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Iterative Analysis for Nonlinear Laminated Rectangular Plates by Finite Difference Method

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Abstract: A new system of equations governing the nonlinear thin laminated plates with large deflections using von Karman equations is derived. The effects of transverse shear in the thin interlayer are included as part of the analysis. The finite difference method is used to perform the geometrically nonlinear behavior of the plate. The resultant equations permit the analysis of the effect of transverse shear stress deformation on the overall behavior of the interlayer using the load incremental method. For the purpose of feasibility and validity of this present method, the numerical results are compared with other available solutions for accuracy as well as efficiency. The solution techniques have been implemented and the numerical results of example problem are discussed and evaluated.

Key words: laminated plate, finite difference method, nonlinear equation, iteration technique, load incremental method

1. Introduction

Today, an increasing number of structural designs, especially in the aerospace industry, are utilizing laminated construction in the fabrication of major structural components. The substantial interest in these new high-strength and low-density materials in laminate construction is evidence of the continual quest for strong light-weight structures. Laminated plate can be manufactured in a wide variety of products to meet demands of building codes for safety, lighting, sound, and color. It is used widely in automobiles and aircraft, as safety plates, and in many architectural applications. A laminated plates unit consists of two plates connected by a thin elastometric interlayer. The interlayer has a very low modulus of elasticity relative to the face plates. Theoretically, this difference in material properties suggests that interlayer may not be capable of transferring shear between the two plates. If no shear transfer capability is assumed, laminated plate units could be analyzed theoretically by considering them to act as a structural system consisting of two plates that are not joined. On the other hand, if the interlayer is assumed to afford complete shear transfer capability between the plates, laminated plate unit behavior could be approximated by that of a monolithic plate [6, 7]. The analysis of laminated plate units is made difficult because of significant differences in material properties within the composite section. When a laminated plate unit is loaded laterally, individual plates are subjected to both bending and membrane effects in the central region, while in regions close to the sides, bending effects are more dominant. Intuitively, the transverse shear affects stresses in regions near the sides of the unit to a larger degree than in regions near the center of the unit. Since the contribution to bending resistance by the interlayer is small, bending resistance of the interlayer is neglected in the proposed model. Reissner [4] was the first to develop large deflection equations for sandwich plates where transverse shear deformations of the core between thin plates was considered. In his formulation, the plates were so thin compared to the core thickness that variations in stress over the thickness of the plates were assumed to be negligible. This formulation is not useful for analysis of laminated glass units. Pister and Dong [3] developed nonlinear equations similar to those of von Karman assumptions that plane sections before bending remain plane after bending. Although it has been used by many [1, 2, 5, 8, 9], this theory ignores shear deformations in materials within the system of plates. Research on laminated glass units revealed that the plates tend to slip with respect to each other, with

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the amount of slip depending on temperature.

2. Laminated Plate Equations for Large Deflections

As the plates deflect, differences in displacements at the top and bottom of the interlayer produce shear strains and shear stresses in the interlayer material. The shear stresses produce distributed forces and moments along the middle planes at the two plates. The distributed forces in the x direction are $-F_x$ in the top plate and $+F_x$ in the bottom plate, along with distributed moments M_x (equal to 0.5 t) along the middle plane of each plate as shown in Fig.1. Similar distributed forces $+F_x$, $-F_y$ and M_y act in the y-direction.

Because distributed in-plane forces are present, the von Karman equations have to be modified. These modifications can be done by inserting the new equilibrium equations with distributed membrane forces

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + F_x = 0 \quad (1)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + F_y = 0 \quad (2)$$

Here F_x and F_y are considered positive in the direction of the axes. The corresponding Airy stress function will be:

$$N_x = \frac{\partial^2 \Phi}{\partial y^2} + S \quad (4)$$

$$N_y = \frac{\partial^2 \Phi}{\partial x^2} + S \quad (3)$$

$$N_{xy} = \frac{\partial^2 \Phi}{\partial x \partial y} \quad (5)$$

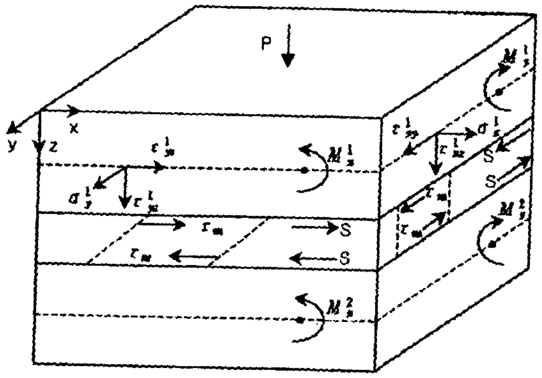

Fig. 1. Laminated model considering forces imparted to plates by shear deformed interlayer

Where S is defined such that

$$F_x = -\frac{\partial S}{\partial x} \text{ and } F_y = -\frac{\partial S}{\partial y} \quad (6)$$

These modifications satisfy Eq. (1) and Eq. (2). When these changes are incorporated into the von Karman equations for a given plate, we get

$$\begin{aligned} D\nabla^4 w &= p + t[N_x w_{xx} + 2N_{xy} w_{xy} + N_y w_{yy}] \\ &= p + t[(\Phi_{yy} - S) w_{xx} - 2\Phi_{xy} w_{xy} - (\Phi_{xx} + S) w_{yy}] \end{aligned} \quad (7)$$

and

$$\nabla^2 \Phi + (1 - \mu)\nabla^4 S = -E[w_{xx} w_{xy} - (w_{xy})^2] \quad (8)$$

Note that N_x, N_y, N_{xy} shown in the above equations are membrane stresses only.

where $D = \frac{Et^3}{12(1-\mu^2)}$ =flexural rigidity of the plate; w=lateral deflection; Φ=Airy stress function; p=lateral pressure; t=plate thickness; E=Young's modulus of elasticity; μ=Poisson's ratio; ∇^4= Biharmonic operator; Here we assume that the two plates of the same thickness t will have a common displacement w from the original middle surface. With the distributed membrane forces acting in opposite directions in the plates, it is clear that the resulting membrane stresses in the plate are different, or in other words, the Airy stress functions are different for the plates. Also, it is to be noted that the lateral pressure p is shared differently by the two plates.

For plate 1, the equations are:

$$D\nabla^4 w = p^1 + t[(\Phi^1_{yy} + S) w_{xx} - 2\Phi^1_{yy} w_{yy} + (\Phi^1_{xx} + S) w_{yy}] \quad (9)$$

and

$$\nabla^4 \Phi^1 + (1 - \mu)\nabla^2 S = -E[w_{xx} w_{yy} - (w_{xy})^2] \quad (10)$$

For plate 2, the equations are :

$$D\nabla^4 w = p^2 + t[(\Phi^2_{yy} + S) w_{xx} - 2\Phi^1_{xy} w_{yy} + (\Phi^2_{xx} + S) w_{yy}] \quad (11)$$

and

$$\nabla^4 \Phi^2 + (1 - \mu)\nabla^2 S = -E[w_{xx} w_{yy} - (w_{xy})^2] \quad (12)$$

The superscripts 1 and 2 for p and Φ represent quantities corresponding to upper and lower plates 1 and 2 respectively. Combining Eqs.(9),(11) and Eqs.(10) ,(12) respectively, we get:

$$2D\nabla^4 w = p + 2t[\Phi_{yy} - w_{yy} - 2\Phi_{yy}w_{yy} + \Phi_{xx}w_{yy}] \quad (13)$$

and

$$\nabla^4\Phi = E[w_{xx}w_{xy} - (w_{xy})^2] \quad (14)$$

where

$$\Phi = \frac{\Phi^1 + \Phi^2}{2} \text{ and } p = p^1 + p^2$$

Upon examination, it can be seen that the above equations are the same as the layered plate equations, but have to be solved for in-plane external loadings such F_x, F_y, M_x and M_y, where :

$$M_x = M_x^1 + M_x^2 \quad (15)$$

$$M_y = M_y^1 + M_y^2 \quad (16)$$

The additional superscripts 1 and 2 on M represent quantities corresponding to plate 1 and 2. Two minor changes are needed to convert from a monolithic plate to a laminated unit of total thickness t, with equal individual plate thickness of t' and of flexural rigidity D. The flexural rigidity D is changed from $\frac{Et^3}{12(1-\mu^2)}$ to $\frac{E(t')^3}{6(1-\mu^2)}$ and t is replaced by $2t'$.

3. Solution of Equations

Solution of von Karman equation employs the finite difference method. Von Karman Eqs. (13) and (14) can be represented by two algebraic functions, using the two central difference equations. Eq.(13) becomes:

$$[R]\{w\} = \{p\} + L_1(w, \Phi) \quad (17)$$

and ϕ Eq. (14) becomes

$$[S]\{\Phi\} = L_2(w) \quad (18)$$

where $[R]$ and $[S]$ =Biharmonic operators
w =vector representing lateral displacement
p =vector representing load
Φ =vector representing Airy function
L_1, L_2 =nonlinear functions representing part of right side of von Karman's equations.

It can be seen that Eq. (17) represents lateral deflection, while Eq. (18) represents the Airy stress function.

4. Iterative Procedure

Like any other iterative technique, new values for the variables are calculated based on values obtained from the previous iteration. Using values of w and Φ from the ith iteration the L_1 function can be calculated numerically from the expression for $L(w, \phi)$. The first von Karman's equation for the $(i+1)$th iteration becomes

$$[R]\{w_{i+1}\} = \{P\} + L_1(w_1, \Phi_i) \quad (19)$$

From this, w_{i+1} can be determined. Now that w_{i+1} is known, it can be substituted into the right hand side of the second von Karman equation such as that Eq. (18) becomes

$$[S]\{\Phi_{i+1}\} = L_2(w_{i+1}) \quad (20)$$

and from this Φ_{i+1} can be obtained. An error term is used to end the iteration when convergence is reached in the computation of w

$$\delta_{i+1} = \frac{\sum_{j=1}^{n} |w_{i+1,j} - w_{i,j}|}{n} \le \alpha(w_{max})_{i+1}$$

where

i = iteration number
j = node number
n = number of nodes in the grid
α = iterative tolerance number

The iterative procedure to be developed is given below:

Step 1. Assume F_x, F_y, M_x, M_y equal to zero
Step 2. Calculate w and Φ
Step 3. Calculate Φ_1 and Φ_2
Step 4. Calculate membrane and bending stresses in both plates
Step 5. Calculate corresponding strains
Step 6. Calculate u and v at the top and bottom of the interlayer
Step 7. Calculate N_{xz} and N_{yz} in the interlayer
Step 8. Calculate F_x, F_y, M_x and M_y
Step 9. Go back to Step 2., until satisfactory convergence is reached for w and Φ

The method is applicable to different thickness of the interlayer, since N_{xz} and N_{yz} in the interlayer are dependent on its thickness.

5. Numerical Implementation and Discussions

To illustrate the formulation of the method and its efficiency the solution of a simply supported uniformly loaded rectangular thin aluminium plate is presented. Due to the symmetry of the problem only one quarter of the plate is considered in the analysis, and it is modelled by 10×15 grid sizes. The dimensions and material properties of the plate used are a=60 cm, b=90 cm, t=1.1 cm, E = 75 GPa and μ = 0.33. The plate is subjected to a uniformly increasing static lateral pressure up to 32Pa. For the purpose of comparing the numerical results of the present method, a monolithic plate with similar conditions is solved first. Fig. 2 shows the load-maximum deflection curve of the monolithic plate for each of loads and corresponding number of iterations required. These results give close agreement to the solution of SAP2000 for each of loads. The variation of the bending stresses along the short axis in the plate due to different lateral pressures is shown in Fig. 3. This demonstrates the nonlinearity of the aluminium plate and the migration of the maximum stresses from the center towards the support. A set of computer generated displacement and maximum principal stress contours are also presented here in Figs. 4-6. These figures show the nonlinearity of the displacement patterns and the migration of the maximum principal stresses from the center towards the corners of the plate. For analysis and design of aluminum the values of maximum principal stresses and their location are valuable information.

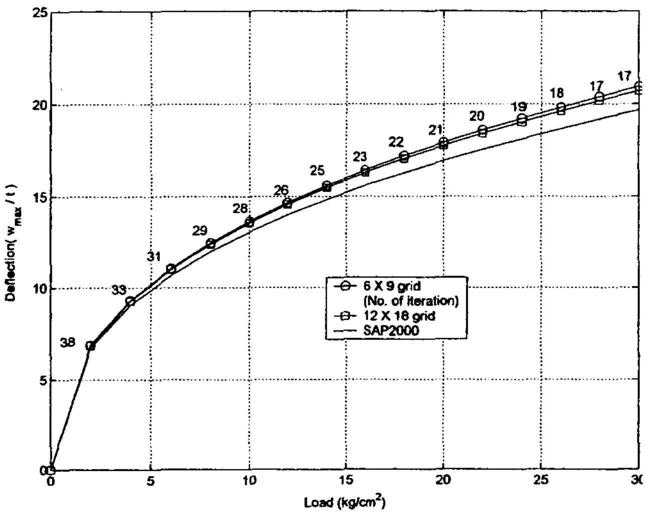

Fig. 2. Maximum deflections for loads and corresponding number of iterations

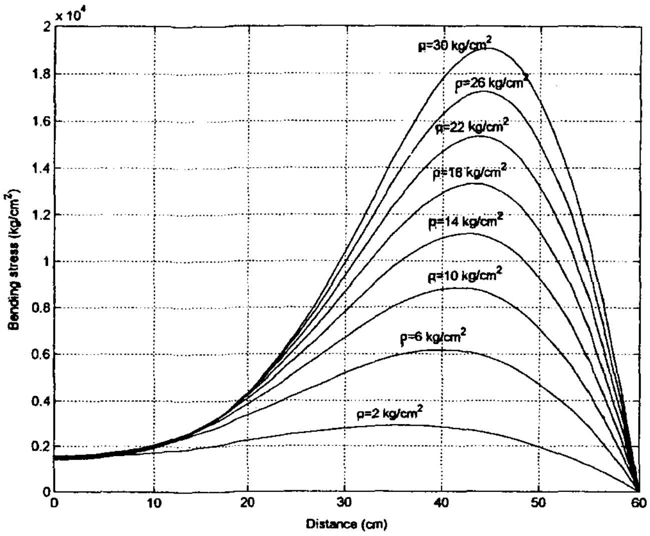

Fig. 4. Bending stresses along the short axis in the plate

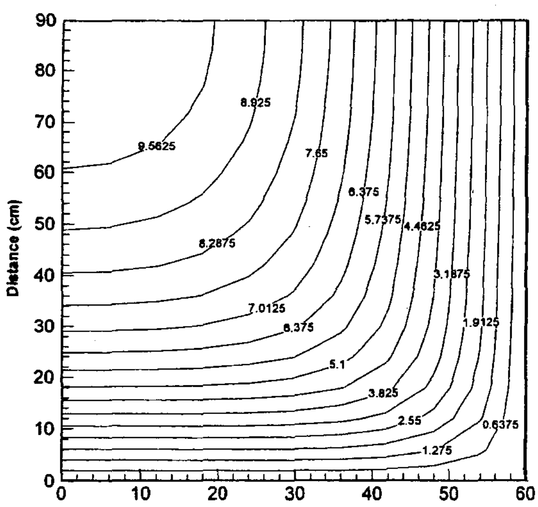

Fig. 3. Lateral displacements due to p=4 kg/cm²

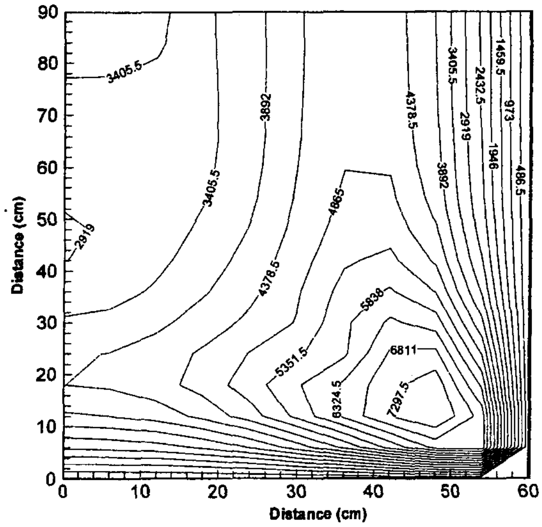

Fig. 5. Maximum principle tensile stresses due to p=14 kg/cm² pressure.

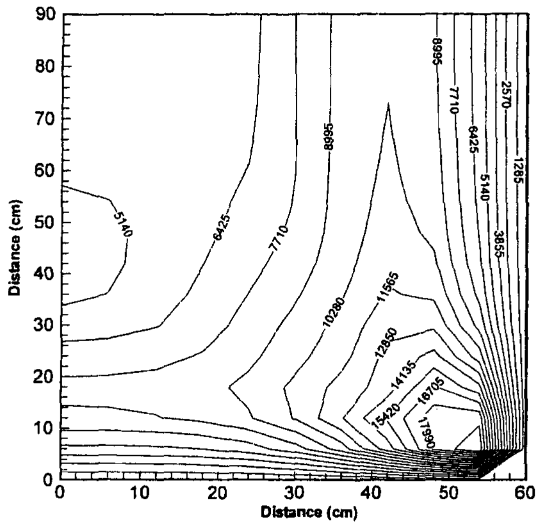

Fig. 6. Maximum principle tensile stresses due to p=14 kg/cm^2

6. Conclusions

A mathematical model for the nonlinear stress analysis of thin rectangular laminated aluminium plates is developed. For the geometrically nonlinear, large deflection behavior of the aluminum plate, the classical von Karman equations are used. These equations are solved numerically by using the finite difference method. An iterative technique is employed to solve these quasi-linear algebraic equations. The following conclusions are advanced:

1) The results from the model developed agree well with previous solutions.,

2) The model developed is very efficient in computer storage requirements and execution time.

3) The iteration converged exactly to the von Karman field equations.

4) Variable mesh size allows analysis of any size of rectangular plate.

5) The solution obtained for any loading does not depend on size or number of increments of load or on previous displacements.

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