A NOTE ON ENERGY MINIMIZING MAP
ON MANIFOLD WITH ISOLATED PEAKS

Heayong Shin

Abstract. In this paper, we consider some homogeneous maps from a cone over 2-spheres and determines whether they become energy minimizing maps or not. In fact, any homogeneous map from a standard cone over 2-sphere of radius smaller than 1 can not be a minimizing harmonic map.

1. Introduction

There has been some efforts to extend the theory of harmonic maps to singular spaces by several mathematicians. Korevaar & Schoen has developed sobolev theory of maps from Riemannian domains into general complete metric spaces and have proved the regularity of harmonic maps into nonpositively curved metric spaces. The similar regularity result has also been proved by Jost independently. But when the domain is not a Riemannian manifold, the harmonic map theory becomes more ambiguous. Jost has defined the energy density of such a map with respect to a given measure on the domain which satisfy certain conditions. But as far as the author knows, the minimum structure of the domain where the theory of harmonic map can be properly considered is not known.

When the domain is a smooth manifold with C^∞ Riemannian metric given except on some isolated points and the tangent cone at the singular points can be given, the regularity of the minimizing harmonic map at those points can be examined by considering the existence of homogeneous minimizing harmonic map on the target cone. In this paper, we consider 3-dimensional cone over 2-spheres and will study some conditions for which the homogeneous harmonic map is or is not minimizing.

2. Homogeneous Minimizing Map in a Cone

Let S be a 2-dimensional topological sphere with a given metric ds^2, and let $K_S = O \cup ((0,1] \times S)$ be a cone over S, with the metric given by $d\rho^2 + \rho^2 ds^2$. For a map $h : S \to N$ with finite Dirichlet energy we consider the homogeneous extension $\tilde{h} : K_S \to N$ given by $\tilde{h}(r,p) = h(p)$. When S is the standard unit sphere S^2, K_S is the unit ball in R^3 and the map $\tilde{h}$ is given by $\tilde{h}(x) = h\left(\frac{x}{|x|}\right)$. We will examine some cases where $\tilde{h}$ can or can not be an energy minimizer.

Theorem 1. *When S is the Euclidean sphere of radius k, $k < 1$, and N is a Riemannian manifold diffeomorphic to a sphere, $\tilde{h}(r,p) = h(p) : K_S \to N$ is not an energy minimizing map unless h is a constant.*

proof. If $\tilde{h}$ is an energy minimizing map, then $h : S \to N$ must be harmonic. Therefore, we may assume that h is a conformal map from S into N. Consider S as the unit sphere S^2 with metric $k^2 ds^2$, where ds^2 is the standard metric in S^2. Then K_S can be identified as the unit ball $B_1^3(0)$ with the metric.

$$\Psi = dr^2 + k^2 r^2 d\theta^2 + k^2 r^2 \sin^2 \theta d\sigma^2,$$

for standard spherical coordinate r, θ, σ centered at the origin and z-axis as the pole. In this coordinate, the Dirichlet energy $E(\tilde{h})$ of $\tilde{h}$ is given by

$$E(\tilde{h}) = \int_{K_S} |\nabla \tilde{h}|^2 = \int_S |\nabla h|^2 = 2 \int_0^{2\pi} d\sigma \int_0^\pi d\psi \left|\frac{\partial h}{\partial \psi}\right|_N^2.$$

Let $0 < a < 1$ and $A = (0,0,a)$. We introduce another polar coordinate ρ, φ, σ centered at A with the z-axis as the pole, so that $\rho(x) = |x - A|$, $\psi(x)$ is the polar angle and $\sigma(x)$ is the azimuthal angle of x from A. Then between ρ, φ, σ and r, θ, σ we have the following relations

$$\rho \sin \varphi = r \sin \theta, \qquad \rho \cos \varphi = r \cos \theta - a .$$

Let $R(\varphi)$ denote the maximum allowed radius in B^3 for given angle φ (i,e, $(R(\varphi), \varphi, \sigma) \in S^2$), and let $\psi(\varphi)$ be the standard polar angle centered at the origin of the point $(R(\varphi), \varphi, \sigma)$.

We define $u_a(\rho, \varphi, \sigma) = h(\psi(\varphi), \sigma)$ with respect to the coordinate (ρ, φ, σ) in K_c and the standard polar coordinate (ψ, σ) on $S = S^2$.

In local coordinate (ρ, φ, σ), the energy density of u_a can be computed as

$$|\nabla u_a|_N^2 = \left|\frac{\partial u_a}{\partial r}\right|_N^2 + \frac{1}{k^2 r^2}\left|\frac{\partial u_a}{\partial \theta}\right|_N^2 + \frac{1}{k^2 r^2 \sin^2 \theta}\left|\frac{\partial u_a}{\partial \sigma}\right|_N^2$$

$$= \left(\left|\frac{\partial \varphi}{\partial r}\right|^2 + \frac{1}{k^2 r^2}\left|\frac{\partial \varphi}{\partial \theta}\right|^2\right)\left|\frac{\partial u_a}{\partial \varphi}\right|_N^2 + \frac{1}{k^2 r^2 \sin^2 \theta}\left|\frac{\partial u_a}{\partial \sigma}\right|_N^2$$

and the volume element is

$$dV = k^2 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\sigma \; .$$

Here, $|*|_N$ is the norm as vectors in N.

Since

$$\frac{\partial \varphi}{\partial r} = \frac{a \sin \varphi}{\rho \sqrt{\rho^2 + 2a\rho \cos \varphi + a^2}} \quad \text{and} \quad \frac{\partial \varphi}{\partial \theta} = \frac{\rho + a \cos \varphi}{\rho^2} \; ,$$

the energy of u_a is

$$E(u_a) = \int_0^{2\pi} d\sigma \int_0^{\pi} d\varphi \int_0^{R(\varphi)} d\rho \; \left\{\left(\frac{k^2 a^2 \sin^2 \varphi + (\rho + a \cos \varphi)^2}{\rho^2 + 2a\rho \cos \varphi + a^2}\right) \sin \varphi \left|\frac{\partial u_a}{\partial \varphi}\right|_N^2 \right.$$

$$\left. + \frac{1}{\sin \varphi}\left|\frac{\partial u_a}{\partial \sigma}\right|_N^2\right\}$$

$$= \int_0^{2\pi} d\sigma \int_0^{\pi} d\varphi \int_0^{R(\varphi)} d\rho \; \left\{\sin \varphi \left|\frac{\partial u_a}{\partial \varphi}\right|_N^2 + \frac{1}{\sin \varphi}\left|\frac{\partial u_a}{\partial \sigma}\right|_N^2\right\}$$

$$+ (k^2 - 1)\int_0^{2\pi} d\sigma \int_0^{\pi} d\varphi \int_0^{R(\varphi)} d\rho \; \frac{a^2 \sin^2 \varphi}{\rho^2 + 2a\rho \cos \varphi + a^2} \sin \varphi \left|\frac{\partial u_a}{\partial \varphi}\right|_N^2$$

$$= E_1(u_a) + E_2(u_a)$$

For $E_2(u_a)$,

$$E_2(u_a) = (k^2 - 1)\int_0^{2\pi} d\sigma \int_0^{\pi} d\varphi \int_0^{R} d\rho \; \frac{a^2 \sin^2 \varphi}{\rho^2 + 2a\rho \cos \varphi + a^2} \sin \varphi \left|\frac{\partial u_a}{\partial \varphi}\right|_N^2$$

$$= (k^2 - 1)\int_0^{2\pi} d\sigma \int_0^{\pi} d\varphi \int_0^{R} d\rho \; \left\{\frac{a^2 \sin^2 \varphi \, \sin \varphi}{\rho^2 + 2a\rho \cos \varphi + a^2}\left|\frac{\partial \psi(\varphi)}{\partial \varphi}\right|^2 \left|\frac{\partial h}{\partial \psi}\right|_N^2\right\}$$

$$= (k^2 - 1)\int_0^{2\pi} d\sigma \int_0^{\pi} d\varphi \; a \sin^2 \varphi \left|\frac{\partial \psi}{\partial \varphi}\right|^2 \left|\frac{\partial h}{\partial \psi}\right|_N^2 \left\{\tan^{-1}\left(\frac{R + a \cos \varphi}{a \sin \varphi}\right)\right.$$

$$\left. - \tan^{-1}\left(\frac{\cos \varphi}{\sin \varphi}\right)\right\} \; .$$

Now, we change the parameter of the above integral by ψ.

Since, $R(\varphi)\sin\varphi = \sin\psi$ and $R(\varphi)\cos\varphi = \cos\psi - a$,

$$R^2 = 1 - 2a\cos\psi + a^2 , \qquad \frac{\partial\psi}{\partial\varphi} = \frac{1 - 2a\cos\psi + a^2}{1 - a\cos\psi} .$$

Hence,

$$E_2(u_a) = (k^2 - 1)\int_0^{2\pi} d\sigma \int_0^\pi d\psi \; \frac{a\sin^2\psi}{(1 - a\cos\psi)}\left|\frac{\partial h}{\partial\psi}\right|_N^2$$
$$\cdot \left\{ tan^{-1}\left(\frac{1 - a\cos\psi}{a\sin\psi}\right) - tan^{-1}\left(\frac{\cos\psi - a}{\sin\psi}\right) \right\} .$$

Therefore, $E_2(u_0) = 0$, and unless h is a constant,

$$\left.\frac{d}{da}\right|_{a=0^+} E_2(u_a) = (k^2 - 1)\int_0^{2\pi} d\sigma \int_0^\pi d\psi \left|\frac{\partial h}{\partial\psi}\right|_N^2 \sin^2\psi \left(\frac{\pi}{2} - tan^{-1}\left(\frac{\cos\psi}{\sin\psi}\right)\right)$$
$$< 0$$

By the same computation,

$$E_1(u_a) = \int_0^{2\pi} d\sigma \int_0^\pi d\psi \left|\frac{\partial h}{\partial\psi}\right|_N^2 \sin\psi \left((1 - a\cos\psi) + \frac{1 - 2a\cos\psi + a^2}{1 - a\cos\psi}\right)$$

So, $E_1(u_0) = E(\tilde{h})$ and by taking z-axis in opposite direction if necessary we may assume that

$$\left.\frac{d}{da}\right|_{a=0} E_1(u_a) = \int_0^{2\pi} d\varphi \int_0^\pi d\psi \left|\frac{\partial h}{\partial\psi}\right|_N^2 \sin\psi(-2\cos\psi) \le 0 .$$

Therefore, for sufficiently small a, we have $E(u_a) = E_1(u_a) + E_2(u_a) < E(\tilde{h})$. This proves that $\tilde{h}$ is not an energy minimizer. $\square$

Theorem 2. *Let S be the Euclidean sphere with radius $k \ge 1$, and N be a Riemannian manifold diffeomorphic to a sphere. Considering $S = (S^2, k^2 ds^2)$ and $N = (S^2, gds^2)$ where S^2 is the unit sphere and $g^2 ds^2$ is the metric on N conformal to that of S^2, the map $u(r, x) = x : K_S \to N$ is an energy minimizer if g satisfies*

$$\int_{s\in S^2} (s \cdot E)\, g^2(s)\, dA_{S^2}(s) = 0$$

for any unit vector E in R^n where $s \cdot E$ is the inner product in R^3.

proof. As in Theorem 1, we may assume that $K_S = (B_1^3, \psi)$ where

$$\psi = dr^2 + k^2 r^2 d\theta^2 + k^2 r^2 \sin^2 \theta d\sigma \text{ and } u(x) = \frac{x}{|x|},$$

Let A be the set of mappings $v \in W^{1,2}(B_1^3, S^2)$ such that $v = id$ on ∂B_1^3 in the sense of trace, and

$$\widetilde{A} = \{v \in W^{1,2}(B_1, S_1)| \ v \text{ is smooth except finite points}, v|_{\partial B_1} = id \ \}.$$

Then, $\widetilde{A}$ is a dense subset of A. [3] So we only need to show that $E(u) \leq E(v)$ for all $v \in \widetilde{A}$. For any $v \in \widetilde{A}$, let $\mathcal{S} = \{p_1, p_2, \ldots, p_k, n_1, n_2, \ldots, n_{k-1}\}$ be the singularities of v where the singular point of positive (negative) degree d are listed $|d|$ times in $p_1, \cdots, p_k(n_1, \cdots, n_{k-1})$. Then, for any $s \in S^2$, $v^{-1}(s)$ contains a union of curves joining n_i to $p_{\sigma(i)}$ and s to $p_{\sigma(k)}$ for some permutation σ of $\{1, 2, \ldots, k\}$. Then, the 1-dimensional Hausdorff measure $\mathcal{H}^1(v^{-1}(s))$ of the set $v^{-1}(s) \subset K_S$ is

$$\mathcal{H}^1(v^{-1}(s)) \geq \Sigma_{i=1}^{k-1} d(n_i, p_{\sigma(i)}) + d(s, p_{\sigma(k)}).$$

Therefore,

$$\mathcal{H}^1(v^{-1}(s)) \geq \min_{\sigma} \left\{ \sum_{i=1}^{k-1} d(n_i, p_{\sigma(i)}) + d(s, p_{\sigma(k)}) \right\} .$$

where the distance d is the distance in K_S.
¿From the coarea formula,

$$\int_{K_S} |\nabla v|_N^2 dV \geq 2 \int_{K_S \setminus \mathcal{S} \cup \{0\}} J(v) dV = 2 \int_N \mathcal{H}^1(v^{-1}(s)) dA_N(s) .$$

where $J(v)$ is the absolute value of the determinant of dv restricted to the space orthogonal to $v^{-1}(s)$ and dV, dA are volume elements of K_S, N, respectively. Hence,

$$\int_{K_S} |\nabla v|_N^2 dV \geq 2 \int_N \min_{\sigma} \left\{ \sum_{i=1}^{k-1} d(n_i, p_{\sigma(i)}) + d(s, p_{\sigma(k)}) \right\} dA_N$$

$$= 2 \int_{S^2} \min_{\sigma} \left\{ \sum_{i=1}^{k-1} d(n_i, p_{\sigma(i)}) + d(s, p_{\sigma(k)}) \right\} d\mu(s) .$$

where $d\mu = dA_N$ is a positive measure on the unit sphere S^2 in R^3.

Now we will show that, for some $X_0 \in B_1^3$,

$$\int_{S^2} \min_\sigma \left\{ \sum_{i=1}^{k-1} d(n_i, p_{\sigma(i)}) + d(s, p_{\sigma(k)}) \right\} d\mu(s) \geq \min_{x \in B^3} \int_{S^2} d(s, X_0) d\mu(s) \ .$$

By approximation, we may assume that $\mu = \sum_{i=1}^{q} \alpha_i \delta_{b_i}$ where $\alpha_i \geq 0$, $\sum \alpha_i = 1$, and δ_{b_i} is the Dirac measure at $b_i \in S^2$. Then, the left side of the above inequality becomes

$$\sum_{l=1}^{q} \{ \alpha_l \sum_{i=1}^{k-1} d(n_i, p_{\sigma_l(i)}) + \alpha_l d(b_l, p_{\sigma_l(k)}) \}$$

for some permutations σ_l of $\{1, 2, \ldots, n\}$. Using the Birkhoff's Theorem inductively (in the exactly same way as in [2] Lemma 7.7 or [11] Lemma 5), we can show that the above summation is bigger than or equal to

$$\min_{X_0 \in B^3} \sum_{l=1}^{q} \alpha_l d(X_0, b_l) \ .$$

for some $X_0 \in N$. Therefore,

$$E(v) \geq 2 \min_{X_0 \in B^3} \int_{S^2} d(s, X_0) d\mu(s) \ .$$

Now we consider $\int_{S^2} d(s, X_0) d\mu(s) = \int_{S^2} d(s, X_0) g^2(s) dA_{S^2}(S)$ for some $X_0 \in B^3$. By rotation, we may assume that $X_0 = (0, 0, l)$, $-1 \leq l \leq 1$, then from the construction of metric d of K_S, we have

$$d(s, X_0) = \sqrt{\sin^2(k\theta) + (\cos k\theta - l)^2} \ , \quad \text{if } \theta \leq \pi/k$$

and $d(s, X_0) = 1 + l$ if $\theta \geq \pi/k$, where θ is the polar angle of s. Since $k \geq 1$, $d(s, X_0) \geq |s - X_0|$ where $|*|$ is the standard norm in R^3. Therefore,

$$\int_{K_S} |\nabla v|^2 dV \geq 2 \min_{X_0 \in B^3} \int_{S^2} d(s, X_0) g^2(s) dA_{S^2} \geq 2 \int_{S^2} |s - X_0| g^2(s) dA_{S^2}$$

and from the condition that $\int_{S^2} \left(s \cdot \frac{X_0}{|X_0|} \right) g^2(s) dA_{S^2}(s) = 0$ we have

$$2 \int_{S^2} |s - X_0| g^2(s) dA_{S^2}(s) \geq 2 \int_{S^2} g^2(s) dA_{S^2}(s) = E(u) \ .$$

This implies the minimizing property of the map u . $\square$

By the above theorems, we may have some informations about the existence of minimizing tangent map at an isolated singularity (peak) of the domain when the tangent cone at the peak is a cone over a standard sphere. When the tangent cone is a cone over a sphere of radius smaller than 1, there doesn't exist nonconstant minimizing tangent map at the peak. On the contrary, where the tangent cone is a cone over a sphere of radius ≥ 1, we may have singularity of minimizing harmonic map at the peak. When the tangent cone is a cone over a general topological 2-sphere, it is not known yet when a nonconstant minimizing tangent map can exist at such a point. The answer to the above question would give crucial information about the regularity of minimizing harmonic maps from singular spaces such as Alexandrove spaces.

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Department of Mathematics
Chung-Ang University, Seoul, Korea 156-756
hshin@cau.ac.kr