

**AN ACCURATE APPROXIMATION FOR THE FIBER
REFRACTIVE INDEX PROFILE VIA TAYLOR'S EXPANSION
FORMULA**

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ABSTRACT. By the use of Taylor's expansion formula with the integral remainder, an accurate approximation for the fiber refractive index profile is given.

1. INTRODUCTION

Transversal interferometry of optical fibers has been established in the last twenty years as one of the most useful and accurate tools for refractive index profiling. As shown in [3], since introduction of accurate data reduction formulae by Sochacki [1], the measurement speed and precision depends mainly on the data acquisition techniques.

This, indeed, may be argued if an accurate numerical approximation of the refractive index representation formula obtained in [1]

$$n(u) := \exp \left[\frac{1}{\pi} \int_u^1 \frac{\psi(s)}{\sqrt{s^2 - u^2}} ds \right], 0 \leq u \leq 1;$$

is assumed.

For some preliminary studies on the numerical approximation of the refractive index, see [4] and the references therein.

In what follows, based on the Taylor's formula with integral remainder, we point out a numerical approach in approximating the refractive index in the assumption that the derivatives up to a certain order of the deflection function ψ are known. Since in experiments only discrete values of ψ may be obtained, we consider as the actual deflection function ψ to be the best fitting curve to data (polynomial etc...), which are differentiable up to a certain order and may be obtained, for example, by classical interpolative techniques [2].

The theoretical error analysis is performed and the uniform convergence of the numerical procedure established.

Some numerical example showing a very good accuracy are also mentioned.

Before we present our numerical approximation for the refractive index, some preliminary facts on the phase-stepping interferometry and optical fiber profiling are necessary. To briefly describe this, we follow [3].

1.1. Basics of the Phase-Stepping Interferometry. The intensity distribution, as observed in the exit plane of the interferometer of any kind, can be generally described as

$$(1) \quad I = I_0 + I_c \cos [\Delta(x, y) - \Phi_B]$$

or, equivalently:

$$(2) \quad I = I_0 + I_c \cos \Delta(x, y) \cos \Phi_B + I_c \sin \Delta(x, y) \sin \Phi_B$$

where I_0 is the background intensity, I_c is the interference pattern amplitude governing the image contrast, $\Delta(x, y)$ is the phase change introduced by investigated object, and Φ_B is the bias (phase difference between the interfering wave fronts, a quantity characteristic for the instrument). When Φ_B is constant, the observed field is an homogeneous one (in absence of $\Delta(x, y)$). When Φ_B is a linear function of space coordinates, a fringe pattern appears.

From (1) follows that the intensity I in any interferogram is a periodic function of the bias Φ_B with 2π period, so it can be alternately represented by a Fourier series in terms of Φ_B :

$$(3) \quad I = a_0 + \sum_{i=1}^{\infty} [a_i \cos (i \cdot \Phi_B) + b_i \sin (i \cdot \Phi_B)]$$

where the Fourier coefficients are, as usual, given by

$$(4) \quad \begin{aligned} a_i &= \frac{1}{2\pi} \int_0^{2\pi} I(\Phi_B) \cos (i \cdot \Phi_B) d\Phi_B, i = 1, 2, \dots \\ b_i &= \frac{1}{2\pi} \int_0^{2\pi} I(\Phi_B) \sin (i \cdot \Phi_B) d\Phi_B, i = 1, 2, \dots \end{aligned}$$

If, for simplicity, we take into consideration only the constant and the first order terms of expansion (3) and relating them to the formula (2) the exit intensity I maybe finally expressed as (see for example [3]):

$$(5) \quad I = a_0 + a_1 \cos (\Phi_B) + b_1 \sin (\Phi_B)$$

with

$$\begin{aligned}
 (6) \quad a_0 &= \frac{1}{2\pi} \int_0^{2\pi} I(\Phi_B) d\Phi_B = I_0 \\
 a_1 &= \frac{1}{2\pi} \int_0^{2\pi} I(\Phi_B) \cos(\Phi_B) d\Phi_B = I_1 \cos \Delta(x, y) \\
 b_1 &= \frac{1}{2\pi} \int_0^{2\pi} I(\Phi_B) \sin(\Phi_B) d\Phi_B = I_1 \sin \Delta(x, y).
 \end{aligned}$$

From this comparison follows that the ratio of the first order coefficients of the Fourier expansion gives the tangent of the phase change $\Delta(x, y)$, hence it can be obtained as (see for example [3]):

$$(7) \quad \Delta(x, y) = \arctan \left(\frac{b_1}{a_1} \right).$$

To simplify the computation of coefficients a_1 and b_1 the integrals in (6) may be replaced by finite sums according to any quadrature rule. For periodic functions the Bessel formulae are recommended. In such a case a_1 and b_1 may be approximated by:

$$\begin{aligned}
 (8) \quad a_1 &= \frac{1}{N} \sum_{k=0}^{2N-1} I_k \cos \left(\frac{k\pi}{N} \right), \\
 b_1 &= \frac{1}{N} \sum_{k=0}^{2N-1} I_k \sin \left(\frac{k\pi}{N} \right),
 \end{aligned}$$

where I_k are the exit intensity distributions observed with bias $\Phi_B = \Phi_k = \frac{k\pi}{N}$, and $2N$ is the number of equidistant points Φ_k here the intensity samples are taken. They have to run over the entire period of I , so that $I(\Phi_0) = I(\Phi_{2N})$.

With the smallest possible number of samples ($2N = 4$), the phase of interest can be retrieved from:

$$(9) \quad \Delta(x, y) = \arctan \left[\frac{I_1(x, y) - I_3(x, y)}{I_0(x, y) - I_2(x, y)} \right]$$

where I_0, I_1, I_2 and I_3 are intensity distributions as observed with $\Phi_B = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ respectively. That means that by introducing a $\frac{\pi}{2}$ step in the bias Φ_B the phase function $\Delta(x, y)$ can be unequivocally reconstructed from four interferograms.

The technique briefly described above following [3], was first applied by Brunning et al. [5] to study optical surfaces in Twyman-Green interferometer and is commonly called the phase stepping technique. It has quickly found applications in a great number of wavefront measuring devices for optical elements control, surface profiling, recognition of shape and deformation, as well as measurement of vibration amplitudes and phases (see for example [6]). Many other modifications of the original algorithm were

also published, see for example the review paper by Creath [7] and the references therein.

1.2. Applications to Optical Fibers Profiling. As shown in [3], the phase stepping technique can also be applied to retrieve the phase information necessary in the refractive index profiling of optical fibers.

We will briefly describe how this can be done.

Suppose that the basic interference formula (1) describes the interference pattern as observed in any transverse wavefront shearing interferometer, where the fiber is investigated. Choosing the coordinate system so that the axis of the fiber image is in the x -axis, the x dependence can be eliminated from formulae (1) – (9) due to the cylindrical symmetry of the object and its homogeneous along the axis. When the wavefront shear in direction perpendicular to the fiber axis exceeds its diameter, two fiber images can be observed and function $\Phi(y) = \frac{\lambda}{2\pi}\Delta(y)$ represents the optical path length through the fiber. If the shear in the same direction is very small (comparing with fiber's diameter), then $\frac{\lambda}{2\pi}\Delta(y)$ is proportional to the derivative of the optical path length $\Phi(y)$ in the shear direction, where the image shear s is the proportionality constant (see for example [8]):

$$(10) \quad \frac{d\Phi(y)}{dy} = \frac{1}{s} \cdot \frac{\lambda}{2\pi} \cdot \Delta(y).$$

In this case the fiber is said to be observed in *differential interference contrast* (DIC).

In any case the derivative (with respect to the radial position in observation plane) of the optical path length $\Phi(y)$ through the fiber can be either directly obtained or numerically constructed from phase data $\Delta(y)$ retrieved from four intensity distribution measurements as required by formula (9). This derivative is essential in the refractive index profile retrieval algorithm proposed in [1]. In the first step of this algorithm it serves to calculate the *deflection function* $\psi(\tilde{y})$

$$(11) \quad \psi(\tilde{y}) = -\arcsin \left[\frac{d\Phi(y)}{dy} \right]$$

where \tilde{y} is the incident ray position in object plane (with illumination perpendicular to the fiber axis) and can be related to the image plane coordinate y by an appropriate mapping relation depending on the experiment conditions [1], [5].

The deflection function calculated in such of way is parametrically related to the fiber refractive index profile

$$(12) \quad n(u) := \exp \left[\frac{1}{\pi} \int_u^1 \frac{\psi(\tilde{y})}{\sqrt{\tilde{y}^2 - u^2}} d\tilde{y} \right]$$

with parameter u defined as

$$(13) \quad u = r/n(u), 0 \leq u \leq 1$$

and r being the radial position measured from the fiber axis, for simplicity normalized to 1 at its edge (as well as the refractive index).

From the above considerations it follows that the refractive index profile of optical fiber can be quite easily reconstructed by means of formulae (9) – (12) from four homogeneous field interferograms obtained with $\pi/2$ stepped bias Φ_B . Such experiment was successfully performed in [1], but we omit the details.

2. A NUMERICAL APPROXIMATION OF THE REFRACTIVE INDEX PROFILE

It is obvious that, in practice, an accurate computation of the fiber refractive index profile provided by the analytic formula:

$$(14) \quad n(u) := \exp \left[\frac{1}{\pi} \int_u^1 \frac{\psi(s)}{\sqrt{s^2 - u^2}} ds \right], 0 \leq u \leq 1$$

depends on the numerical accuracy in approximating the integral

$$(15) \quad I(\psi, u) := \frac{1}{\pi} \int_u^1 \frac{\psi(s)}{\sqrt{s^2 - u^2}} ds, 0 \leq u \leq 1.$$

A natural approach, if information on the derivatives of ψ are available, is to compute (15) by the use of Taylor's formula

$$(16) \quad \psi(s) = \psi(u) + \sum_{k=1}^n \frac{(s-u)^k}{k!} \psi^{(k)}(u) + \frac{1}{n!} \int_u^s (s-t)^n \psi^{(n+1)}(t) dt$$

for any $s \in [u, 1]$, $u \in [0, 1]$, where $n \geq 1$ is a natural number. Then we get

$$(17) \quad \begin{aligned} I(\psi, u) &= \frac{1}{\pi} \psi(u) \int_u^1 \frac{ds}{\sqrt{s^2 - u^2}} \\ &+ \frac{1}{\pi} \sum_{k=1}^n \frac{\psi^{(k)}(u)}{k!} \int_u^1 (s-u)^{k-1} \sqrt{\frac{s-u}{s+u}} ds \\ &+ \frac{1}{\pi n!} \int_u^1 \frac{ds}{\sqrt{s^2 - u^2}} \left(\int_u^s (s-t)^n \psi^{(n+1)}(t) dt \right) \\ &= \frac{1}{\pi} M_0(u) \psi(u) + \frac{1}{\pi} \sum_{k=1}^n M_k(u) \psi^{(k)}(u) + R_n(\psi, u) \end{aligned}$$

where

$$(18) \quad M_0(u) : = \int_u^1 \frac{ds}{\sqrt{s^2 - u^2}},$$

$$(19) \quad M_k(u) : = \frac{1}{k!} \int_u^1 (s-u)^{k-1} \sqrt{\frac{s-u}{s+u}} ds$$

and $R_n(\psi, u)$ is the remainder in formula (17), *i.e.*,

$$R_n(\psi, u) := \frac{1}{\pi n!} \int_u^1 \frac{ds}{\sqrt{s^2 - u^2}} \left(\int_u^s (s-t)^n \psi^{(n+1)}(t) dt \right).$$

We note, using Maple 6, that we may compute the functions $M_0(u)$, $M_k(u)$, for $k = 1, 2, \dots$. Since the expressions of $M_k(u)$, for $k = 2, 3, \dots$ are quite complicate we will not present them here explicitly. We mention only the expressions for $M_0(u)$ and $M_1(u)$:

$$M_0(u) = \ln \left(1 + \sqrt{1 - u^2} \right) - \ln(u),$$

$$\begin{aligned} M_1(u) & : = [\sqrt{(1-u)/(u+1)} \times \sqrt{1-u^2} \\ & - \sqrt{(1-u)/(u+1)} \times u \times \ln(1 + \sqrt{1-u^2}) \\ & + \sqrt{(1-u)/(u+1)} \times u \times \sqrt{1-u^2} \\ & - \sqrt{(1-u)/(u+1)} \times u^2 \times \ln(1 + \sqrt{1-u^2}) \\ & + \sqrt{1/u} \times u^{3/2} \times \ln(u) \times \sqrt{1-u^2} / (\sqrt{1-u^2})]. \end{aligned}$$

The functions $M_k(u)$ for $k = 2, 3, \dots, 9$; which were completely computed by a Maple program will be used in the computer implementation of the approximation proposed for the refractive index profile $n(u)$.

3. ERROR ANALYSIS

Using the properties of the integral, we have

$$\begin{aligned} (20) \quad |R_n(\psi, u)| & \leq \frac{1}{\pi n!} \int_u^1 \left| \int_u^s (s-t)^n \psi^{(n+1)}(t) dt \right| \frac{ds}{\sqrt{s^2 - u^2}} \\ & = : B_n(\psi, u). \end{aligned}$$

If $\psi^{(n+1)}$ is bounded in the interval $[0, 1]$ and if we denote

$$\left\| \psi^{(n+1)} \right\|_{[u,s],\infty} := \sup_{t \in [u,s]} \left| \psi^{(n+1)}(t) \right| < \infty$$

then we have

$$\begin{aligned} (21) \quad \left| \int_u^s (s-t)^n \psi^{(n+1)}(t) dt \right| & \leq \sup_{t \in [u,s]} \left| \psi^{(n+1)}(t) \right| \int_u^s (s-t)^n dt \\ & \leq \frac{(s-u)^{n+1}}{n+1} \left\| \psi^{(n+1)} \right\|_{[u,1],\infty}. \end{aligned}$$

If $\psi^{(n+1)}$ is p -integrable ($p > 1$) in the interval $[0, 1]$ and if we denote

$$\left\| \psi^{(n+1)} \right\|_{[u,s],p} := \left(\int_u^s \left| \psi^{(n+1)}(t) \right|^p dt \right)^{\frac{1}{p}}, p \geq 1;$$

then by Hölder's integral inequality for $p > 1$ ($\frac{1}{p} + \frac{1}{q} = 1$), we deduce

$$(22) \quad \left| \int_u^s (s-t)^n \psi^{(n+1)}(t) dt \right| \leq \left(\int_u^s (s-t)^{nq} dt \right)^{\frac{1}{q}} \times \left(\int_u^s |\psi^{(n+1)}(t)|^p dt \right)^{\frac{1}{p}}$$

$$\leq \frac{(s-u)^{n+\frac{1}{q}}}{(nq+1)^{1/q}} \left\| \psi^{(n+1)} \right\|_{[u,1],p}.$$

If $\psi^{(n+1)}$ is integrable in the interval $[0, 1]$ and if we denote

$$\left\| \psi^{(n+1)} \right\|_{[u,s],1} := \int_u^s |\psi^{(n+1)}(t)| dt;$$

then we have

$$(23) \quad \left| \int_u^s (s-t)^n \psi^{(n+1)}(t) dt \right| \leq \sup_{t \in [u,s]} (s-t)^n \int_u^s |\psi^{(n+1)}(t)| dt$$

$$\leq (s-u)^n \left\| \psi^{(n+1)} \right\|_{[u,1],1}.$$

Using (21) – (23) we may deduce the following upper bounds for $B_n(\psi, u)$

$$(24) \quad B_n(\psi, u)$$

$$(25) \quad \leq \frac{1}{\pi} \times \begin{cases} \frac{1}{(n+1)!} \int_u^1 (s-u)^n \sqrt{\frac{s-u}{s+u}} \left\| \psi^{(n+1)} \right\|_{[u,s],\infty} dt \\ \frac{1}{n!(nq+1)^{1/q}} \int_u^1 (s-u)^{n+\frac{1}{q}-1} \sqrt{\frac{s-u}{s+u}} \left\| \psi^{(n+1)} \right\|_{[u,s],p} dt \\ \frac{1}{n!} \int_u^1 (s-u)^{n-1} \sqrt{\frac{s-u}{s+u}} \left\| \psi^{(n+1)} \right\|_{[u,s],1} dt \end{cases}$$

$$(26) \quad \leq \frac{1}{\pi} \times \begin{cases} \frac{1}{(n+1)!} \left\| \psi^{(n+1)} \right\|_{[u,1],\infty} \int_u^1 (s-u)^n \sqrt{\frac{s-u}{s+u}} dt \\ \frac{1}{n!(nq+1)^{1/q}} \left\| \psi^{(n+1)} \right\|_{[u,1],p} \int_u^1 (s-u)^{n+\frac{1}{q}-1} \sqrt{\frac{s-u}{s+u}} dt \\ \frac{1}{n!} \left\| \psi^{(n+1)} \right\|_{[u,1],1} \int_u^1 (s-u)^{n-1} \sqrt{\frac{s-u}{s+u}} dt \end{cases}.$$

Since, the following type of integrals occur

$$N(\alpha, u) := \int_u^1 (s-u)^\alpha \sqrt{\frac{s-u}{s+u}} dt, \alpha \geq 0, u \in [0, 1]$$

and these type of integrals can not be exactly computed for general α (note, however, that for particular small α Maple 6 may provide the exact value), some evaluations of $N(\alpha, u)$ are highly desirable.

Having this in mind, we observe that, the simplest way to obtain an upper bound for $N(\alpha, u)$, is to take into account that

$$0 \leq \frac{s-u}{s+u} \leq 1, \text{ for } u \in [0, 1], u \leq s \leq 1$$

giving

$$(27) \quad N(\alpha, u) \leq \int_u^1 (s-u)^\alpha dt = \frac{(1-u)^{\alpha+1}}{(\alpha+1)}, \alpha \geq 0, u \in [0, 1].$$

Using (23) and (24) we get the following simple estimate for the absolute value of the remainder $R_n(\psi, u)$ in approximating the integral $I(u)$ with the analytic expression

$$(28) \quad A_n(\psi, u) := \frac{1}{\pi} M_0(u) \psi(u) + \frac{1}{\pi} \sum_{k=1}^n M_k(u) \psi^{(k)}(u)$$

where $M_0(u), M_k(u)$ were defined in the previous section,

$$(29) \quad |R_n(\psi, u)| \leq \frac{1}{\pi} \times \begin{cases} \frac{(1-u)^{n+1}}{(n+1)!(n+1)} \left\| \psi^{(n+1)} \right\|_{[u,1],\infty} \\ \frac{(1-u)^{n+\frac{1}{q}}}{n!(nq+1)^{1/q+1}} \left\| \psi^{(n+1)} \right\|_{[u,1],p} \\ \frac{(1-u)^n}{n!n} \left\| \psi^{(n+1)} \right\|_{[u,1],1} \end{cases}.$$

The equation (29) shows that the remainder $R_n(\psi, u)$ is (rapidly) uniformly convergent to 0 as $n \rightarrow \infty$, meaning that the approximation of $I(u)$ by $A_n(\psi, u)$ is accurate for enough large n .

4. NUMERICAL EXAMPLES

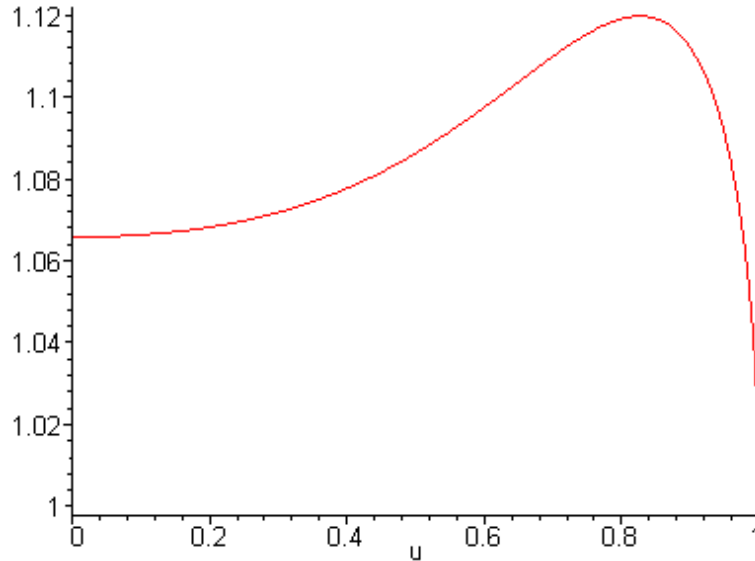


Figure 1.

In what follows, we will numerically compare the approximate value for the refractive index profile

$$L_9(\psi, u) := \exp \left[\frac{1}{\pi} \left(M_0(u)\psi(u) + \sum_{k=1}^9 M_k(u)\psi^{(k)}(u) \right) \right], 0 \leq u \leq 1;$$

with the exact value

$$n(u) := \exp \left[\frac{1}{\pi} \int_u^1 \frac{\psi(s)}{\sqrt{s^2 - u^2}} ds \right], 0 \leq u \leq 1.$$

If we take for example $\psi(u) = u^5, u \in [0, 1]$, then the exact value computed by Maple 6 is

$$\begin{aligned} n(u) &= ((1/5 * \text{sqrt}(1 - u^2) + 4/15 * u^2 * \text{sqrt}(1 - u^2) \\ &\quad + 8/15 * u^4 * \text{sqrt}(1 - u^2))/Pi), \\ u &\in [0, 1]. \end{aligned}$$

The plot of this function $n(u)$ is embodied in the Figure 1.

The plot of the error in approximating $n(u)$ by $L_9(\psi, u)$ is embodied in Figure 2 The

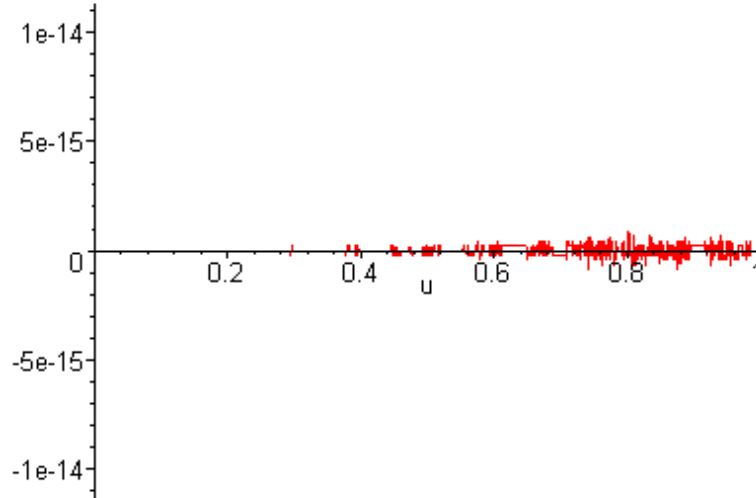


Figure 2.

accuracy is of order 10^{-15} .

If we take for example the function

$$\psi(u) = \arctan(u + 1), u \in [0, 1]$$

then Maple 6 is unable to compute exactly the value of the refractive index $n(u)$. An approximation provided by $L_9(u)$ with a good accuracy (of order 10^{-14}) is embodied in Figure 3.

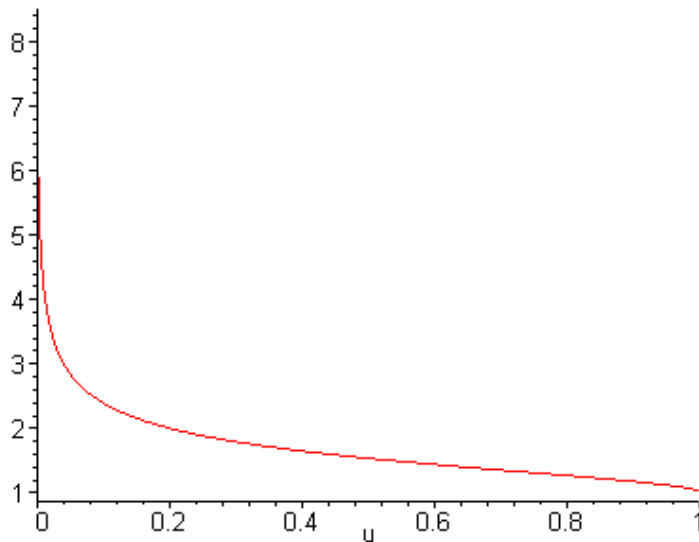


Figure 3.

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