

# Free Vibrations of Arches in Cartesian Coordinates

## 直交座標系에 의한 아치의 자유振動

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**Key Words** : Cartesian Coordinates(直交座標系), Free Vibration(自由振動), Harmonic Motion(調和振動), Arch(아치), Mode Shape(振動形), Natural Frequency(固有振動數), Rotatory Inertia(回轉慣性), Unsymmetric Axis(非對稱軸), Variable Curvature(變化曲率).

### ABSTRACT

The differential equations governing free vibrations of the elastic arches with unsymmetric axis are derived in Cartesian coordinates rather than in polar coordinates, in which the effect of rotatory inertia is included. Frequencies and mode shapes are computed numerically for parabolic arches with both clamped ends and both hinged ends. Comparisons of natural frequencies between this study and SAP 2000 are made to validate theories and numerical methods developed herein. The convergent efficiency is highly improved under the newly derived differential equations in Cartesian coordinates. The lowest four natural frequency parameters are reported, with and without the rotatory inertia, as functions of three non-dimensional system parameters: the rise to chord length ratio, the span length to chord length ratio, and the slenderness ratio. Also typical mode shapes of vibrating arches are presented.

### 요약

이제까지 아치의 자유振動에 관한 연구들은 모두 極座標系에서 해석한 연구들이다. 이 논문은 極座標系에서 아치의 해석을 止揚하고, 直交座標系에서 아치의 자유振動을 해석한 연구이다. 아치의 자유振動을 支配하는 微分方程式을 直交座標系에서 誘導하고, 이를 수치해석하여 固有振動數와 振動形을 산출하였다. 미분방정식에는 回轉慣性 효과를 고려하고, 아치의 線型은 拋物線으로 채택하였다. 실제 구조물에 대한 適用을 위하여 非對稱軸을 갖는 아치를 數值解析하였다. 본 연구와 SAP 2000의 結果를 비교하여 본 연구의 妥當性を 檢證하였다. 수치해석의 結果로 아치의 無次元 變數들이 無次元 固有振動數에 미치는 影響을 분석하고, 典型的인 振動形의 例를 그림에 나타내었다.

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## 1. INTRODUCTION

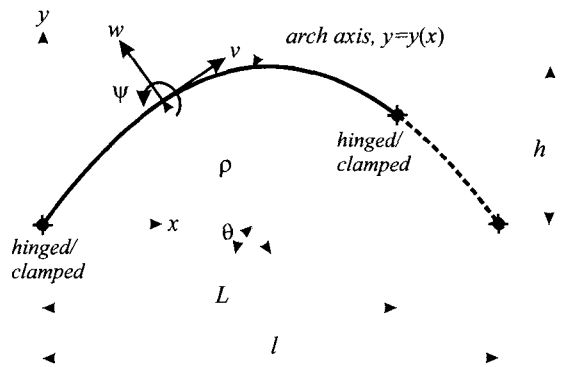
Arches are one of the most important basic structural units as well as the beams, columns and plates. Most complicated structures consist of only these basic units and therefore it is very attractive research subject to analysis both the static and dynamic behavior of such units including the arches.

The problems of free vibrations of arches have been the subject of much work due to their many practical applications. Furthermore, characteristics of free vibrations of structures including arches are definitely unique, which are consequently used as an assessment index in evaluating the soundness of structures.

The governing equations and its significant historical literature on the free in-plane vibrations of elastic arches have been reported in many references for more than three decades. Background material for the current study was critically reviewed by Lee and Wilson.<sup>(1)</sup> Briefly, such works included studies of the non-circular arches with predictions of only the lowest frequency in flexure by Romanelli and Laura,<sup>(2)</sup> and in extension by Wang,<sup>(3)</sup> and Wang and Moore<sup>(4)</sup>; studies of circular arches with predictions of the higher frequencies by Wolf<sup>(5)</sup> and Veletsos et al.,<sup>(6)</sup>; studies of arches with variable curvature of the higher frequencies in flexure by Lee and Wilson,<sup>(1)</sup> Kang and Bert,<sup>(7)</sup> Lee et al.,<sup>(8)</sup> and Oh et al.<sup>(9)</sup>; and the effects of transverse shear and rotatory inertia on free vibration frequencies of arches by Irie et al.,<sup>(10)</sup> Davis et al.<sup>(11)</sup> and Wilson and Lee.<sup>(12)</sup>

This paper has three main purposes : (1) to present the differential equations for free, planar vibrations of arches with variable curvature and unsymmetric axis, where all equations are derived in Cartesian coordinates rather than in polar coordinates ; (2) to include the effect of rotatory inertia in the differential equations; and (3) to illustrate the numerical solutions to the newly derived equations for a broad class of parabolic arches.

In most previous works on arch vibrations, the polar coordinates were employed and also the effect of unsymmetric axis was excluded in the parametric studies of vibration problems of arches. The results presented herein extent significantly previous works. That is, using the Cartesian



- $L$  : Span length
- $l$  : Chord length
- $h$  : Rise
- $v$  : Tangential displacement
- $w$  : Radial displacement
- $\psi$  : Rotation of cross-section
- $\rho$  : Radius of curvature
- $\theta$  : Inclination of  $\rho$  with  $x$ -axis

Fig. 1 Geometry of object arch and its variables

formulation together with highly efficient and convergent numerical methods, the free vibration frequencies and mode shapes, with and without the rotatory inertia, are investigated for parabolic arches with unsymmetric axis. Such numerical results are presented for both clamped ends and both hinged ends. The lowest four non-dimensional frequency parameters are shown as functions of three system parameters: the rise to chord length ratio, the span length to chord length ratio, and the slenderness ratio.

The following assumptions are inherent in this theory : the arch is linearly elastic; both tangential and radial displacements are considered and the small deflection theory is governed. In addition, the arch is assumed to be in harmonic motion.

## 2. MATHEMATICAL MODEL

The geometry and nomenclature of the arch, placed in the Cartesian coordinates  $(x, y)$ , with

variable curvature and unsymmetric axis are shown in Fig. 1. The arch is supported by either both clamped ends or both hinged ends. The geometric variables are defined as follows.

The shape of parabolic arch, which is chosen as the object arch with variable curvature herein, is expressed in terms of  $(l, h)$  and the coordinate  $x$  in the range from  $x = 0$  to  $x = L$ . That is,

$$y = -(4hl^{-2})x(x-l), \quad 0 \leq x \leq L \quad (1)$$

A small element of the arch is shown in Fig. 2 in which are defined the positive directions for the axial force  $N$ , the shear force  $Q$ , the bending moment  $M$ , the tangential inertia force  $F_v$ , the radial inertia force  $F_w$ , and the rotatory inertia couple  $C_\psi$ . Treating the inertia forces and the inertia couple as equivalent static quantities, the three equations for "dynamic equilibrium" of the element are

$$N' + Q + \rho F_v = 0 \quad (2)$$

$$Q' - N + \rho F_w = 0 \quad (3)$$

$$\rho^{-1}M' - Q - C_\psi = 0 \quad (4)$$

where  $(\prime)$  is the operator  $d/d\theta$ .

The equations that relate  $N$ ,  $M$  and  $\psi$  to the displacements  $v$  and  $w$ <sup>(13)</sup> are

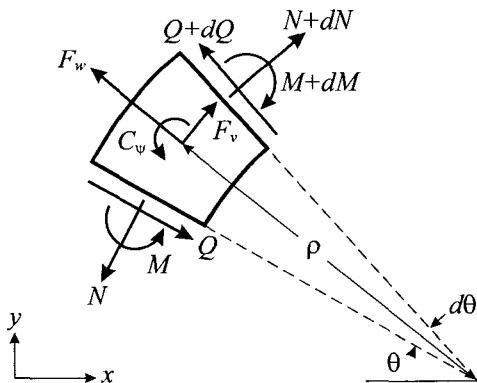


Fig. 2 Element subjected to stress resultants and inertia forces

$$N = EA\rho^{-1}[(v' + w) + r^2\rho^{-2}(w'' + w)] \quad (5)$$

$$M = EA r^2 \rho^{-2}(w'' + w) \quad (6)$$

$$\psi = \rho^{-1}(w' - v) \quad (7)$$

where  $E$  is Young's modulus,  $A$  is the cross-sectional area and  $r$  is the radius of gyration of cross-section.

The arch is assumed to be in harmonic motion, or each coordinate is proportional to  $\sin(\omega_i t)$  where  $\omega_i$  is the  $i$  th circular frequency and  $t$  is time. Then the tangential and radial inertia forces, and rotatory inertia couple per unit arc length are, respectively,

$$F_v = m\omega_i^2 v \quad (8)$$

$$F_w = m\omega_i^2 w \quad (9)$$

$$C_\psi = m\omega_i^2 r^2 \psi = m\omega_i^2 r^2 \rho^{-1}(w' - v) \quad (10)$$

where  $m$  is the mass per unit arc length.

When Eqs. (5) and (6) are differentiated once, the results are

$$N' = EA\rho^{-1}[(v'' + w') + r^2\rho^{-2}\rho'(w''' + w')] - \rho^{-1}\rho'(v' + w) - 3r^2\rho^{-3}\rho'(w'' + w) \quad (11)$$

$$M' = -EA r^2 \rho^{-2}[(w''' + w') - 2\rho^{-1}\rho'(w'' + w)] \quad (12)$$

When Eqs. (10) and (12) are substituted into Eq. (4), then

$$Q = \rho^{-1}M' - RC_\psi = -EA r^2 \rho^{-3}[(w''' + w') - 2\rho^{-1}\rho'(w'' + w)] - Rm\omega_i^2 r^2 \rho^{-1}(w' - v) \quad (13)$$

where the index  $R$  is defined as follows.

$$R = 1 \text{ if } C_\psi \text{ is included,} \quad (14.1)$$

$$R = 0 \text{ if } C_\psi \text{ is excluded.} \quad (14.2)$$

The following equation is obtained by differentiating Eq. (13).

$$Q' = -EA r^2 \rho^{-3} [(w'''' + w'') - 5\rho^{-1} \rho' (w''' + w')] + 2\rho^{-1} (4\rho^{-1} \rho'^2 - \rho'') (w'' + w) - Rm\omega_i^2 r^2 \rho^{-1} [(w' - v') - \rho^{-1} \rho' (w' - v)] \quad (15)$$

From Fig. 1, it is seen that the inclination  $\theta$  is related to the coordinate  $x$ . By the mathematical definition,

$$\theta = \pi/2 - \tan^{-1}(dy/dx) = \pi/2 - \tan^{-1}[-(4hl^2)(2x-l)] \quad (16)$$

When Eq. (16) is differentiated, the result is

$$d\theta = (8hl^2)[l^4 + 16h^2(2x-l)^2]^{-1} dx \quad (17)$$

Define the following arch parameters.

$$g_1 = 1/(8hl^2)[l^4 + 16h^2(2x-l)^2] \quad (18.1)$$

$$g_2 = (8h/l^2)(2x-l) \quad (18.2)$$

$$g_3 = 16h/l^2 \quad (18.3)$$

From Eq. (17), and with Eqs. (18.1)~(18.3), the following differential operators are obtained.

$$\frac{d}{d\theta} = g_1 \frac{d}{dx} \quad (19)$$

$$\frac{d^2}{d\theta^2} = g_1^2 \frac{d^2}{dx^2} + g_1 g_2 \frac{d}{dx} \quad (20)$$

$$\frac{d^3}{d\theta^3} = g_1^3 \frac{d^3}{dx^3} + 3g_1^2 g_2 \frac{d^2}{dx^2} + g_1(g_1 g_3 + g_2^2) \frac{d}{dx} \quad (21)$$

$$\frac{d^4}{d\theta^4} = g_1^4 \frac{d^4}{dx^4} + 6g_1^3 g_2 \frac{d^3}{dx^3} + g_1^2(4g_1 g_3 + 7g_2^2) \frac{d^2}{dx^2} + g_1 g_2(4g_1 g_3 + g_2^2) \frac{d}{dx} \quad (22)$$

The radius of curvature  $\rho$  at any point of the parabolic arch is expressed as Eq. (23). Also, its derivatives  $\rho'$  and  $\rho''$  can be expressed in terms of  $x$  by using Eq. (23) with Eqs. (19) and (20)

as Eqs. (24) and (25), respectively. That is,

$$\rho = [1 + (dy/dx)^2]^{3/2} (d^2y/dx^2)^{-1} = (1/\sqrt{2}) g_1^{3/2} g_3^{1/2} \quad (23)$$

$$\rho' = (3\sqrt{2}/4) g_1^{3/2} g_2 g_3^{1/2} \quad (24)$$

$$\rho'' = (3\sqrt{2}/8) g_1^{3/2} g_3^{1/2} (2g_1 g_3 + 3g_2^2) \quad (25)$$

Now cast the differential equations of free vibration for the arch into non-dimensional form by introducing the non-dimensional parameters as follows.

$$\zeta = x/l \quad (26)$$

$$\eta = y/l \quad (27)$$

$$f = h/l \quad (28)$$

$$e = L/l \quad (29)$$

$$\lambda = v/l \quad (30)$$

$$\delta = w/l \quad (31)$$

$$s = l/r \quad (32)$$

Here the coordinates  $(x, y)$ , the rise  $h$ , the span length  $L$ , and the displacements  $v$  and  $w$  are normalized by the chord length  $l$ , and  $S$  is the slenderness ratio.

When Eqs. (5), (9), and (15) together with Eqs. (18)~(32) are used in Eq. (3), the result is Eq. (33). Also, when Eqs. (8), (11) and (13) are combined with Eqs. (2), the result is Eq. (34). That is,

$$\delta^{iv} = a_1 \delta^{iii} + (a_2 + Rc_i^2 a_3) \delta^{ii} + (a_4 + Rc_i^2 a_5) \delta^i + (a_6 + c_i^2 a_7) \delta + c_i^2 (a_8 + Ra_9) \lambda^i + Rc_i^2 a_{10} \lambda \quad (33)$$

$$\lambda^{ii} = a_{11} \delta^{ii} + (a_{12} + Rc_i^2 a_{13}) \delta^i + a_{14} \delta + a_{15} \lambda^i + c_i^2 (a_{16} + Ra_{17}) \lambda \quad (34)$$

where  $(^i)$  is the operator  $d/d\xi$ , and the constants of  $a_1$  through  $a_{17}$  are as follows.

$$a_1 = 1.5b_1^{-1} b_2 \quad (35.1)$$

$$a_2 = -b_1^{-2} (64fb_1 + 2.5b_2^2 - b_3 + 2) \quad (35.2)$$

$$a_3 = -8fs^{-2}b_1 \quad (35.3)$$

$$a_4 = b_1^{-3}b_2(56fb_1 - 11.5b_2^2 + b_3 + 5.5) \quad (35.4)$$

$$a_5 = 4fs^{-2}b_2 \quad (35.5)$$

$$a_6 = -b_1^{-4}(8fs^2b_1^3 + 18b_2^2 - b_3) \quad (35.6)$$

$$a_7 = 64f^2b_1^2 \quad (35.7)$$

$$a_8 = -8fs^2 \quad (35.8)$$

$$a_9 = 8fs^{-2} \quad (35.9)$$

$$a_{10} = -12fs^{-2}b_1^{-2}b_2 \quad (35.10)$$

$$a_{11} = 0.1875f^{-1}s^{-2}b_1^{-3}b_2 \quad (35.11)$$

$$a_{12} = 0.1875f^{-1}s^{-2}b_1^{-4}b_2^2 - b_1^{-1} \quad (35.12)$$

$$a_{13} = s^{-4}b_1^{-1} \quad (35.13)$$

$$a_{14} = 1.5b_1^{-2}b_2(0.125f^{-1}s^{-2}b_1^{-3} + 1) \quad (35.14)$$

$$a_{15} = 0.5b_1^{-1}b_2 \quad (35.15)$$

$$a_{16} = -8fs^{-2}b_1 \quad (35.16)$$

$$a_{17} = -s^{-4}b_1^{-2} \quad (35.17)$$

where,

$$b_1 = 0.125f^{-1}[1 + 16f^2(2\xi - 1)^2] \quad (36.1)$$

$$b_2 = 8f(2\xi - 1) \quad (36.2)$$

$$b_3 = 6[1 + 64f^2(2\xi - 1)^2] \quad (36.3)$$

The non-dimensional frequency parameter is defined as

$$c_i = \omega_i r^{-1} l^2 \sqrt{m/(EA)} \\ = \omega_i l^2 \sqrt{\gamma A/(EI)}, \quad i = 1, 2, 3, 4, \dots \quad (37)$$

where  $\gamma$  is the mass density.

Now consider the boundary conditions. At a clamped end ( $x=0$  or  $x=L$ ), the boundary conditions are  $v=w=\Psi=0$  and these relations can be expressed in the non-dimensional form as

$$\lambda = 0 \quad \text{at } \xi = 0 \quad \text{or } \xi = e \quad (38)$$

$$\delta = 0 \quad \text{at } \xi = 0 \quad \text{or } \xi = e \quad (39)$$

$$\delta^i = 0 \quad \text{at } \xi = 0 \quad \text{or } \xi = e \quad (40)$$

Here, the latest Eq. (40) implies that the rotation of cross-section  $\Psi$  expressed in Eq. (7) is zero.

At a hinged end ( $x=0$  or  $x=L$ ), the boundary conditions are  $v=w=M=0$  and these relations can be expressed in the non-dimensional form as

$$\lambda = 0 \quad \text{at } \xi = 0 \quad \text{or } \xi = e \quad (41)$$

$$\delta = 0 \quad \text{at } \xi = 0 \quad \text{or } \xi = e \quad (42)$$

$$\delta^{ii} + b_1^{-1} b_2 \delta^i = 0 \quad \text{at } \xi = 0 \quad \text{or } \xi = e \quad (43)$$

Also, the latest Eq. (43) implies that the bending moment  $M$  expressed in Eq. (6) is zero.

### 3. NUMERICAL METHODS AND DISCUSSION

Based on the above analysis, a general FORTRAN computer program was written to calculate the frequency parameters  $c_i$  and the corresponding mode shapes  $\lambda = \lambda_i(\xi)$ ,  $\delta = \delta_i(\xi)$  and  $\Psi = \Psi_i(\xi)$ . The numerical methods described by Lee and Wilson,<sup>(1)</sup> and Lee et al.<sup>(14)</sup> were used to solve the differential Eqs. (33) and (34), subjected to the end constraint Eqs. (38)~(40) or Eqs. (41)~(43). First, the determinant search method combined with the Regula-Falsi method<sup>(15)</sup> was used to obtain the frequency parameter  $c_i$ , and then the Runge-Kutta method<sup>(15)</sup> was used to calculate the mode shapes  $\lambda$ ,  $\delta$  and  $\Psi$ .

Prior to executing the numerical studies, the convergence analysis, for which  $f=0.3$ ,  $e=0.8$ ,  $s=50$ , and  $R=1$ , was conducted to determine the appropriate step size  $\Delta\xi$  in the Runge-Kutta method. Figure 3 shows  $1/\Delta\xi$  versus  $c_i$  curves, in which a step size of  $1/\Delta\xi=20$  is found to give convergence for  $c_i$  to within three significant figures. It is noted that the convergence efficiency herein is highly promoted, under same convergence criteria, comparing the appropriate  $1/\Delta\xi=50$  obtained by Oh<sup>(16)</sup> in polar coordinates. However, the step size of  $\Delta\xi=1/50$  was used herein in

order to increase accuracy of numerical solutions.

Four lowest values of  $c_i$  ( $i=1, 2, 3, 4$ ) and the corresponding mode shapes were calculated in this study. Numerical results, given in Table 1, Table 2 and Figs. 4 through 7, are now discussed. The first series of numerical results are shown in Table 1. These studies served as an approximate check on the analysis presented herein. For comparative purposes, finite element solutions based on the commercial packages SAP 2000 were used to compute the first four frequency parameters  $c_i$

for both clamped ends and both hinged ends. See the geometry, mechanical and cross-sectional properties of arches used in comparisons in the appendix. The results showed that the 100 finite elements were necessary to match within a tolerance of about 2.5 % values of  $c_i$  computed by solving the governing differential equations.

All of numerical results that follow are based on the analysis reported herein. The effects of rotatory inertia on natural frequencies are shown in Table 2. It is apparent that the effect of rotatory inertia is to always depress the natural frequencies, in which these depressions are less than about 3 %. Further, the frequencies of both clamped ends are always greater than those of

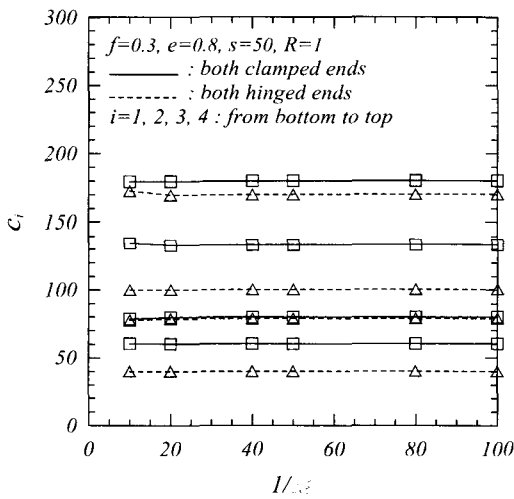


Fig. 3 Convergence analysis

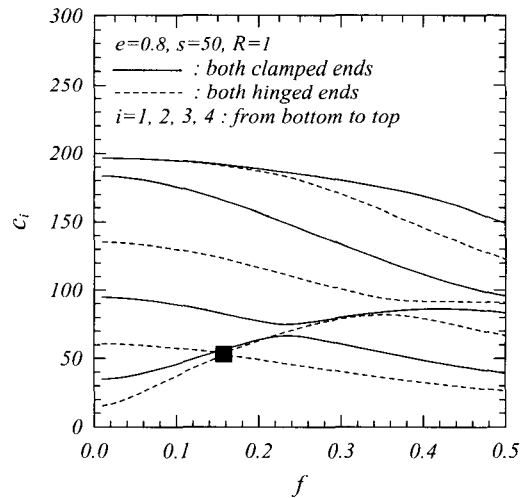


Fig. 4  $c_i$  versus  $f$  curves

Table 1 Comparisons of  $c_i$  between this study and SAP 2000\*

Geometry	$i$	Frq. parameter $c_i$		Ratio**
		This study	SAP2000	
Both clamped ends, $f = 0.3, e = 0.8,$ $s = 50, R = 1$	1	60.13	60.30	0.997
	2	80.12	80.96	0.990
	3	133.5	136.9	0.975
	4	180.4	181.2	0.996
Both hinged ends, $f = 0.3, e = 0.8,$ $s = 50, R = 1$	1	40.34	40.92	0.986
	2	79.07	80.42	0.983
	3	100.6	100.9	0.997
	4	170.5	173.6	0.982

\* The three-dimensional frame element is used in SAP 2000.

\*\* Ratio = (this study)/(SAP 2000)

Table 2 The effect of rotatory inertia on frequencies

Geometry	$i$	Frq. parameter $c_i$		Ratio*
		$R = 0$	$R = 1$	
Both clamped ends, $f = 0.3, e = 0.8,$ $s = 50$	1	61.05	60.13	0.985
	2	80.44	80.12	0.996
	3	136.0	133.5	0.982
	4	181.4	180.4	0.994
Both hinged ends, $f = 0.3, e = 0.8,$ $s = 50$	1	40.59	40.34	0.993
	2	79.35	79.07	0.996
	3	102.7	100.6	0.980
	4	174.5	170.5	0.977

\* Ratio = ( $R = 1$ )/( $R = 0$ )

both hinged ends, other parameters remaining the same.

It is shown in Fig. 4, for which  $e = 0.8$ ,  $s = 50$  and  $R = 1$ , that each frequency curve of second modes of both clamped ends and both hinged ends reaches a peak as the horizontal rise to chord length ratio  $f$  is increased while the other frequency parameters decrease as  $f$  is increased. Further, it is observed for these unsymmetric arch configurations that two mode shapes can exist at a single frequency, a phenomena that was previously observed only for symmetric arch configurations.<sup>(1)</sup> For both hinged ends, the first and second modes have the same frequency  $c_1 = c_2 = 53.06$  at  $f = 0.153$  (marked as ■). However, the frequency curves of first and second modes for both clamped ends come close each other but not close.

It is shown in Fig. 5, for which  $f = 0.3$ ,  $s = 50$  and  $R = 1$ , that the frequency parameters  $c_i$  decrease as the span length to chord length ratio  $e$  is increased. Particularly, it is noted that the frequency parameters of third and fourth modes are more significantly decreased as  $e$  gets smaller value.

It is shown in Fig. 6, for which  $f = 0.3$ ,  $e = 0.8$  and  $R = 1$ , that the frequency parameters  $c_i$

increase, and in most cases approach a horizontal asymptote, as the slenderness ratio  $s$  is increased. Further, it is seen from all of Figures mentioned above that frequencies of both clamped ends are always greater than those of both hinged ends, other parameters remaining the same.

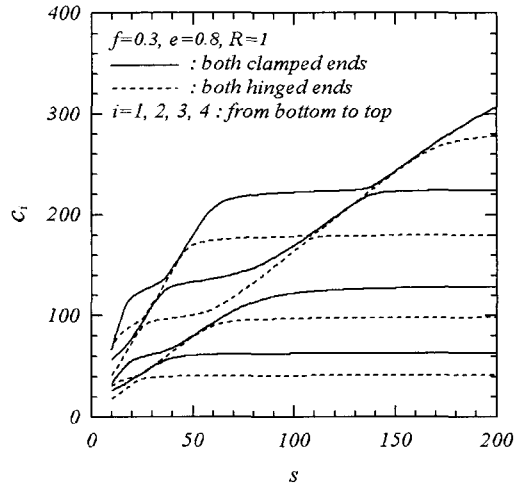


Fig. 6  $c_i$  versus  $s$  curves

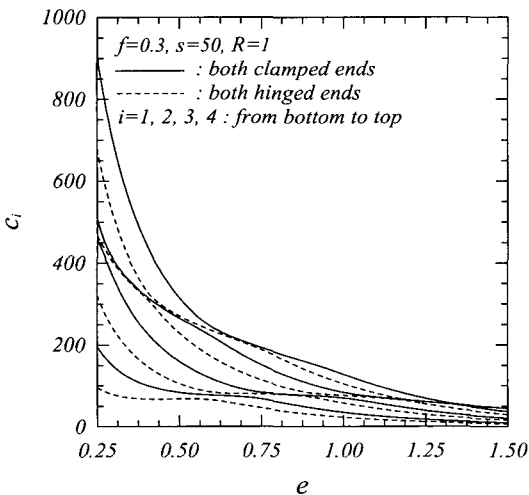


Fig. 5  $c_i$  versus  $e$  curves

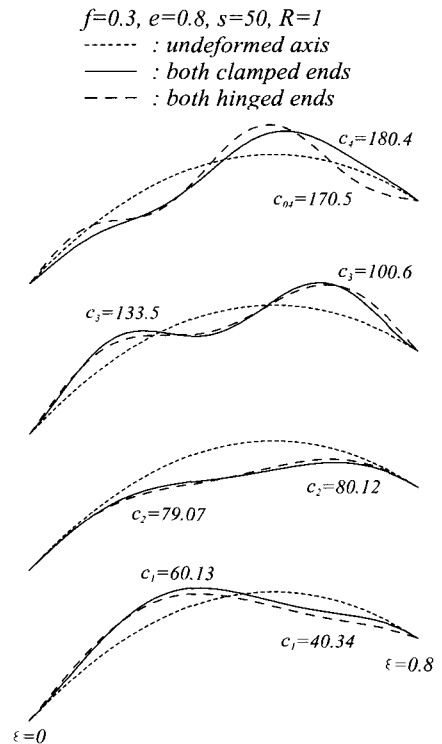


Fig. 7 Example of mode shapes

Figure 7 shows the computed mode shapes with  $f=0.3$ ,  $e=0.8$ ,  $s=50$  and  $R=1$  for both clamped ends and both hinged ends. From these figures, the amplitude and the positions of maximum amplitude and nodal points of each mode can be obtained, which is widely used in the fields of vibration control.

#### 4. CONCLUDING REMARKS

This study deals with the free vibrations of arches with unsymmetric axis. The governing differential equations are derived in Cartesian coordinates rather than in polar coordinates, in which the effect of rotatory inertia on the natural frequency is included. The differential equations, subjected to parabolic arches, newly derived herein were solved numerically to calculate both natural frequencies and mode shapes. For validating the theories and numerical methods presented herein, frequency parameters obtained in this study are compared to those of SAP 2000. The convergent efficiency of numerical methods developed herein is highly improved under the differential equations in Cartesian coordinates. As the numerical results, the relationships between the frequency parameters and the various non-dimensional arch parameters are reported, and typical mode shapes are presented. It is expected that results obtained herein can be practically utilized in the fields of vibration control.

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## APPENDIX

The geometry, mechanical and cross-sectional properties of arches used in comparisons are

$$l=0.5 \text{ m}, L=0.4 \text{ m}, h=0.15 \text{ m}, \gamma=2.68e9 \text{ kg/m}^3, \\ E=6.89e10 \text{ N/m}^2, A=12e-4 \text{ m}^2 \text{ and } I=12e-8 \text{ m}^4.$$

By using these values, the angular frequencies  $\omega_i$  of arches are obtained in SAP 2000 as follows.

Both clamped: 12.23, 16.41, 27.77, 36.73 rad/s

Both hinged: 8.30, 16.31, 20.46, 35.21 rad/s

The non-dimensional frequencies  $c_i$  of SAP 2000 are obtained by Eq. (37) and above angular frequencies as follows.

Both clamped: 60.30, 80.96, 136.9, 181.2

Both hinged: 40.92, 80.43, 100.9, 173.6