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Estimation of the parameter in an Exponential Distribution using a LINEX Loss

Jungsoo Woo[1], Hwajung Lee, Kabsook Eun[2]

Abstract

A Bayes estimator of the scale parameter in an exponential distribution will be considered by a LINEX error, then the risk of the Bayes estimator using a LINEX loss will be compared with that of a Bayes estimator using a square error.

Keywords : LINEX error, Square error, Prior pdf, Bayes risk.

1. A LINEX loss and a standard exponential prior

For certain types of loss functions, expressions for the Bayes estimator can be determined. As usual, symmetric loss function of square error loss and, absolute error loss are used in Bayes analysis, so their risks can be compared each other.

But LINEX asymmetric loss fuction has been useful when a given positive estimation error is regarded as more serious than a negative error. So we can choose asymmetric LINEX loss functon that has been usually applied to econometrics. LINEX loss function was extensively discussed by Zellner(1986).

Many authors have cosidered asymmetric linear loss functions, specially Zellner(1986) studied a very useful asymmetric LINEX loss function. it rises approximatedly exponentially on one side of zero and approximately linearly on the other side.

So far, we don't have comparison between two Bayes risks of parameter in an exponential distribution by LINEX loss and square loss, so we need to compare

1. Professor, Department of Math. & Stat., Yeungnam University, Kyongsan, 712-749, Korea
2. Graduate Student, Department of Math. & Stat., Yeungnam University, Kyongsan, 712-749, Korea

their Bayes risks.

We shall consider the following exponential pdf(Rohatgi(1976)):

$$f(x;\lambda) = \lambda e^{-\lambda x}, \ x>0, \quad \text{where } \lambda>0. \tag{1.1}$$

The Bayes estimator of the parameter λ will be considered by a LINEX error loss and a standard exponential prior:

$$\pi(\lambda) = e^{-\lambda}, \quad \lambda>0. \tag{1.2}$$

We shall compare the Bayes risk of a Bayes estimator of the parameter in an exponential distribution using a LINEX error loss and a standard exponential prior with that of a Bayes estimator of the parameter in an exponential distribution using a square error loss and a standard exponential prior which we have well known.

1-1. LINEX error loss

In this section a standard exponential prior density will be used as (1.2), and a LINEX error loss will be used as follow:

$$L(d,\tau(\theta)) = e^{d-\tau(\theta)} - (d-\tau(\theta)) - 1 \text{(see Sadooghi-Alvandi(1990))}$$

Assume $X_1, X_2, \ldots, X_n$ be a simple random sample from the density (1.1).

Then the joint pdf of $X_1, X_2, \ldots, X_n$ is $f(x_1, x_2, \ldots, x_n \mid \lambda) = \lambda^n e^{-\lambda\sum_{i=1}^{n} x_i}$.

For the standard exponential prior, the joint pdf of $X_1, X_2, \ldots, X_n$ and λ is

$$f(x_1, x_2, \ldots, x_n, \lambda) = \lambda^n e^{-\lambda(1+\sum_{i=1}^{n} x_i)},$$

and hence from the formula 3.381(4) of Gradshteyn & Ryzhik(1965), the marginal joint pdf of $X_1, X_2, \ldots, X_n$ is

$$f(x_1, x_2, \ldots, x_n) = \frac{n!}{(1+\sum_{i=1}^{n} x_i)^{n+1}}.$$

Therefore, the posterior pdf of λ can be obtained :

$$h(\lambda \mid x_1, \ldots, x_n) = (1+\sum_{i=1}^{n} x_i)^{n+1} \cdot \lambda^n \cdot e^{-\lambda(1+\sum_{i=1}^{n} x_i)} / \Gamma(n+1). \tag{1.3}$$

which follows a gamma distribution with parameters n+1 and $\frac{1}{1+\sum_{i=1}^{n} x_i}$.

By the posterior pdf (1.3) and the result in Sadooghi-Alvandi(1990), under the LINEX error loss, we can obtain a Bayes estimator of λ:

$$\lambda_l^* \equiv -\ln E(e^{-\lambda} \mid X_1, \ldots, X_n)$$

$$= (n+1)\ln \frac{2+\sum_{i=1}^{n} X_i}{1+\sum_{i=1}^{n} X_i}. \qquad (1.4)$$

The Bayes risk of λ_l^* under a LINEX error loss can be obtained as followings:

$$R_l(\pi, \lambda_l^*) \equiv E_\lambda[E(e^{\lambda_l^* - \lambda} - (\lambda_l^* - \lambda) - 1) \mid \lambda]$$

$$= E_\lambda(\lambda - 1) + E_\lambda[e^{-\lambda} \cdot E((\frac{2+\sum_{i=1}^{n} X_i}{1+\sum_{i=1}^{n} X_i})^{n+1} \mid \lambda)] - (n+1)E_\lambda[E(\ln \frac{2+\sum_{i=1}^{n} X_i}{1+\sum_{i=1}^{n} X_i} \mid \lambda)] \qquad (1.5)$$

By the standard exponential prior, the Ist term in 2nd equality (1.5) is zero,
by the formulas 3.381(4) and 3.914(4) of Gradshteyn & Ryzhik(1965), the expectation of the 2nd term in 2nd equality (1.5) is 1,
by formula 3.381(4) of Gradshteyn & Ryzhik(1965), the expectation of the 3rd term in 2nd equality (1.5) can be obtained as:

$$n\int_0^\infty \frac{y^{n-1}}{(1+y)^{n+1}} \ln \frac{2+y}{1+y}\, dy.$$

Therefore, the Bayes risk is 1- n(n+1) $\int_0^\infty \frac{y^{n-1}}{(1+y)^{n+1}} \ln \frac{2+y}{1+y}\, dy.$ (1.6)

which the integral in (1.6) converges by the advanced calculation.

The integral in (1.6) = $\int_0^\infty \frac{y^{n-1}}{(1+y)^{n+1}} \ln(2+y)dy - \int_0^\infty \frac{y^{n-1}}{(1+y)^{n+1}} \ln(1+y)dy$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{n+1}} \ln(2+y)dy - B(n,1) \cdot (\psi(n+1) - \psi(1))$$

by 4.293(14) of Gradshteyn & Ryzhik(1965),

$$= \int_0^1 (1-x)^{n-1} \ln(\frac{x+1}{x})\, dx - B(n,1)(\psi(n+1) - \psi(1))$$

by $x = \frac{1}{y+1}$ and B(x,y) is a Beta function ,

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k [\int_0^1 x^k \ln(1+x)dx - \int_0^1 x^k \ln x\, dx]$$

$$- n^{-1}(\psi(n+1) - \psi(1))$$

by the Binomial expansion and L'Hospital rule,

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k [\frac{1}{k+1} \ln 2 + \int_0^1 \frac{1}{k+1} x^k dx$$

$$- \frac{1}{k+1} \int_0^1 \frac{x^{k+1}}{1+x} dx] - n^{-1}(\psi(n+1) - \psi(1)),$$

by the partial integration.

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k [\frac{1}{k+1} \ln 2 + \frac{1}{(k+1)^2},$$

$$- \frac{1}{k+1} \int_1^2 \frac{(y-1)^{k+1}}{y} dy] - n^{-1}(\psi(n+1) - \psi(1)), \qquad (1.7)$$

by a transformation 1+x=y ,

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\psi(x)$ is the psi-function.

The integral in the last equality (1.7),

$$\int_1^2 \frac{(y-1)^{k+1}}{y} dy = \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \int_1^2 y^{j-1} dy$$

$$= (-1)^{k+1} \ln 2 + \sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \frac{(2^j - 1)}{j} \qquad (1.8)$$

From the results (1.6), (1.7) and (1.8),
the Byes risk of λ_l^* under a LINEX error loss can be obtained:

$$R_l(\pi, \lambda_l^*) = 1 - n(n+1)\{ \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k [\frac{1}{k+1} \ln 2 + \frac{1}{(k+1)^2}$$

$$- \frac{1}{k+1} (-1)^{k+1} \ln 2 - \frac{1}{k+1} \sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \frac{(2^j-1)}{j}]$$

$$- n^{-1}[\psi(n+1) - \psi(1)]\} \qquad (1.9)$$

1-2. Square error loss

On the other hand, by using a square error loss and a standard exponential prior, a Bayes estimator of the parameter in an exponential distribution with the pdf (1.1) and its Bayes risk are well known as

the Bayes estimator of λ and its Bayes risk are

$$\lambda_s^* = \frac{n+1}{1 + \sum_{i=1}^{n} X_i}. \qquad (1.10)$$

and $R_s(\pi, \lambda_s^*) = \frac{2}{n+2}$, respectively. (1.11)

Furthermore, from the posterior pdf (1.3), we can obtain the generalized MLE(mode) of λ:

$$\lambda_m^* \equiv \frac{n}{1+\sum_{i=1}^{n} X_i} \tag{1.12}$$

Based on the mode (1.12) and the formulas 3.381(4) and 3.914(3) of Gradshteyn & Ryzhik (1965), then the Bayes risk of the mode λ_m^* under a square error loss is :

$$R_s(\pi, \lambda_m^*) = \frac{2}{n+1} \tag{1.13}$$

From the Bayes risks (1.11) and (1.13), it's no wonder that the Bayes estimator has less Bayes risk than the mode under the square error loss. And from the results (1.10) and (1.12), the mode locates at the left side of the Bayes estimator under the posterior density curve.

On the other hand, by the posterior pdf (1.3),

$2(1+\sum_{i=1}^{n} X_i)\lambda \mid (x_1, \ldots, x_n)$ follows a chi-square distribution with a df 2(n+1),

and hence, we can obtain a Bayes confidence interval:

$\left(\frac{\chi^2_{2(n+1),\alpha/2}}{2(1+\sum_{i=1}^{n} X_i)}, \frac{\chi^2_{2(n+1),1-\alpha/2}}{2(1+\sum_{i=1}^{n} X_i)}\right)$ is a $(1-\alpha)100\%$ Bayes confidence interval of λ.

2. A LINEX loss and a gamma prior

The Bayes estimator of the parameter λ will be considered by a LINEX error loss and a gamma prior:

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}, \quad \lambda > 0 \tag{2.1}$$

We shall compare the Bayes risk of a Bayes estimator of the parameter in an exponential distribution using a LINEX error loss and a gamma prior with that of a Bayes estimator of the parameter in an exponential distribution using a square error loss and a gammal prior which we have well known.

2-1. LINEX error loss

The Bayes estimator of the parameter λ will be considered by a LINEX error loss:

$$L(d,\tau(\theta)) = e^{d-\tau(\theta)} - (d-\tau(\theta)) - 1 \text{(see Sadooghi-Alvandi(1990))}$$

and a gamma prior density (2.1).

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Assume $X_1, X_2, \ldots, X_n$ be a simple random sample from the density (1.1). Then the joint pdf of $X_1, X_2, \ldots, X_n$ is $f(x_1, x_2, \ldots, x_n \mid \lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$.

For the gamma prior, the joint pdf of $X_1, X_2, \ldots, X_n$ and λ is

$$f(x_1, x_2, \ldots, x_n, \lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{n+\alpha-1} e^{-\lambda(\sum_{i=1}^{n} x_i + \beta)} ,$$

and the marginal joint pdf of $X_1, X_2, \ldots, X_n$ is

$$f(x_1, x_2, \ldots, x_n) = \frac{\Gamma(n+\alpha)\beta^\alpha}{\Gamma(\alpha)(\sum_{i=1}^{n} x_i + \beta)^{n+\alpha}} .$$

Therefore, the posterior pdf of λ can be obtained :

$$h(\lambda \mid x_1, \ldots, x_n) = \frac{(\sum_{i=1}^{n} x_i + \beta)^{n+\alpha}}{\Gamma(n+\alpha)} \lambda^{n+\alpha-1} e^{-\lambda(\sum_{i=1}^{n} x_i + \beta)} . \tag{2.2}$$

which follows a gamma distribution with parameters $n+\alpha$ and $\frac{1}{\sum_{i=1}^{n} x_i + \beta}$.

By the posterior pdf (2.2) and the result in Sadooghi-Alvandi(1990), under the LINEX error loss we can obtain a Bayes estimator of λ:

$$\begin{aligned} \lambda_l^* &\equiv -\ln E(e^{-\lambda} \mid X_1, \ldots, X_n) \\ &= -(n+\alpha)\ln\left(\frac{\sum_{i=1}^{n} X_i + \beta}{\sum_{i=1}^{n} X_i + \beta + 1}\right) \end{aligned} \tag{2.3}$$

from the formula 3.381(4) of Gradshteyn & Ryzhik(1965).

The Bayes risk of λ_l^* under a LINEX error loss can be obtained as followings:

$$R_l(\pi, \lambda_l^*) \equiv E_\lambda[E(e^{\lambda_l^* - \lambda} - (\lambda_l^* - \lambda) - 1) \mid \lambda]$$

$$= \frac{\Gamma(n+\alpha)\beta^{\alpha}}{\Gamma(n)\Gamma(\alpha)} \int_0^{\infty} \frac{y^{n-1}}{(y+\beta)^{n+\alpha}}\, dy - \frac{\Gamma(n+\alpha)(n+\alpha)\beta^{\alpha}}{\Gamma(n)\Gamma(\alpha)}$$

$$\cdot \int_0^{\infty} \frac{y^{n-1}}{(y+\beta)^{n+\alpha}} \ln \frac{y+\beta+1}{y+\beta}\, dy + \frac{\alpha}{\beta} - 1$$

from the formula 3.381(4) of Gradshteyn & Ryzhik(1965).

$$= \frac{\alpha}{\beta} - \frac{n+\alpha}{B(n,\alpha)} \int_0^{1} (1-t)^{n-1} t^{\alpha-1} \ln\left(1+\frac{t}{\beta}\right) dt,$$

by integration(1.4) and $\frac{\beta}{y+\beta} \equiv t$,

$$= \frac{\alpha}{\beta} - \frac{n+\alpha}{B(n,\alpha)} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{1}{k+\alpha} \left[\ln\left(1+\frac{1}{\beta}\right) - \frac{1}{\beta(k+\alpha+1)} \cdot \right.$$

$$\left. F\left(1; k+\alpha+1; k+\alpha+2; -\frac{1}{\beta}\right) \right], \tag{2.4}$$

from the formula 3.381(4) of Gradshteyn & Ryzhik(1965), where B(x,y) and F(a;b;c;x) are the beta function and the hypergeometric function, respectively , in Gradshteyn & Ryzhik(1965),

Here, although a standard exponential prior is a special case of a gamma prior, it is not easy for us to show that the Bayes risk (2.4) can represent by the risk (1.9) when α=1 and β=1 in the gamma prior. Therefore, we have considered two separated sections.

1-2. Square error loss

On the other hand, by using a square error loss and a gamma prior, a Bayes estimator of the parameter in an exponential distribution with the pdf (1.1) and its Bayes risk are well known as

the Bayes estimator of λ is

$$\lambda_s^* = \frac{n+\alpha}{\sum_{i=1}^{n} X_i + \beta}. \tag{2.5}$$

To find its Bayes risk,

$$\text{Let } I(n; n+\alpha+2) \equiv \int_0^{\infty} \frac{y^{n-1}}{(y+\beta)^{n+\alpha+2}}\, dy = \frac{1}{\beta^{\alpha+2}} B(\alpha+2, n), \tag{2.6}$$

from integration by substitution, $t \equiv \frac{\beta}{y+\beta}$, where B(x,y) is a beta function.

From the integration (2.6) and the formula 3.381(4) of Gradshteyn & Ryzhik(1965), we can obtain the Bayes risk of the Bayes estimator λ_s^* under a square error loss:

$$R_s(\pi, \lambda_s^*) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha+1} e^{-\beta\lambda} d\lambda - \frac{2(n+\alpha)}{\Gamma(n)} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{y^{n-1}}{y+\beta}$$

$$\cdot \int_0^\infty \lambda^{n+\alpha-1} e^{-\lambda(y+\beta)} d\lambda dy + \frac{(n+\alpha)^2}{\Gamma(n)} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{y^{n-1}}{(y+\beta)^2}$$

$$\cdot \int_0^\infty \lambda^{n+\alpha-1} e^{-\lambda(y+\beta)} dy d\lambda$$

$$= \frac{\alpha(\alpha+1)}{\beta^2} - \frac{(n+\alpha)^2 \Gamma(n+\alpha)\beta^\alpha}{\Gamma(n)\Gamma(\alpha)} \cdot I(n; n+\alpha+2),$$

by integral(1.5),

$$= \frac{\alpha(\alpha+1)}{\beta^2(n+\alpha+1)}. \tag{2.7}$$

From the posterior pdf (2.2), the generalized MLE of λ is $\frac{n+\alpha-1}{\sum_{i=1}^n X_i + \beta}$, which locates at a left of the Bayes estimate of λ under the posterior pdf (2.2) .

For a given $0<\gamma<1$, $\int_0^a \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx = \gamma$,

define $a \equiv \Gamma^*(\alpha\,;\gamma)$.

From the posterior pdf (2.2), $2(\sum_1^n X_i + \beta)\lambda$ follows a gamma distribution with a shape parameter n+α and a scale parameter 2, from definition of Γ^* and the posterior pdf(2.2) of λ, $\left(\frac{\Gamma^*(\alpha\,;\frac{\gamma}{2})}{2(\sum_{i=1}^n X_i + \beta)}, \frac{\Gamma^*(\alpha\,;1-\frac{\gamma}{2})}{2(\sum_{i=1}^n X_i + \beta)}\right)$ can be obtained as an (1- γ)100% Bayes confidence interval for λ. (2.8)

By the Bayes risks (1.9) and (1.11) and Tables of the psi-function in Abramowitz & Stegun(1970), and the Bayes risks (2.4) and (2.7) and the recursion formula of hypergeometric function in Abramowitz & Stegun(1970), Table 1 shows Bayes risks of two Bayes estimators under two different error losses .

Table 1. Bayes risks of two Bayes estimators λ_s^* and λ_l^* based on a gamma prior and a standard prior

n	$\alpha=\frac{1}{2},\beta=1$		$\alpha=1,\beta=1$		$\alpha=2,\beta=1$	
	λ_s^*	λ_l^*	λ_s^*	λ_l^*	λ_s^*	λ_l^*
5	0.11538	0.047776	0.28571	0.11574	0.75000	0.29321
10	0.06522	0.028943	0.16667	0.07270	0.46154	0.19533
15	0.04545	0.020824	0.11765	0.05315	0.33333	0.14686
20	0.03488	0.016276	0.09091	0.04193	0.26087	0.11778
25	0.02830	0.013364	0.07407	0.03463	0.21429	0.09836
30	0.02381	0.011338	0.06250	0.02951	0.18182	0.08445

(to continue)

n	$\alpha=\frac{1}{2},\ \beta=2$		$\alpha=2,\ \beta=2$		$\alpha=\frac{1}{2},\ \beta=\frac{1}{2}$		$\alpha=2,\ \beta=\frac{1}{2}$	
	λ_s^*	λ_l^*	λ_s^*	λ_l^*	λ_s^*	λ_l^*	λ_s^*	λ_l^*
5	0.028846	0.013029	0.18750	0.082007	0.46154	0.16528	3.00000	0.97633
10	0.016304	0.007656	0.11538	0.052796	0.26087	0.10488	1.84615	0.68354
15	0.011364	0.005429	0.08333	0.038989	0.18182	0.07724	1.33333	0.52843
20	0.008721	0.004208	0.06522	0.030922	0.13953	0.06124	1.04348	0.43151
25	0.007075	0.003435	0.05357	0.025626	0.11321	0.05077	0.85714	0.36494
30	0.005952	0.002903	0.04545	0.021881	0.09524	0.04337	0.72727	0.31632

Throughout numerical evaluations in Table 1, for a gamma prior and a standard prior, the Bayes risks of a Bayes estimator λ_l^* of the scale parameter in an exponential distribution under the LINEX error loss are less than those of a Bayes estimator λ_s^* of the scale parameter in an exponential distribution under the square error loss when $\alpha=\frac{1}{2}$, $\alpha=1, \alpha=2$ and $\beta=\frac{1}{2}$, 1, 2, and n=5(5)30.

References

1. Abramowitz, M & Stegun, I.A.(1970), Handbook of Mathematical Functions, Dover Publications, Inc., New York
2. Gradshteyn, I.S.& Ryzhik, I.M.(1965), Table of Integrals, Series, and Products, Academic Press, New York
3. Rohatgi, V.K.(1976), An Introduction to Probability Theory and Mathematical Statistics, John Wiley & Sons, New York
4. Sadooghi-Alvandi, S.M.(1990), Estimation of the parameter of a Poisson Distribution using a LINEX Loss Function, Austral, J. Statist., 32(3), 393-398
5. Zellner, A.(1986). Bayesian estimation and prediction using asymmetric loss function. J. Amer. Statist. Assoc. 81, 446-451.

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