, $\int_a^b \left\{ f(t,\cdot,\cdot,y,y') + (\cdot)^T B(t)z(t) \right\} dt$ is invex in x and x', and $-\int_a^b \left\{ f(t,x,x',\cdot,\cdot) - (\cdot)^T C(t)w(t) \right\} dt$ is invex in y and y', with $\eta(x,u) + u(t) \geqq 0$ and $\xi(v,y) + y(t) \geqq 0$ (except perhaps at corners of $(x'(t), y'(t))$ or $(u'(t), v'(t))$). Then we have*

$$inf(P) \geqq sup(D).$$

Proof. Let (x,y,w) be feasible for (P) and (u,v,z) be feasible for (D). Then

$$\begin{aligned} &\int_a^b \left\{ f(t,x,x',y,y') - y(t)^T \left[f_y(t,x,x',y,y') - \frac{d}{dt} f_{y'}(t,x,x',y,y') \right] \right. \\ &\left. + \left(x(t)^T B(t)x(t) \right)^{\frac{1}{2}} \right\} dt \end{aligned}$$

$$
\begin{aligned}
&-\int_a^b \Big\{ f(t,u,u',v,v') - u(t)^T \Big[f_x(t,u,u',v,v') - \frac{d}{dt} f_{x'}(t,u,u',v,v') \Big] \\
&\quad - \big(v(t)^T C(t) v(t)\big)^{\frac{1}{2}} \Big\}\, dt \\
&\geqq \int_a^b \Big\{ f(t,x,x',y,y') - y(t)^T \Big[f_y(t,x,x',y,y') - \frac{d}{dt} f_{y'}(t,x,x',y,y') \Big] \\
&\quad + \big(x(t)^T B(t) x(t)\big)^{\frac{1}{2}} \big(z(t)^T B(t) z(t)\big)^{\frac{1}{2}} \Big\}\, dt \\
&-\int_a^b \Big\{ f(t,u,u',v,v') - u(t)^T \Big[f_x(t,u,u',v,v') - \frac{d}{dt} f_{x'}(t,u,u',v,v') \Big] \\
&\quad - \big(v(t)^T C(t) v(t)\big)^{\frac{1}{2}} \big(w(t)^T C(t) w(t)\big)^{\frac{1}{2}} \Big\}\, dt \\
&\quad\ \text{(From (4) and (2))} \\
&\geqq \int_a^b \Big\{ f(t,x,x',y,y') - y(t)^T \Big[f_y(t,x,x',y,y') - \frac{d}{dt} f_{y'}(t,x,x',y,y') \Big] \\
&\quad + x(t)^T B(t) x(t) \Big\}\, dt \\
&-\int_a^b \Big\{ f(t,u,u',v,v') - u(t)^T \Big[f_x(t,u,u',v,v') - \frac{d}{dt} f_{x'}(t,u,u',v,v') \Big] \\
&\quad - v(t)^T C(t) w(t) \Big\}\, dt
\end{aligned}
$$

(Using the generalized Schwarz inequality)

$$
\begin{aligned}
&\geqq \int_a^b \Big\{ [\eta(x,u) + u(t)]^T \Big[f_x(t,u,u',v,v') - \frac{d}{dt} f_{x'}(t,u,u',v,v') + B(t) z(t) \Big] \\
&- [\xi(v,y) + y(t)]^T \Big[f_y(t,x,x',y,y') - \frac{d}{dt} f_{y'}(t,x,x',y,y') - C(t) w(t) \Big] \Big\}\, dt
\end{aligned}
$$

(since $\int_a^b \big\{ f(t,\cdot,\cdot,v,v') + (\cdot)^T B(t) z(t) \big\}\, dt$ is invex in x and x',

and $-\int_a^b \big\{ f(t,x,x',\cdot,\cdot) + (\cdot)^T C(t) w(t) \big\}\, dt$ is invex in y and y')

$\geqq 0$

(from (3) and (1) together with $\eta(x,u) + u(t) \geqq 0$ and $\xi(v,y) + y(t) \geqq 0$).

Hence the result of Theorem 3.1 holds.

Before presenting the Wolfe type symmetric dual to (P), we state the following Fritz John necessary optimality conditions [Lemma 3.1] for the nondifferentiable problem

(P_0) and it can be easily derived by invoking the result of Valentine [14] or those of Zhang and Mond [15].

Lemma 3.1. If x^* is an optimal solution of (P_0), then there exist $\gamma \in R$, piecewise smooth $\rho : [a,b] \to R^m$ and $z : [a,b] \to R^n$ such that for $t \in [a,b]$

$$\begin{aligned} \gamma \left(f_x(t, x^*, x^{*'}) + B(t)z(t) \right) &= \rho^T g_x(t, x^*, x^{*'}), \\ \rho^T g(t, x^*, x^{*'}) &= 0, \\ z(t)^T B(t) z(t) &\leqq 1, \\ \left(x^*(t)^T B(t) x^*(t) \right)^{\frac{1}{2}} &= x^*(t)^T B(t) z(t), \\ (\gamma, \rho) &\geqq 0, \\ (\gamma, \rho) &\neq 0. \end{aligned}$$

The following the generalized Schwarz inequality [12, p.262] is required in the sequel:

$$x^T B w \leqq (x^T B x)^{\frac{1}{2}} (w^T B w)^{\frac{1}{2}},$$

for all x, $w \in R^n$ and equality holds if $Bx = \gamma Bw$ for some $\gamma \geqq 0$.

In the following theorems and proofs, f^* represents $f(t, x^*, x^{*\prime}, y^*, y^{*\prime})$ and partial derivatives are similarly denoted.

Theorem 3.2 (Strong Duality). *Let (x^*, y^*, w^*) be an optimal solution for (P). Suppose that the system*

$$\begin{aligned} \Bigg[p(t)^T \left(f^*_{yy} - \frac{d}{dt} f^*_{y'y} \right) + \frac{d}{dt} \left(p(t)^T \frac{d}{dt} f^*_{y'y'} \right) \\ + \frac{d^2}{dt^2} \left(- p(t)^T f^*_{y'y'} \right) \Bigg] p(t) = 0 \end{aligned} \tag{5}$$

only has the solution $p(t) = 0$ for all $t \in [a,b]$. Assume that for all $t \in [a,b]$

$$\int_a^b \left\{ f(t, \cdot, \cdot, y, y') + (\cdot)^T B(t) z(t) \right\} dt \ \textit{ is invex in } x \textit{ and } x',$$

and

$$-\int_a^b \left\{ f(t, x, x', \cdot, \cdot) + (\cdot)^T C(t) w(t) \right\} dt \ \textit{ is invex in } y \textit{ and } y'$$

(except perhaps at corners of $(x'(t), y'(t))$ or $(u'(t), v'(t))$). Then (x^*, y^*, w^*) is an optimal solution for (D), and the objective values of (P) and (D) are equal.

Proof. since (x^*, y^*, w^*) is an optimal solution of the primal problem (P), by Lemma 3.1, there exist $\alpha \in R$, $\beta : [a, b] \to R^m$ and $\gamma \in R$ such that

$$\begin{aligned} H^* \equiv & \alpha \left\{ f^* - y^{*T}[f_y^* - \frac{d}{dt} f_{y'}^*] + (x^{*T}Bx^*)^{\frac{1}{2}} \right\} \\ & + \beta^T \left(f_y^* - \frac{d}{dt} f_{y'}^* - Cw^* \right) + \gamma(w^{*T}Cw^* - 1) \end{aligned}$$

satisfies

$$H_x^* - \frac{d}{dt} H_{x'}^* + \frac{d^2}{dt^2} H_{x''}^* = 0, \tag{6}$$

$$H_y^* - \frac{d}{dt} H_{y'}^* + \frac{d^2}{dt^2} H_{y''}^* = 0, \tag{7}$$

$$-\beta^T C + 2\gamma C w^* = 0, \tag{8}$$

$$\beta^T (f_y^* - \frac{d}{dt} f_{y'}^* - Cw^*) = 0, \tag{9}$$

$$\gamma(w^{*T}Cw^* - 1) = 0, \tag{10}$$

$$z^{*T}Bz^* \leqq 1, \tag{11}$$

$$\left(x^{*T}Bx^*\right)^{\frac{1}{2}} = x^{*T}Bz^*, \tag{12}$$

$$(\alpha, \beta, \gamma) \geqq 0, \tag{13}$$

$$(\alpha, \beta, \gamma) \neq 0, \tag{14}$$

throughout $[a, b]$ (except at corners of $({x^*}'(t), {y^*}'(t))$ where (6) and (7) hold for unique right-and left-hand limits). $\alpha, \beta(t)$ and γ cannot be simultaneously zero at any $t \in [a, b]$, and β is continuous except perhaps at corners of $({x^*}'(t), {y^*}'(t))$.

Equation (6) now becomes

$$\begin{aligned} & \alpha(f_x^* - \frac{d}{dt} f_{x'}^*) + (\beta - \alpha y^*)^T \left(f_{yx}^* - \frac{d}{dt} f_{y'x}^* \right) + \alpha B x^* \left(x^{*T}Bx^*\right)^{-\frac{1}{2}} \\ & - \frac{d}{dt} \left\{ (\beta - \alpha y^*)^T \left(f_{yx'}^* - f_{y'x}^* - \frac{d}{dt} f_{y'x'}^* \right) \right\} + \frac{d^2}{dt^2} \left\{ - (\beta - \alpha y^*)^T f_{y'x'}^* \right\} \\ & = 0. \end{aligned} \tag{15}$$

Equation (7) gives

$$(\beta-\alpha y^*)^T\left(f^*_{yy}-\frac{d}{dt}f^*_{y'y}\right)$$
$$+\frac{d}{dt}\left\{(\beta-\alpha y^*)^T\frac{d}{dt}f^*_{y'y'}\right\}+\frac{d^2}{dt^2}\left\{-(\beta-\alpha y^*)^T f^*_{y'y'}\right\} = 0. \tag{16}$$

Multiplying (16) by $\beta-\alpha y^*$ yields

$$\left[(\beta-\alpha y^*)^T(f^*_{yy}-\frac{d}{dt}f^*_{y'y})+\frac{d}{dt}\left\{(\beta-\alpha y^*)^T\frac{d}{dt}f^*_{y'y'}\right\}\right.$$
$$\left.+\frac{d^2}{dt^2}\left\{-(\beta-\alpha y^*)^T f^*_{y'y'}\right\}\right](\beta-\alpha y^*)=0.$$

Thus by the assumption (5),

$$\beta=\alpha y^*. \tag{17}$$

Now multiplying (8) by $w^*(t)^T=w^{*T}$, we get

$$w^{*T}C\beta=2\gamma w^{*T}Cw^*. \tag{18}$$

This gives $\alpha\neq 0$, since if $\alpha=0$, then by (17), (18), and (10), $\beta=\gamma=0$, everywhere, contradicting the necessary condition (14). Hence $\alpha>0$ from (13). Now equation (18) with the aid of (17) and the fact $\alpha>0$ gives,

$$y^{*T}C=\left(2\frac{\gamma}{\alpha}\right)Cw^*. \tag{19}$$

Thus

$$y^*Cw^*=\left(y^{*T}Cy^*\right)^{\frac{1}{2}}\left(w^{*T}Cw^*\right)^{\frac{1}{2}}. \tag{20}$$

If $\gamma>0$, then (10) gives $w^{*T}Cw^*=1$, and so (20) yields

$$y^*Cw^*=\left(y^{*T}Cy^*\right)^{\frac{1}{2}}.$$

If $\gamma=0$, then (19) gives $y^*C=0$. So we will get $y^{*T}Cw^*=\left(y^{*T}Cy^*\right)^{\frac{1}{2}}$. Therefore in either case, we obtain

$$y^{*T}Cw^*=\left(y^{*T}Cy^*\right)^{\frac{1}{2}}. \tag{21}$$

Equation (15) with (17) and (12) together with $\alpha>0$ now becomes

$$f_x{}^* - \frac{d}{dt} f^*_{x'} + Bz^* = 0. \tag{22}$$

By (22) and (11), (x^*, y^*, z^*) is feasible for (D).

Multiplying (15) by $x^*(t)$, and using (17) and $\alpha > 0$ in succession, we get

$$-x^{*T}\left(f_x{}^* - \frac{d}{dt} f^*_{x'}\right) - x^{*T} Bz^* = 0. \tag{23}$$

Hence

$$\int_a^b \{f(t, x^*, x^{*\prime}, y^*, y^{*\prime})$$
$$-y^*(t)^T\left[f_y(t, x^*, x^{*\prime}, y^*, y^{*\prime}) - \frac{d}{dt} f_{y'}(t, x^*, x^{*\prime}, y^*, y^{*\prime})\right] + \left(x^*(t)^T B(t) x^*(t)\right)^{\frac{1}{2}}\Big\}\, dt$$
$$= \int_a^b \{f(t, x^*, x^{*\prime}, y^*, y^{*\prime}) - y^*(t)^T C(t) w^*(t) + x^*(t)^T B(t) z^*(t)\}\, dt$$

(using (9) and (17) with $\alpha > 0$, and then (12))

$$= \int_a^b \left\{f(t, x^*, x^{*\prime}, y^*, y^{*\prime}) - \left(y^{*T} C y^*\right)^{\frac{1}{2}} - x^{*T}\left(f^*_x - \frac{d}{dt} f^*_{x'}\right)\right\} dt$$

(using (21) and (23))

$$= \int_a^b \{f(t, x^*, x^{*\prime}, y^*, y^{*\prime})$$
$$-x^*(t)^T\left[f_x(t, x^*, x^{*\prime}, y^*, y^{*\prime}) - \frac{d}{dt} f_{x'}(t, x^*, x^{*\prime}, y^*, y^{*\prime})\right] - \left(y^*(t)^T C(t) y^*(t)\right)^{\frac{1}{2}}\Big\}\, dt.$$

If the invexity conditions of Theorem 3.1 are satisfied, then, by weak duality, (x^*, y^*, z^*) is optimal for (D), and the extreme values of (P) and (D) are equal.

A converse duality theorem may be stated; the proof would be analogous to that of Theorem 3.2.

Theorem 3.3 (Converse Duality). *Let (x^*, y^*, z^*) be an optimal solution for (D). Suppose that the system*

$$\left[p(t)^T\left(f^*_{xx} - \frac{d}{dt} f^*_{x'x}\right) + \frac{d}{dt}\left(p(t)^T \frac{d}{dt} f^*_{x'x'}\right)\right.$$

$$+\ \frac{d^2}{dt^2}\left(-\ p(t)^T f^*_{x'x'}\right)\Bigg]\, p(t) = 0$$

only has the solution $p(t) = 0$ *for all* $t \in [a, b]$*. Assume that for all* $t \in [a, b]$

$$\int_a^b \left[f(t, \cdot, \cdot, y, y') + (\cdot)^T B(t)z(t)\right] dt \ \text{ is invex in } x \text{ and } x',$$

and

$$-\int_a^b \left[f(t, x, x', \cdot, \cdot) - (\cdot)^T C(t)w(t)\right] dt \ \text{ is invex in } y \text{ and } y'$$

(except perhaps at corners of $(x'(t), y'(t))$ *or* $(u'(t), v'(t))$*). Then* (x^*, y^*, w^*) *is an optimal solution for (P), and the objective values of (P) and (D) are equal.*

Now we establish the self duality of (P).

Assume that $m = n$, $C = B$, $z = w$ and $f(t, x, x', y, y') = -f(t, y, y', x, x')$ (i.e., f is skew-symmetric) for all $(x(t), y(t)), t \in [a, b]$ such that $(x'(t), y'(t))$ is piecewise smooth on $[a, b]$ and that $x_0 = y_0, x_1 = y_1$.

It follows that (D) may be rewritten as a minimization problem:
(D$'$)

$$\begin{aligned}
\text{Minimize } & \int_a^b \Big\{f(t, y, y', x, x') \\
& -x(t)^T \left[f_y(t, y, y', x, x') - \frac{d}{dt} f_{y'}(t, y, y', x, x')\right] + \left(y(t)^T B(t) y(t)\right)^{\frac{1}{2}} \Big\}\, dt \\
\text{subject to } & x(a) = x_0,\ \ x(b) = x_1,\ \ y(a) = x_0,\ \ y(b) = x_1, \\
& f_y(t, y, y', x, x') - \frac{d}{dt} f_{y'}(t, y, y', x, x') - C(t)w(t) \leqq 0, \\
& w(t)^T C(t) w(t) \leqq 1.
\end{aligned}$$

(D$'$) is formally identical to (P); that is, the objective and constraint functions and initial conditions of (P) and (D$'$) are identical. This problem is said to be self dual.

It is easily seen that whenever (x, y, z) is feasible for (P), then (y, x, z) is feasible for (D), and vice versa.

Theorem 3.4 (Self Duality). *Assume that (P) is self dual and that the invexity conditions of Theorem 3.1 are satisfied. If* (x^*, y^*, z^*) *is an optimal solution for (P), and the system (5) only has a zero solution, then* (y^*, x^*, z^*) *is an optimal solution for both (P) and (D), and the common optimal value is* 0*.*

Proof. By Theorem 3.2, (x^*, y^*, z^*) is an optimal solution for (D), and the optimal values of (P) and (D) are equal to $F(x^*, y^*, z^*)$.

From self duality, (y^*, x^*, z^*) is feasible for both (P) and (D), so Theorems 3.1 and 3.2 give optimality in both problems, and thus objective values of $F(y^*, x^*, z^*)$.

Now it remains to show that $F(x^*, y^*, z^*) = 0$.
Since f is skew-symmetric, we have

$$\int_a^b f(t, y^*, y^{*\prime}, x^*, x^{*\prime})dt = -\int_a^b f(t, x^*, x^{*\prime}, y^*, y^{*\prime})dt.$$

So

$$\begin{aligned}
&F(x^*, y^*, z^*) \\
&= \int_a^b \Big\{ f(t, x^*, x^{*\prime}, y^*, y^{*\prime}) - y^*(t)^T \Big[f_y(t, x^*, x^{*\prime}, y^*, y^{*\prime}) - \frac{d}{dt} f_{y'}(t, x^*, x^{*\prime}, y^*, y^{*\prime}) \Big] \\
&\quad + \big(x^*(t)^T B(t) x^*(t)\big)^{\frac{1}{2}} \Big\} dt \\
&= \int_a^b \Big\{ f(t, x^*, x^{*\prime}, y^*, y^{*\prime}) - y(t)^{*T} C(t) w^*(t) + \big(x^*(t)^T B(t) x^*(t)\big)^{\frac{1}{2}} \Big\} dt \\
&= \int_a^b \Big\{ f(t, x^*, x^{*\prime}, y^*, y^{*\prime}) - (y(t)^{*T} C(t) y^*(t))^{\frac{1}{2}} + \big(x^*(t)^T B(t) x^*(t)\big)^{\frac{1}{2}} \Big\} dt
\end{aligned}$$

and

$$F(x^*, y^*, z^*) = F(y^*, x^*, z^*).$$

Hence

$$\begin{aligned}
&F(x^*, y^*, z^*) \\
&= F(y^*, x^*, z^*) \\
&= \int_a^b \Big\{ f(t, y^*, y^{*\prime}, x^*, x^{*\prime}) - \big(x^*(t)^T B(t) x^*(t)\big)^{\frac{1}{2}} + \big(y^*(t)^T B(t) y^*(t)\big)^{\frac{1}{2}} \Big\} dt \\
&= \int_a^b \Big\{ -f(t, x^*, x^{*\prime}, y^*, y^{*\prime}) - \big(x^*(t)^T B(t) x^*(t)\big)^{\frac{1}{2}} + \big(y^*(t)^T C(t) y^*(t)\big)^{\frac{1}{2}} \Big\} dt \\
&= -\int_a^b \Big\{ f(t, x^*, x^{*\prime}, y^*, y^{*\prime}) + \big(x^*(t)^T B(t) x^*(t)\big)^{\frac{1}{2}} - (y^*(t)^T C(t) y^*(t)) \Big\} dt \\
&= -F(x^*, y^*, z^*).
\end{aligned}$$

Thus

$$F(x^*, y^*, z^*) = -F(x^*, y^*, z^*).$$

Therefore

$$F(x^*, y^*, z^*) = 0.$$

4. The Static Case of Nondifferentiable Symmetric Duality

In the time dependency of programs (P) and (D) is removed and f is considered to have domain $R^n \times R^m$, we obtain the nondifferentiable symmetric dual pair given by

$$\begin{aligned} \text{(SP)} \quad \text{Minimize} \; & f(x,y) - y^T f_y(x,y) + \left(x^T Bx\right)^{\frac{1}{2}} \\ \text{subject to} \; & f_y(x,y) - Cw \leqq 0, \\ & w^T Cw \leqq 1, \end{aligned}$$

$$\begin{aligned} \text{(SD)} \quad \text{Maximize} \; & f(u,v) - u^T f_x(u,v) + \left(v^T Cv\right)^{\frac{1}{2}} \\ \text{subject to} \; & f_x(u,v) + Bz \geqq 0, \\ & z^T Bz \leqq 1. \end{aligned}$$

The following duality theorems can be proved along the lines of Theorems 3.1, 3.2 and 3.3.

Theorem 4.1 (Weak Duality). *Let (x, y, w) be feasible for (SP) and (u, v, z) be feasible for (SD). Assume that $f(\cdot, y) + (\cdot)^T Cw$ is invex in x and $-\left\{f(x,\cdot) - (\cdot)^T Cw\right\}$ is invex in y with $\eta(x,u) + u \geqq 0$ and $\xi(v,y) + y \geqq 0$. Then we have*

$$inf(SP) \geqq sup(SD).$$

Theorem 4.2 (Strong Duality). *Let (x^*, y^*, w^*) be an optimal solution for (SP). Suppose that f^*_{yy} is positive or negative definite.*

If, in addition, the invexity conditions of Theorem 4.1 are satisfied, then (x^*, y^*, z^*) is an optimal solution for (SD), and the objective values of (SP) and (SD) are equal.

Theorem 4.3 (Converse Duality). *Let* (x^*, y^*, z^*) *be an optimal solution for (SD). Suppose that* f^*_{xx} *is positive or negative definite.*

If, in addition, the invexity conditions of Theorem 4.1 are satisfied, then (x^*, y^*, w^*) *is an optimal solution for (SP), and the objective values of (SP) and (SD) are equal.*

The pair (SP) and (SD) will be self dual when $m = n$, $C = B$, $z = w$ and $f(x, y) = -f(y, x)$ (i.e. f is skew-symmetric for all $x, y \in R^n$).

We state without proof a static version of Theorem 3.4.

Theorem 4.4 (Self Duality). *Assume that (SP) is self dual and that the invexity conditions of Theorem 4.1 are satisfied. If* (x^*, y^*, z^*) *is an optimal solution for (SP), and* f^*_{yy} *is positive or negative definite, then* (y^*, x^*, z^*) *is an optimal solution for both (SP) and (SD), and the common optimal value is* 0*.*

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Department of Applied Mathematics
Pukyong National University
Pusan 608-737
Korea
e-mail: icmom1@hanmail.net