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FINITE VOLUME ELEMENT METHODS FOR NONLINEAR PARABOLIC PROBLEMS

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Abstract. In this paper, finite volume element methods for nonlinear parabolic problems are proposed and analyzed. Optimal order error estimates in $W^{1,p}$ and L_p are derived for $2 \le p \le \infty$. In addition, superconvergence for the error between the approximation solution and the generalized elliptic projection of the exact solution (or and the finite element solution) is also obtained.

1. Introduction

Consider the following nonliear parabolic initial boundary value problem

$$
\begin{array}{llrcll}
 & (a) & \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}(a(x,u)\dfrac{\partial u}{\partial x}) & = & f(x,t,u), \quad (x,t) & \in \ (a,b)\times(0,T], \\
(1.1) & (b) & u(x,0) & = & u_0(x), \quad x & \in \ [a,b], \\
 & (c) & u(a,t)=0, \quad \dfrac{\partial u}{\partial x}(b,t) & = & 0, \quad t & \in \ [0,T],
\end{array}
$$

where the functions a, f and u_0 are smooth enough to ensure the analysis validity and $a(x,u)$ is bounded from above and below:

$$0 < a_0 \le a(x,u) \le M, \qquad (x,u) \in [a,b]\times R. \tag{1.2}$$

Since we shall show that the approximate solution is uniformly convergent to the exact solution of (1.1), the above assumptions only need to hold in a neighborhood of the exact solution.

Throughout the paper, C denotes a generic constant independent of discrete parameter h and can have different values at different places. Let $\|\cdot\|_{m,p,G}$ and $|\cdot|_{m,p,G}$ denote norm and seminorm in the Sobolev space $W^{m,p}(G)$ with G and p often omitted when $G = I = [a,b]$ and $p = 2$, respectively. And write $\|\cdot\| = \|\cdot\|_0$.

It is well known that finite difference methods (FDM) and finite element methods (FEM) are two kinds of important numerical methods for solving partial defferential equations. However there exist some obvious defects in both methods. For a long time

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people have spent much effort to modefy the classical numerical methods. Finite volume element (FVE) methods are numerical technique that lie somewhere between FDM and FEM. FVE methods have a flexibility similar to that of FEM for handling complicated solution geometries and boundary conditions, and have a comparable simiplicity for implimentation like FDM when partitions have simply structures. More importantly, numerical solution generated by FVE methods usually maintains mass conservation features, which are desirable in many applications. However, the analysis for FVE methods is far behind that for FDM and FEM. The readers are referenced to [1, 3–6, 8, 10, 11] for elliptic problems, [2, 4, 5, 8, 9] for linear parabolic problems, [7, 11] for linear hyperbolic problems, and [11] for the other partial differential equations.

Our main goal is to discuss FVE methods of one-dimensional nonlinear parabolic problems. We derive the optimal error estimates in $W^{1,p}$ and L_p for $2 \leq p \leq \infty$. Moreover, some superconvergence is also obtained.

The rest of this paper is organized. In Section 2, FVE approximation schemes are formulated in piecewise linear finite element spaces. Some important lemmas are introduced in Section 3, which are essential in our analysis. Main results of this paper are given in Section 4.

2. Finite Volume Element Methods

In this paper we will follow the notations and symboles in [5]. For examples, $T_h = \{I_i; I_i = [x_{i-1}, x_i], 1 \leq i \leq n\}$, and $T_h^* = \{I_i^*; I_i^* = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], 1 \leq i \leq n-1, I_n^* = [x_{n-\frac{1}{2}}, x_n]\}$ denote the primal partition and its dual partition, respectively. The trial function space $U_h \subset H_E^1(I) \equiv \{u \in H^1(I); u(0) = 0\}$ is defined as a piecewise linear function space over T_h and $U_h = span\{\phi_i(x), 1 \leq i \leq n\}$. The test function space $V_h \subset L^2(I)$ is defined as a piecewise constant function space over T_h^* and $V_h = span\{\chi_i(x); 1 \leq i \leq n\}$. Let $h_i = x_i - x_{i-1}$, $h = \max\{h_i; 1 \leq i \leq n\}$. The partitions are assumed to be regular, that is, there exists a constant $\mu > 0$ such that $h_i \geq \mu h$, $i = 1, 2, \cdots, n$. Let Π_h and Π_h^* be interpolation opetators into U_h and V_h, respectively, i.e. $\Pi_h w(x) = \sum_{i=1}^n w(x_i)\phi_i(x)$, $\Pi_h^* w(x) = \sum_{i=1}^n w(x_i)\chi_i(x)$. We have the following properties:

$$\begin{array}{llll} (a) & |w - \Pi_h w|_{m,p} \leq Ch^{k-m}|w|_{k,p}, & m = 0, 1, \quad k = 1, 2, & 1 \leq p \leq \infty, \\ (b) & \|w - \Pi_h^* w\|_{0,p} \leq Ch|w|_{1,p}, & 1 \leq p \leq \infty. & \end{array} \tag{2.1}$$

We now define a bilinear form as follows.

$$a^*(z; u, v) = \sum_{j=1}^n v_j a^*(z; u, \chi_j), \quad v \in V_h, \tag{2.2}$$

where

$$a^*(z; u, \chi_j) = \begin{cases} a(z)_{j-\frac{1}{2}} u'(x_{j-\frac{1}{2}}) - a(z)_{j+\frac{1}{2}} u'(x_{j+\frac{1}{2}}), & u \in H_E^1(I) \cup U_h, \\ a(z)_{j-\frac{1}{2}} \dfrac{u_j - u_{j-1}}{h_j} - a(z)_{j+\frac{1}{2}} \dfrac{u_{j+1} - u_j}{h_{j+1}}, & u \in U_h, \end{cases}$$

with $u' = \frac{\partial u}{\partial x}$, $u_j = u(x_j)$, $v_j = v(x_j)$, $x_{j-\frac{1}{2}} = \frac{1}{2}(x_{j-1} + x_j)$, $a(z)_{j-\frac{1}{2}} = a(z(x_{j-\frac{1}{2}}))$, $u_0 = 0$, $a(z)_{n+\frac{1}{2}} = 0$. For error estimates we next introduce the generalized elliptic projection operator $R_h^* : H_E^1(I) \longrightarrow U_h$ defined by

$$a^*(u; R_h^* w - w, v_h) = 0, \quad v_h \in V_h, \tag{2.3}$$

where u is the solution of (1.1).

Then, the semi-discrete finite volume element approximation scheme is to find a map $u_h(t) : [0,T] \longrightarrow U_h$ such that

$$\begin{array}{lrcl} (a) & (u_{h,t}, v_h) + a^*(u_h; u_h, v_h) & = & (f(t,u_h), v_h), \quad v_h \in V_h, \\ (b) & u_h(0) & = & u_{h,0}, \end{array} \tag{2.4}$$

where $u_{h,0} \in U_h$ is determined by

$$\begin{array}{rcl} A(\xi(0), v_h) & = & a^*(u_0; \xi(0), v_h) + b^*(\xi(0); R_h^* u_0, v_h) + \lambda(\xi(0), v_h) \\ & = & -b^*(\eta(0); R_h^* u_0, v_h), \qquad v_h \in V_h, \end{array} \tag{2.5}$$

here λ is a constant which will be determined in §3, $\xi = u_h - R_h^* u$, $\eta = R_h^* u - u$, and

$$b^*(z; u, v_h) = \sum_{j=1}^{n} (a_u(u_0) z)_{j-\frac{1}{2}} \frac{(u_j - u_{j-1})(v_j - v_{j-1})}{h_j}. \tag{2.6}$$

3. Lemmas

In this section, we will give some lemmas for the error analysis later. We first have for any $u_h \in U_h$,

$$\begin{array}{rcl} |u_h|_{1,p} & = & (\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |u_h'|^p dx)^{\frac{1}{p}} \\ & = & \{\sum_{i=1}^{n} h_i (\frac{u_i - u_{i-1}}{h_i})^p\}^{\frac{1}{p}}. \end{array}$$

Next, we define some discrete norms in U_h. Let

$$\begin{array}{rcl} \|u_h\|_{0,h} & = & \{\sum_{i=1}^{n} h_i (u_{i-1}^2 + u_i^2)\}^{\frac{1}{2}}, \\ \|u_h\|_{1,h} & = & (\|u\|_{0,h}^2 + |u_h|_{1,h}^2)^{\frac{1}{2}}, \\ |||u_h||| & = & (u_h, \Pi_h^* u_h)^{\frac{1}{2}}. \end{array}$$

Lemma 3.1. (See [8, 11]) There exist two positive constants C_1 and C_2, independent of h, such that for any $u_h \in U_h$,

$$\begin{array}{c} C_1 \|u_h\|_{0,h} \le \|u_h\| \le C_2 \|u_h\|_{0,h}, \\ C_1 |||u_h||| \le \|u_h\| \le C_2 |||u_h|||, \\ C_1 \|\Pi_h^* u_h\| \le \|u_h\| \le C_2 \|\Pi_h^* u_h\|, \\ C_1 \|u_h\|_{1,h} \le \|u_h\|_1 \le C_2 \|u_h\|_{1,h}. \end{array}$$

Noting $u_0 = 0$ and $a(z)_{n+\frac{1}{2}} = 0$ in (2.2), we can rewrite the bilinear form $a^*(z;\cdot,\Pi_h^*\cdot)$ as

(3.1)
$$\begin{aligned} a^*(z;u,\Pi_h^* w) &= \sum_{j=1}^{n} a(z)_{j-\frac{1}{2}} u'(x_{j-\frac{1}{2}})(w_j - w_{j-1}), \qquad (u,w) \in H_E^1(I) \cup U_h \times U_h, \\ &= \sum_{j=1}^{n} a(z)_{j-\frac{1}{2}} \frac{(u_j - u_{j-1})(w_j - w_{j-1})}{h_j}, \quad (u,w) \in U_h \times U_h. \end{aligned}$$

Then, according to technique given in [8, 11], we easily derive the following conclusions.

Lemma 3.2. For any u_h, $w_h \in U_h$, we have

$$\begin{aligned} &(a) \quad (u_h, \Pi_h^* w_h) = (w_h, \Pi_h^* u_h), \\ &(b) \quad a^*(z; u_h, \Pi_h^* w_h) = a^*(z; w_h, \Pi_h^* u_h). \end{aligned} \tag{3.2}$$

Lemma 3.3. There exist two positive constants M and α_0, independent of h, and $h_0 > 0$ such that for all $0 < h \le h_0$,

$$\begin{aligned} &(a) \quad |a^*(z; u_h, \Pi_h^* w_h)| \le M \|a(z)\|_{0,\infty} \|u_h\|_1 \|w_h\|_1, \quad u_h, w_h \in U_h, \\ &(b) \quad |a^*(z; u_h, \Pi_h^* u_h)| \ge \alpha_0 \|u_h\|_1^2, \qquad u_h \in U_h. \end{aligned} \tag{3.3}$$

For R_h^* we have the following results.

Lemma 3.4. (See [7]) For $2 \le p \le \infty$, we have

$$\begin{aligned} &(a) \quad \|w - R_h^* w\|_{1,p} \le Ch \|w\|_{2,p}, \\ &(b) \quad \|w - R_h^* w\|_{0,p} \le Ch^2 \|w\|_{3,1}. \end{aligned} \tag{3.4}$$

For any $w \in H_0^1(I)$ we introduce its elliptic projection $R_h w$ defined by

$$a(u; R_h w - w, \chi) = 0, \qquad \chi \in U_h, \tag{3.5}$$

where

$$a(u; w, \chi) = \int_a^b a(u) w' \chi' dx.$$

Then we have, by the well known estimates,

$$\begin{aligned} &(a) \quad \|w - R_h w\|_{0,p} + h \|w - R_h w\|_{1,p} \le Ch^2 \|w\|_{2,p}, 2 \le p \le \infty, \\ &(b) \quad \|(w - R_h w)_t\| + h \|(w - R_h w)_t\|_1 \le Ch^2 (\|w\|_2 + \|w_t\|_2). \end{aligned} \tag{3.6}$$

The following lemma comes from [5].

Lemma 3.5.

$$\|R_h^* w - R_h w\|_{1,p} \le Ch^2 \|w\|_{3,p}, \quad 2 \le p \le \infty. \tag{3.7}$$

We now present a very useful lemma.

Lemma 3.6. (See [5, 8]) For any u_h, $w_h \in U_h$, we have

$$\begin{aligned} &(a) \quad |d(z; u-u_h, w_h)| \le C\|a(z)\|_{1,\infty} h(|u-u_h|_{1,p}|w_h|_{1,p'} + h|u|_{3,q}|w_h|_{1,q'}), \\ &(b) \quad |d(z; u-u_h, w_h)| \le C\|a(z)\|_{1,\infty} h(|u-u_h|_{1,p}|w_h|_{1,p'} + |u|_{2,q}|w_h|_{1,q'}), \end{aligned} \tag{3.8}$$

where

$$d(z; u-u_h, w_h) = a(z; u-u_h, w_h) - a^*(z; u-u_h, \Pi_h^* w_h), \tag{3.9}$$

$$1 \le p, \quad q \le \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Remark 3.1. If $a(z)$ in the bilinear form $a(z;\cdot,\cdot)$ and $a^*(z;\cdot,\Pi_h^*\cdot)$ is replaced by the other function, the inequalities (3.3a) and (3.8) are still valid.

Lemma 3.7. For $k = 0,\ 1,\ 2$ we have

$$\begin{aligned} &(a) \quad \|D_t^k(R_h^* w - w)\|_1 \le Ch \sum_{l=0}^{k} \|D_t^k w\|_2, \\ &(b) \quad \|D_t^k(R_h^* w - w)\| \le Ch^2 \sum_{l=0}^{k} \|D_t^k w\|_{3,p}, \quad p > 1. \end{aligned} \tag{3.10}$$

Proof. Only the case of $k = 1$ will be proved. For simplicity, we set $\zeta = R_h^* w - w$. Differentiating (2.3) with respect to t, we see that

$$a^*(u; \zeta_t, v_h) + a_t^*(u; \zeta, v_h) = 0, \qquad v_h \in V_h, \tag{3.11}$$

where the coefficients of $a_t^*(\cdot;\cdot,\cdot)$ are obtained from differentiating the corresponding coefficients of $a^*(\cdot;\cdot,\cdot)$ with respect to t. Then, by Lemma 3.3 and ε-inequality,

$$\begin{aligned} \frac{\alpha_0}{2}\|\zeta_t\|_1^2 - C\|\zeta\|_1^2 &\le a^*(u; \zeta_t, \Pi_h^*\zeta_t) + a_t^*(u; \zeta, \Pi_h^*\zeta_t) \\ &= a^*(u; \zeta_t, \Pi_h^*(\Pi_h w_t - w_t)) + a_t^*(u; \zeta, \Pi_h^* w_t - w_t) \\ &\le \frac{\alpha_0}{4}\|\zeta_t\|_1^2 + C(\|\Pi_h w_t - w_t\|_1^2 + \|\zeta\|_1^2). \end{aligned}$$

The first conclusion follows from (2.1a) and (3.4a).

To estimate $\|\zeta_t\|$, we can make use of duality argument. For $\phi \in L_2(\Omega)$, let $\Phi \in H_E^1(\Omega)$ be the solution of the auxiliary problem

$$a(u; v, \Phi) = (v, \phi), \qquad v \in H_E^1(\Omega), \tag{3.12}$$

and

$$\|\Phi\|_2 \le C\|\phi\|. \tag{3.13}$$

We differentiate (3.5) with respect to t to get

$$a(u; (R_h w - w)_t, \chi) + a_t(w; R_h w - w, \chi) = 0, \qquad \chi \in U_h, \tag{3.14}$$

and therefor, by (3.14) and (3.11),

$$(3.15)\quad \begin{aligned}(\zeta_t,\phi) &= a(u;\zeta_t,\Phi-R_h\Phi)+d(u;\zeta_t,R_h\Phi)\\ &\quad -a_t^*(u;R_h^*w-R_hw,\Pi_h^*R_h\Phi)+[a_t(u;R_hw-w,R_hw)\\ &\quad -a_t^*(u;R_hw-w,\Pi_h^*R_h\Phi)]+a(u;(R_hw-w)_t,R_hw)\\ &\equiv J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

From (3.6a) and (3.10a), we have

$$|J_1|\le C\|\zeta_t\|_1\|\Phi-R_h\Phi\|_1\le Ch^2(\|w\|_2+\|w_t\|_2)\|\Phi\|_2.$$

To bound J_2, we apply Lemma 3.6 to obtain

$$\begin{aligned}|J_2| &\le Ch\{|\zeta_t|_1|R_h\Phi|_1+h|w_t|_{3,p}|R_h\Phi|_{1,p'}\}\\ &\le Ch^2(\|w\|_2+\|w_t\|_{3,p})\|\Phi\|_2,\quad p>1,\quad \tfrac{1}{p'}+\tfrac{1}{p}=1,\end{aligned}$$

where the boundedness of the elliptic projection operator, $\|R_h\Phi\|_{1,q}\le C\|\Phi\|_{1,q}$, and Sobolev's imbedding inequalities have been used. Similarly, by (3.6a),

$$|J_4|\le Ch^2\|w\|_{3,p}\|\Phi\|_2,\qquad p>1.$$

For J_3, it follows from (3.3a) and (3.7) that

$$|J_3|\le C\|R_h^*w-R_hw\|_1\|R_h\Phi\|_1\le Ch^2\|w\|_3\|\Phi\|_2.$$

Finally, it is easy to see, by integration by parts and (3.6), that

$$\begin{aligned}|J_5| &= |a(u;(R_hw-w)_t,R_h\Phi-\Phi)+a(u;(R_hw-w)_t,\Phi)|\\ &\le C\{\|(R_hw-w)_t\|_1\|R_h\Phi-\Phi\|_1+\|(R_hw-w)_t\|\|\Phi\|_2\}\\ &\le Ch^2(\|w\|_2+\|w_t\|_2)\|\Phi\|_2.\end{aligned}$$

Combining the estimates obtained about J_1–J_5 with (3.15), we have, by (3.13),

$$(\zeta_t,\phi)\le Ch^2(\|w\|_{3,p}+\|w_t\|_{3,p})\|\phi\|,\quad \forall\phi\in L_2(\Omega),$$

which implies (3.10b). this completes the proof.

The following lemma gives another key character of the bilinear form $a^*(\cdot;\cdot,\Pi_h^*\cdot)$.

Lemma 3.8. For $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$, we have

$$(3.16)\quad \begin{aligned}&|a^*(u;v,\Pi_h^*w_h)-a^*(u_h;v,\Pi_h^*w_h)|\\ &\le C|v|_{1,\infty}(\|u-u_h\|_{0,p}+h|u-u_h|_{1,p})\|w_h\|_{1,p'},\\ &\qquad\qquad u_h,\ w_h\in U_h.\end{aligned}$$

Proof. It follows from (2.2) and Hőlder inequality that

$$
\begin{aligned}
& |a^*(u;v,\Pi_h^* w) - a^*(u_h;v,\Pi_h^* w)| \\
= \ & |\sum_{j=1}^n (a(u)-a(u_h))_{j-\frac{1}{2}} v_{j-\frac{1}{2}} (w_j - w_{j-1})| \\
\le \ & C|v|_{1,\infty} \sum_{j=1}^n |(u-u_h)_{j-\frac{1}{2}}||w_j - w_{j-1}| \\
\le \ & C|v|_{1,\infty} \{\sum_{j=1}^n h_j |(u-u_h)_{j-\frac{1}{2}}|^p\}^{\frac{1}{p}} \{\sum_{j=1}^n h_j |\tfrac{w_j - w_{j-1}}{h_j}|^{p'}\}^{\frac{1}{p'}} \\
= \ & C|v|_{1,\infty} \{\sum_{j=1}^n h_j |\phi_{j-\frac{1}{2}}|^p\}^{\frac{1}{p}} |w_h|_{1,p'},
\end{aligned}
\tag{3.17}
$$

where $\phi = u - u_h$.

To obtain the desired estimates, we introduce the affine mapping $\xi = \dfrac{x - x_j}{h_j}$ which maps each element $I_j = [x_{j-1}, x_j]$ into reference element $\hat{J} = [0,1]$ with correspondence $\phi(x) = \hat{\phi}(\xi)$. Thus by the imbedding property $W^{1,p}(\hat{J}) \hookrightarrow C(\hat{J})$, $p \ge 1$,

$$
\begin{aligned}
|\phi(x_{j-\frac{1}{2}})|^p &= |\hat{\phi}(\frac{1}{2})|^p \ \le \ C\|\hat{\phi}\|^p_{1,p,\hat{J}} \\
&= C(\int_0^1 |\hat{\phi}|^p d\xi + \int_0^1 |\hat{\phi}'|^p d\xi).
\end{aligned}
$$

Since

$$
\int_0^1 |\hat{\phi}(\xi)|^p d\xi = \int_{x_{j-1}}^{x_j} |\phi|^p dx h_j^{-1} = h_j^{-1} \|\phi\|^p_{0,p,I_j},
$$

$$
\int_0^1 |\hat{\phi}'(\xi)|^p d\xi = \int_{x_{j-1}}^{x_j} |\phi'(x)|^p h_j^p dx h_j^{-1} = h_j^{p-1} |\phi|^p_{1,p,I_j},
$$

so that

$$
\{\sum_{j=1}^n h_j |\phi_{j-\frac{1}{2}}|^p\}^{\frac{1}{p}} \le C(\|u-u_h\|_{0,p} + h_j |u-u_h|_{1,p}).
$$

This together with (3.17) completes the proof.

Remark 3.2. The technique of lemma 3.8 is easily adapted to give

$$
|b^*(\xi(0); R_h^* u_0, \Pi_h^* w_h)| \le C|R_h^* u_0|_{1,\infty} \|\xi(0)\| \|w_h\|_1.
$$

Then we now can select λ large enough to ensure the coercivity of the bilinear form $A(\cdot, \Pi_h^* \cdot)$ in (2.5) over $H_E^1(I)$.

Writing $\xi = u_h - R_h^* u,\ \tau = R_h^* u - u$ as in §2, we will now turn to the estimates for the initial value errors $\xi(0)$ and $\xi_t(0)$.

Lemma 3.9. Assume that u_0 and $u_{h,0}$ are the initial values problems (1.1) and (2.4), respectively. Then we have

$$\begin{aligned} &(a) \quad \|\xi(0)\|_1 \le Ch^2\|u_0\|_{3,1}, \\ &(b) \quad \|\xi_t(0)\| \le Ch^2\{\|u_0\|_{3,p} + \|u_t(0)\|_{3,p}\}, \quad p > 1, \end{aligned} \tag{3.18}$$

where $u_t(0) = \dfrac{\partial}{\partial x}(a(x,u_0)\dfrac{\partial u_0}{\partial x}) + f(x,0,u_0)$.

Proof. It follows from (2.5) that

$$\begin{aligned} \|\xi(0)\|_1^2 &\le CA(\xi(0),\Pi_h^*\xi(0)) \\ &= C|b^*(\zeta(0);R_h^*u_0,\Pi_h^*\xi(0))| \\ &= C|\sum_{j=1}^{n}(a_u(u_0)\eta(0))_{j-\frac{1}{2}}(R_h^*u_0)'_{j-\frac{1}{2}}(\xi_j-\xi_{j-1})(0)|. \end{aligned}$$

Following a similar argument used in the proof of Lemma 3.8, we have

$$\|\xi(0)\|_1^2 \le C|R_h^*u_0|_{1,\infty}\{\|(R_h^*u-u)(0)\| + h|(R_h^*u-u)(0)|_1\}\|\xi(0)\|_1,$$

which together with Lemma 3.4 derives (3.18a).

To show (3.18b), apply (1.1a), (2.3) and (2.4a) to get the error equation

$$\begin{aligned} &(\xi_t,v_h) + a^*(u_h;\xi,v_h) \\ = \; &(f(u_h)-f(u)-\eta_t,v_h) + a^*(u;R_h^*u,v_h) \\ &-a^*(u_h,R_h^*u,v_h), \qquad v_h\in V_h. \end{aligned} \tag{3.19}$$

Subtracting (2.5) from (3.19) with $t=0$ and taking $v_h = \Pi_h^*\xi_t(0)$, we find (with the argument $t=0$ omitted)

$$\begin{aligned} (\xi_t,\Pi_h^*\xi_t) \;=\; &(f(u_h)-f(u)-\eta_t+\lambda\xi,\Pi_h^*\xi_t) \\ &+[a^*(u;\xi,\Pi_h^*\xi_t) - a^*(u_h;\xi,\Pi_h^*\xi_t)] \\ &+[a^*(u;R_h^*u,\Pi_h^*\xi_t) - a^*(u_h;R_h^*u,\Pi_h^*\xi_t) \\ &+b^*(u_h-u;R_h^*u,\Pi_h\xi_t)] \;\equiv\; J_1+J_2+J_3. \end{aligned} \tag{3.20}$$

Obviously

$$|J_1| \le C(\|\xi\| + \|\eta\| + \|\eta_t\|)\|\xi_t\|.$$

For J_2, we have, by (3.3a), the imbedding theorems and inverse properties,

$$\begin{aligned} |J_2| &\le C\|a(u)-a(u_h)\|_{0,\infty}\|\xi\|_1\|\xi_t\|_1 \\ &\le C(\|\xi\|_{0,\infty}+\|\eta\|_{0,\infty})\|\xi\|_1\|\xi_t\|_1 \\ &\le Ch^{-1}(\|\xi\|_1+\|\eta\|_1)\|\xi\|_1\|\xi_t\|. \end{aligned}$$

As for J_3, we note that

$$\begin{aligned} J_3 &= \sum_{j=1}^{n}[a_u(u)(u_h-u)-(a(u_h)-a(u))]_{j-\frac{1}{2}}(R_h^*u)'_{j-\frac{1}{2}}(\xi_{t,j}-\xi_{t,j-1}) \\ &= \sum_{j=1}^{n}\int_0^1 [a_u(u)-a_u(u+s(u_h-u))]_{j-\frac{1}{2}}ds(u_h-u)_{j-\frac{1}{2}}(R_h^*u)'_{j-\frac{1}{2}}(\xi_{t,j}-\xi_{t,j-1}), \end{aligned}$$

then

$$\begin{aligned} |J_3| &\leq C\sum_{j=1}^{n}(\xi+\eta)^2_{j-\frac{1}{2}}(R_h^*u)'_{j-\frac{1}{2}}(\xi_{t,j}-\xi_{t,j-1}) \\ &\leq C|R_h^*u|_{1,\infty}\|u_h-u\|_{0,\infty}\sum_{j=1}^{n}|u_h-u|_{j-\frac{1}{2}}|\xi_{t,j}-\xi_{t,j-1}|. \end{aligned}$$

The argument in the proof of Lemma 3.8 yields

$$\begin{aligned} |J_3| &\leq C\|u_h-u\|_1(\|u-u_h\|+h|u-u_h|_1)\|\xi_t\|_1 \\ &\leq Ch^{-1}(\|\xi\|_1+\|\eta\|_1)\{\|\xi\|+\|\eta\|+h(\|\xi\|_1+\|\eta\|_1)\}\|\xi_t\|. \end{aligned}$$

Collecting the estimates above with (3.20), we have

$$|||\xi_t|||^2 \leq C\{1+h^{-1}(\|\xi\|_1+\|\eta\|_1)\}(\|\xi\|_1+\|\eta_t\|+h\|\eta\|_1)\|\xi_t\|.$$

Then applying Lemmas 3.4, 3.7 and (3.18a) leads to the second result of this lemma.

Now, let us consider estimates for ξ and ξ_t.

Lemma 3.10. Assume that u and u_h are the solutions of (1.1) and (2.4), respectively, then we have

$$\begin{aligned} &\|\xi_t\|+\|\xi\|_1+(\int_0^t\|\xi_t\|_1^2d\tau)^{\frac{1}{2}} \\ &\leq Ch^2\{\|u_0\|_{3,p}+\|u_t(0)\|_{3,p}+\sum_{l=0}^{2}\int_0^t\|D_t^lu\|_{3,p}d\tau\}, p>1, \quad 0\leq t\leq T. \end{aligned} \tag{3.21}$$

Proof. We differentiate (3.19) with respect to t to get

$$\begin{aligned} &(\xi_{tt},v_h)+a^*(u_h;\xi_t,v_h) \\ =\ &((f(u_h)-f(u)-\eta_t)_t,v_h) \\ &+[a^*(u;(R_h^*u)_t,v_h)-a^*(u_h;(R_h^*u)_t,v_h)] \\ &+[a_t^*(u;R_hu,v_h)-a_t^*(u_h;R_hu,v_h)] \\ &-a_t^*(u_h;\xi,v_h), \qquad v_h\in V_h. \end{aligned}$$

Setting $v_h = \Pi_h^* \xi_t$ and using Lemmas 3.2, 3.3 and 3.8, we have, by the boundedness of $\|R_h^* u\|_{1,\infty}$ and $\|(R_h^* u)_t\|_{1,\infty}$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|||\xi_t|||^2 + \alpha_0 \|\xi_t\|_1^2 \\
\leq\ & C\{\|\xi\|_1 + \|\eta\| + \|\xi_t\| + \|\eta_t\| + \|\eta_{tt}\| + h(\|\eta\|_1 + \|\xi_t\|_1 + \|\eta_t\|_1)\}\|\xi_t\|_1 \\
\leq\ & (\varepsilon + Ch)\|\xi_t\|_1^2 + C\{\|\xi\|_1^2 + \|\eta\|^2 + \|\xi_t\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2\}.
\end{aligned}$$

Hence, by elliminating the first term on the right hand side and applying Lemmas 3.1, 3.4, 3.7 and 3.9, we obtain

$$\begin{aligned}
&\|\xi_t\|^2 + \int_0^t \|\xi_t\|_1^2 d\tau \\
\leq\ & C\{h^2(\|u_0\|_{3,p} + \|u_t(0)\|_{3,p} + \sum_{l=0}^{2}\int_0^t \|D_t^l u\|_{3,p} d\tau) + \int_0^t (\|\xi\|_1^2 + \|\xi_t\|^2) d\tau\}.
\end{aligned}$$

Observing that

$$\begin{aligned}
\|\xi\|_1^2 &= \|\xi(0)\|_1^2 + \int_0^t \frac{d}{dt}[(\xi,\xi) + (\xi',\xi')] d\tau \\
&\leq \|\xi(0)\|_1^2 + \varepsilon \int_0^t \|\xi_t\|_1^2 d\tau + C\int_0^t \|\xi\|_1^2 d\tau,
\end{aligned}$$

then, by (3.18a), we have

$$\begin{aligned}
&\|\xi_t\|^2 + \|\xi\|_1^2 + \int_0^t \|\xi_t\|_1^2 d\tau \\
\leq\ & C\{h^2(\|u_0\|_{3,p} + \|u_t(0)\|_{3,p} + \sum_{l=1}^{2}\int_0^t \|D_t^l u\|_{3,p} d\tau) \\
& + \int_0^t (\|\xi_t\|^2 + \|\xi\|_1^2) d\tau\}.
\end{aligned}$$

The result of this lemma follows by applying Gronwall's Lemma.

4. Main results

In this section u and u_h denote the solutions of (1.1) and (2.4), respectively. We will employ lemmas given in §3 to derive optimal $W^{1,p}$ and L_p error estimates of $u - u_h$ and some $W^{1,p}$ superconvergence results.

Writing $\xi = u_h - R_h^* u$ and $\zeta = R_h^* u - u$ as before, we shall begin by demonstrating a superconvergence result of ξ.

Theorem 4.1. Under the conditions of Lemmas 3.6, 3.7 and 3.10, we have

$$\|\xi\|_{1,p} \leq Ch^2\{\|u_0\|_{3,p} + \|u_t(0)\|_{3,p} + \sum_{i=0}^{2}\int_0^t \|D_t^i u\|_{3,p} d\tau\}, \quad 2 \leq p \leq \infty. \tag{4.1}$$

proof: (i) Let us first consider the case of $2 \leq p < \infty$.

We find from Lemma 3.10 that

$$\begin{aligned}\|\nabla u_h\|_{0,\infty} &\leq \|\nabla \xi\|_{0,\infty} + \|\nabla R_h^* u\|_{0,\infty} \\ &\leq Ch^{-1}\|\nabla \xi\| + \|\nabla R_h^* u\|_{0,\infty} \leq C,\end{aligned}$$

and hence $a(u_h) \in W^{1,\infty}(\Omega)$. In order to show (4.1) we now introduce the auxiliary problem. For $\psi \in L^2(\Omega)$, let $\Psi \in H_0^1(\Omega)$ be the solution of

$$a(u_h; v, \Psi) = -(v, \psi_x), \qquad v \in H_0^1(\Omega), \tag{4.2}$$

and

$$\|\Psi\|_{1,p'} \leq C\|\psi\|_{0,p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1. \tag{4.3}$$

We then know from (3.5), (3,19), Lemmas 3.6 and 3.8 that

$$\begin{aligned}(\xi_x, \psi) &= a(u_h; \xi, \Psi) \\ &= d(u_h; u - u_h, R_h\Psi) + d(u_h; u - R_h^* u, R_h\Psi) \\ &\quad +(f(u_h) - f(u) - \xi_t - \eta_t, \Pi_h^* R_h\Psi) \\ &\quad +[a^*(u; R_h^* u, \Pi_h^* R_h\Psi) - a^*(u_h; R_h^* u, \Pi_h^* R_h\Psi] \\ &\leq C\{h(\|\xi\|_{1,p} + \|\eta\|_{1,p} + h\|u\|_{3,p}) + \|\xi\|_{0,p} + \|\eta\|_{0,p} + \|\xi_t\| + \|\eta_t\|\}\|\Pi_h^* R_h\Psi\|_{1,p'}.\end{aligned}$$

Then, by (4.3) and the imbedding property $W^{1,2}(I) \hookrightarrow L_p(I)$,

$$\begin{aligned}\|\xi\|_{1,p} &\leq C|\xi|_{1,p} = C \sup_{\psi \in L_{p'}(I)} \frac{(\xi_x, \psi)}{\|\psi\|_{0,p'}} \\ &\leq Ch\|\xi\|_{1,p} + C(\|\xi\|_1 + \|\xi_t\| + h\|\eta\|_{1,p} + \|\eta\|_{0,p} + \|\eta_t\|).\end{aligned}$$

After eliminating the first term on the right hand side, the results (4.1) for $2 \leq p < \infty$ now follows by Lemmas 3.7 and 3.10.

(ii) Let us next consider the case of $p = \infty$.

For this purpose, we need to apply the Green function. Following [12], the discrete Green function $g_z^n \in U_h$ associated with $a(u_h; \cdot, \cdot)$ satisfies

$$a(u_h; w, g_z^h) = w_z(z), \quad z \in I, \quad w \in U_h.$$

Then

$$\xi_z(z) = a(u_h; \xi, g_z^h).$$

Consequently, upon replacing $R_h\Psi$, p and p' by g_z^h, ∞ and 1 in part (i), respectively, we can easily derive the conclusion by applying $\|g_z^h\|_{1,1} \leq C^{[12]}$. The proof is completed.

$W^{1,p}$ and L_p norms error estimates for $u-u_h$ are then an immediate consequence of Theorem 4.1 combined with Lemma 3.4.

Theorem 4.2. Under the same conditions of Theorem 4.1, we have

$$\begin{aligned}\|u-u_h\|_{0,p}+h\|u-u_h\|_{0,p} &\leq Ch^2\{\|u_0\|_{3,p}+\|u_t(0)\|_{3,p}+\sum_{i=0}^{2}\int_0^t\|D_t^iu\|_{3,p}dt\},\\ &\qquad 2\leq p\leq\infty.\end{aligned}\tag{4.4}$$

We will now turn to superconvergence estimates between the finite volum element solution and finite element solution. Let U be the finite element solution of (1,1), i.e., the map $U(t):[0,T]\longrightarrow U_h$ satisfies

$$\begin{aligned}(a)\quad (U_t,\chi)+a(U;U,\chi) &= (f(U),\chi),\quad \chi\in U_h,\\ (b)\qquad\qquad\qquad U(0) &= U_0,\end{aligned}\tag{4.5}$$

where $U_0\in U_h$ is determined by

$$\begin{aligned}&(a(u_0)\nabla(U_0-R_hu_0),\nabla\chi)+(a_u(u_0)(U_0-R_hu_0)\nabla u_0,\nabla\chi)\\ &+\mu(U_0-R_hu_0,\chi)=-(a_u(u_0)(R_hu_0-u_0)\nabla u_0,\nabla\chi),\quad \chi\in U_h,\end{aligned}\tag{4.6}$$

with some constant μ.

From [9], it holds in R^1 that

$$\begin{aligned}\|U-R_hu\|_{1,p} &\leq Ch^2\{\|u_0\|_2+\|u_0\|_{2,4}^2+\|u_t(0)\|_2\\ &\quad+\sum_{i=0}^{2}\int_0^t\|D_t^iu\|_{2,p}d\tau\},\quad 2\leq p\leq\infty.\end{aligned}\tag{4.7}$$

We will prove the following estimates.

Theorem 4.3. If, in addition the hypotheses of Theorem 4.1, $u_0\in W^{2,4}(I)$, then we have

$$\|U-u_h\|_{1,p}\leq Ch^2,\quad 2\leq p\leq\infty.\tag{4.8}$$

proof: We know from (4.2), (3.5) and (3.19) that

$$\begin{aligned}&((U-u_h)_x,\psi)=a(u_h;U-u_h,R_h\Psi)\\ =\ &a(u_h;U-R_hu,R_h\Psi)+a(u_h;u-u_h,R_h\Psi)\\ =\ &a(u_h;U-R_hu,R_h\Psi)+d(u_h;u-u_h,R_h\Psi)\\ &+(\xi_t+\eta_t+f(u)-f(u_h),\Pi_h^*R_h\Psi)\\ &+a^*(u_h;R_h^*u,\Pi_h^*R_h\Psi)-a^*(u;R_h^*u,\Pi_h^*R_h\Psi).\end{aligned}$$

Then the conclusion follows by applying (4.7) and the technique in the proof of Theorem 4.1.

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