Post-buckling of Non-uniform Cantilever Column Subjected to a Combined Load

결합하중을 받는 임의단면 기둥의 좌굴후 해석

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ABSTRACT

This paper presents the application of the technique of differential transformation to the post-buckling problem of non-uniform cantilever column subjected to a combined load. Numerical calculations are carried out and compared with previously published results to validate the results of the present method. The results obtained by this method agree very well with those reported in the previous works. The results obtained by the present method are presented for both various non-uniform columns and loads.

요 약

강도와 작용하중의 분포 및 설계조건 등에 의해 단면이 길이에 따라 임의로 된 외팔 기둥이 구조물 등에 많이 사용되고 있어서 이들에 대한 좌굴에 관한 해석이 구조물의 설계시에는 매우 중요하다. 본 논문에서는 분포하중, 집중하중을 받는 가변단면 기둥의 좌굴해석후 해석 문제를 differential transformation이라는 새로운 변환방법을 적용하여 해석하여 기존의 해석결과와 비교, 검토하였고, 또한 임의의 가변단면의 외팔 기둥에 대한 좌굴후 해석의 결과를 제시하였다.

1. Introduction

Non-uniform columns are widely used in many structural applications in order to optimize the distribution of weight and strength and to satisfy special architectural and functional requirements.(1,2) Therefore, the post-buckling analysis of non-uniform columns is of interest to many mechanical, offshore technological and civil engineers.(3,4)

The post-buckling analysis of uniform columns under concentrated load at the free end has been performed by using elliptic integrals.(5) The post-buckling analysis of a uniform cantilever column subjected to uniform axial load has been analyzed by using Runge - Kutta integration scheme.(6) The post-buckling of a uniform column subjected to uniform axial load has been analyzed using finite element method.(7) Lee(8) presented the post-buckling analysis of uniform column under a combined load of axial load and concentrated load

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by using numerical integration procedure, and carried out the linear buckling analysis under a combined load by using power series method.

In this paper, the general equation of the post-buckling for non-uniform cantilever column under general load is derived and approximate equation for post-buckling is presented. A new transformation called differential transform is introduced to solve the post-buckling problem of non-uniform column subjected to a combined load consisting of a non-uniformly distributed axial load and concentrated load at the free end. The calculated results are compared with those obtained by other analytical methods.[5,8]

2. Formulation for Post-buckling

A non-uniform column that is loaded by a combined load consisting of a uniformly distributed axial load w and concentrated load p at the free end is considered.

Taking the coordinate axes as shown in Fig.1 and measuring the distance s along the axis of the column from the origin O placed at the free end, the differential equation of the deflection curve is obtained as follows.[8]

$$E_oI_o\frac{d}{ds}\left[\gamma(s)\frac{d\varphi}{ds}\right]=-(p+ws)\sin\varphi \tag{1}$$

where s is the arc length from the free end, $\varphi(s)$ is the slope of column along the arc length, E_oI_o is the flexural rigidity of the column at $s=0$, and $EI(s)=E_oI_o\gamma(s)$ is the variable flexural rigidity of the column at $s=s$.

The boundary conditions of the column under study are represented as

$$\frac{d\varphi}{ds}=0 \text{ at } s=0 \tag{2}$$

$$\varphi=0 \text{ at } s=L \tag{3}$$

where L is the length of the column.

Introducing a new non-dimensional variable $\xi=s/L$, equation (1) can be rewritten in terms of ξ as

$$\frac{d}{d\xi}\left[\gamma(\xi)\frac{d\varphi}{d\xi}\right]=-(P-\gamma\xi)\sin\varphi \tag{4}$$

where $P=\frac{pL^2}{E_oI_o}$ and $\gamma=\frac{wL^3}{E_oL_o}$

The boundary conditions are

$\frac{d\varphi}{d\xi}=0$at $\xi=0$ and $\varphi=0$ at $\xi=1$.

The $\gamma(\xi)$ in equation (4) is taken to have the following polynomial variation along the column.

$$\gamma(\xi)=\frac{EI}{E_oI_o}=\sum_{i=0}^{q}R_i\xi^i \tag{5}$$

Here q is a integer representing the number of terms in series.

In this way, $\gamma(\xi)$ for non-uniform column can be represented exactly or up to any desired accuracy. If r is equal to 1, equation (4) is for the uniform column subjected to a combined load.

Since the constant slope φ_0 of the column at $\xi=0$ must be considered as an additional boundary condition for the post-buckling analysis, the non-dimensional differential equation of the

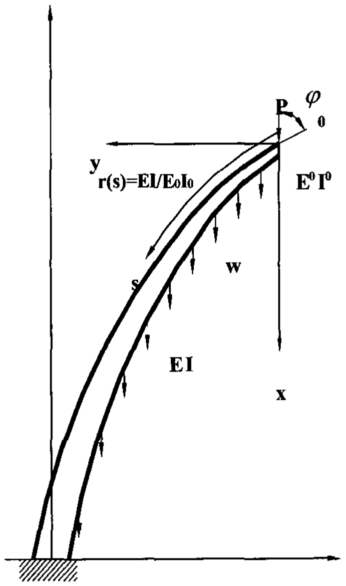

Fig. 1 Non-uniform cantilever column under a combined load

deflection curve can be represented as follows

$$\frac{d}{d\xi}\left[r\frac{d\varphi}{d\xi}\right]=-(P+\gamma\xi)\sin\varphi \quad (6)$$

with boundary conditions

$\varphi=\varphi_0$, $\frac{d\varphi}{d\xi}=0$ at $\xi=0$ and $\varphi=0$ at $\xi=1$.

The values of the buckling loads, P and γ, can be determined from equation (6).

Assuming that φ is small and $\sin\varphi=\varphi$, the linearized differential equation of the deflection curve for non-uniform column subjected to a combined load can be written as

$$\frac{d}{d\xi}\left[r\frac{d\varphi}{d\xi}\right]+(P+\gamma\xi)\varphi=0 \quad (7)$$

with the boundary conditions

$$\varphi=\varphi_0,\ \frac{d\varphi}{d\xi}=0 \text{ at } \xi=0 \quad (8)$$

$$\varphi=0 \text{ at } \xi=1. \quad (9)$$

Then the approximate buckling load of the column can be obtained from equation (7).

Equation (6) is a nonlinear differential equation, the solution of which is attempted by using differential transformation in this study.

The procedure described above can be used for non-uniform cross section column with any continuous distributed axial load.

3. Differential Transformation

Let $y(x)$ be analytic in domain D and $x=0$ be a point in D. Then there exists precisely one power series with center at $x=0$ which represents $y(x)$; this series, the Maclaurin series of the function $y(x)$, is of the form[8]

$$y(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \quad \text{for } \forall x\subset D \quad (10)$$

If we define differential transformation of function $y(x)$ as follows

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \quad (11)$$

and substitute equation (11) into equation (10), equation (10) becomes

$$y(x)=\sum_{k=0}^{\infty}x^k Y(k) \quad (12)$$

$Y(k)$ is the differential transformation for the original function $y(x)$, and equation (12) is the differential inverse transformation of $Y(k)$. In this study we use the lower case letters to represent the original functions and the upper case letters stand for the transformed functions (T-functions).

From the above definition of the differential transformation of the function, we can derive the rules of transformational operations. Some of these, which are useful in the following analysis, are as follows:

Original function	T-function
$w(x)=y(x)\pm z(x)$	$W(k)=Y(k)\pm Z(k)$
$z(x)=\lambda y(x)$	$Z(k)-\lambda Y(k)$
$w(x)=\frac{d^n y(x)}{dx^n}$	$W(x)=(k+1)(k+2)\cdots(k+n)Y(k+n)$
$w(x)=y(x)z(x)$	$W(x)=\sum_{l=0}^{k}Y(l)Z(k-1)$
$w(x)=x^m$	$W(k)=\delta(k-m)\begin{cases}1, & k=m\\ 0, & k\neq m\end{cases}$

In actual applications, the function $y(x)$ may be expressed by a finite series and equation (12) can be written as

$$y(x)=\sum_{k=0}^{n}x^k Y(k) \quad (13)$$

Equation (13) implies that $\sum_{k=n+1}^{\infty}x^k Y(k)$ is neglected. Generally, n is decided by the desired accuracy of the problem.

4. Application of Differential Transformation to Buckling Problem

Taking differential transformation of equation (6) and using the transformational operations mentioned above, we obtain

$$\sum_{l=0}^{k}(l+1)R(l+1)(k-l+1)\Phi(k-l+1)$$
$$+\sum_{l=0}^{k}R(l)(k-l+1)(k-l+2)\Phi(k-l+2)$$
$$+P\Phi_s(k)+\gamma\sum_{l=0}^{k}\Phi_s(k-l)\delta(l-1)=0 \qquad (14)$$

where $R(k)$, $\Phi(k)$ and $\Phi_S(k)$ are T-functions of $r(\xi)$, $\varphi(\xi)$ and $\sin(\varphi(\xi))$, respectively, and $\delta(k)$ is the Dirac delta function.

The boundary conditions (equations (8) and (9)) can be transformed as follows:

$$\Phi(0)=\varphi_0,\ \Phi(1)=0 \quad \text{at} \quad \xi=0 \qquad (15)$$

$$\sum_{k=0}^{\infty}\Phi(k)=0 \quad \text{at} \quad \xi=1 \qquad (16)$$

Equation (14) is a recursive nonlinear algebraic equation which is suitable for symbolic computer implementation.

Substituting $k=0$ into equation (14), we have

$$\Phi(2)=-\frac{P}{2R(0)}\Phi_s(0) \qquad (17)$$

For $k\geq1$, following the recursive procedure we calculate up to the kth term $\Phi(k)$ by using equation (14)

$$\Phi(k+2)=$$
$$-[\sum_{l=0}^{k}(l+1)R(l+1)(k-l+1)\Phi(k-l+1)]$$
$$\Phi(k-l+1)+P\Phi_s(k)+\gamma\Phi_s(k-1)$$
$$+\sum_{l=1}^{k}R(l)(k-l+1)(k-l+2)\Phi(k-l+2)]$$
$$/[R(0)(k+1)(k+2)] \qquad (18)$$

Substituting $\Phi(0)\sim\Phi(k)$ into equation (16), we have the following buckling load equation.

$$f^{(k)}(Q^{(k)})=0 \qquad (19)$$

where $Q^{(k)}$ is a polynomial of both P and γ corresponding to k.

The solution of equation (19) yields the desired buckling load parameter $Q=Q_i^{(k)}$, $i=1,2,\ldots$ of the column. is the i-th estimated buckling load corresponding to k, with k being decided by the following equation.

$$|Q_i^{(k)}-Q_i^{(k-1)}|\leq\varepsilon \qquad (20)$$

where $Q_i^{(k)}$ is the i-th estimated buckling load corresponding to $k-1$ and ε is a tolerance parameter.

Substituting $Q_i^{(k)}$ into $\Phi(0)$ ~ $\Phi(k)$ and substituting these $\Phi(0)$ ~ $\Phi(k)$ into equation (13), we obtain

$$\varphi_i(\xi)=\sum_{k=0}^{n}\xi^k\Phi(k) \quad \text{for } Q_i^{(k)} \qquad (21)$$

$\varphi_i(\Xi)$ $\varphi_i(\xi)$ is the slope of the i-th buckling mode of the column corresponding to non-dimensional buckling load $Q_i^{(k)}$.

5. Numerical Analyses and Discussions

To validate the method and compare the calculated results with the previous works,[5,8] the post-buckling analyses of uniform column under a combined load of axial distributed load and concentrated load is considered.

Table 1 presents the buckling load P of uniform column under a concentrated load at the free end for various slopes of free end and Table 2 gives the buckling load γ of uniform column subjected to uniformly distributed axial load. In Table 3, there is shown the comparison of buckling loads of $P=\gamma$ uniform column subjected to combined load obtained by the present method with those by reference.[8] For comparison purpose, the values reported in references[5,8] are also included in these tables. A glance at Tables 1 to 3 reveals that the agreement between the results obtained from the present method and reference[5,8] is very good

The post-buckling analyses of non-uniform column subjected to three types of loads, a tip load p at the free end, a uniformly distributed axial load w per unit length and a combined load consisting of axial load w and a tip load p applied at the free end, have been performed by differential transformation.

Table 4 presents the buckling loads P of two non-uniform columns ($r(s)=1+0.5s$ and $r(s)=1+0.1s+0.1s^2$) under a concentrated load at the free end for various slopes of free end. Table 5 gives the buckling loads γ of two non-uniform columns ($r(s)=1+0.5s$ and $r(s)=1+0.1s+0.1s^2$) subjected to uniformly distributed axial load.

Tables show that the buckling loads of the column with $r(s)=1+0.5s$ are higher than those of the column with $r(s)=1+0.1s+0.1s^2$, as supposed. It can be seen from tables that the buckling loads of the column increases gradually as the free end slope of the column increases

For the post-buckling of non-uniform column under combined load, $wL=P$ is considered. Then equation (6) reduces to

$$\frac{d}{d\xi}\left[r(\xi)\frac{d\varphi}{d\xi}\right]=-P(1+\xi)\sin\varphi \qquad (22)$$

with boundary conditions

$$\varphi=\varphi_0,\ \frac{d\varphi}{d\xi}=0 \text{ at } \xi=0 \quad \varphi=0 \text{ at } \xi=1.$$

By applying the same procedure as mentioned

Table 1 Comparison of buckling loads $P=pL^2/E_0I_o$ of uniform column under a concentrated load at the free end obtained by the present method with those of elliptic integral method.[5,8]

φ_0 (deg.)	Present method	Elliptic integral method[5,8]
	$P=pL^2/E_0I_o$	$P=pL^2/E_0I_o$
0	2.4674	2.4674
20	2.5054	2.5044.
40	2.6245	2.6228
60	2.8418	2.8428
80	3.1925	3.1903
100	3.7456	3.7455

Table 2 Comparison of buckling loads $\gamma=wL^3/E_0I_0$ of uniform column subjected to uniformly distributed axial load obtained by the present method with those by Lee.[8]

φ_0(deg.)	Present method	Lee's result[8]
0	7.8373	7.8356
20	7.9471	7.9448
40	8.2899	8.2877
60	8.9108	8.9101
80	9.9130	9.9070
100	11.4956	11.4638

Table 3 Comparison $P=\gamma$ of buckling loads of uniform column subjected to combined load obtained by the present method with those by Lee.[8]

φ_0(deg.)	Present method	Lee's result[8]
	$P=pL^2/E_0I_o=\gamma=wL^3/E_0I_o$	
0	1.8960	1.8957
20	1.9189	1.9237
40	1.9923	2.0120
60	2.1328	2.1726
80	2.3783	2.4309
100	2.8189	2.8368

Table 4 Comparison of buckling loads P of non-uniform columns under a concentrated load at the free end with those of uniform column.

φ_0(deg.)	Present method	
	$r(s)=1+0.5s$	$r(s)=1+0.1s+0.1s^2$
0	3.3112	2.7675
20	3.3617	2.8102
40	3.5272	2.9500
60	3.8582	3.2300
80	4.4794	3.7610
100	6.0250	4.8674

Table 5 Comparison of buckling loads γ of non-uniform columns subjected to uniformly distributed axial load with those of uniform column.

φ_0(deg.)	Present method	
	$r(s)=1+0.5s$	$r(s)=1+0.1s+0.1s^2$
0	10.8883	8.9514
20	11.0367	9.0753
40	11.5022	9.4623
60	12.3912	10.1620
80	13.1543	11.2682
100	14.7529	12.9314

Table 6 Buckling loads P of non-uniform columns subjected to a combined load consisting of concentrated load and uniformly distributed axial load.

φ_0(deg.)	Present method	
	$r(s)=1+0.5s$	$r(s)=1+0.1s+0.1s^2$
0	2.5679	2.1367
20	2.5986	2.1624
40	2.6972	2.2449
60	2.8865	2.4030
80	3.2191	2.6801
100	3.8195	3.1784

above, buckling load were obtained and presented in Table 6.

6. Conclusions

In this paper, the post-buckling of non-uniform cantilever column subjected to a combined load is investigated by using differential transformation. The results obtained by the present method agree very well with those reported in the previous works. Numerical analysis are performed for both various non-uniform columns and loads. The present analysis shows the usefulness of differential transformation in solving nonlinear problem arising in the post-buckling problem.

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