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TORSION IN THE HOMOLOGY OF THE DOUBLE LOOP SPACES OF COMPACT SIMPLE LIE GROUPS

YOUNGGI CHOI AND SEONHEE YOON

ABSTRACT. We study the torsions in the integral homology of the double loop space of the compact simple Lie groups by determining the higher Bockstein actions on the homology of those spaces through the Bockstein lemma and computing the Bockstein spectral sequence.

1. Introduction

It is well-known from the Morse theory that the integral homology of the single loop space of any compact simple Lie group is concentrated on even dimensions and of torsion free [1]. However the situation is different when we consider the double loop spaces of compact simple Lie groups. Here we study p-torsions in the integral homology of the double loop space of the compact simple Lie groups.

The main tool of the computation is the Bockstein lemma by which we determine the higher Bockstein actions on the homology of those spaces. The Bockstein spectral sequence is applied to the study of torsions in the integral homology of those spaces.

In this paper we get following results. For classical compact Lie groups, $G = SO(n)$, $SU(n)$, $Sp(n)$, the order of p-torsions in $\Omega^2 G$ depends on n for an odd prime p. If $p = 2$, the order of 2-torsions in $\Omega^2 SU(n)$ depends on n and the order of 2-torsions in $\Omega^2 Sp(n)$ and $\Omega^2 SO(n)$ is bounded by 2 or 2^2. Results are described in Theorem 3.2-3.8. For the exceptional Lie groups G, the order of p-torsions in $\Omega^2 G$ is at most 2^3 if $p = 2$, and at most p^2 if p is an odd prime.

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These are summarized in the following two tables.

$H_*(\Omega^2 X; Z)$	condition	order of 2–torsion
$H_*(\Omega^2 SO(8n+i); Z)$	$i \neq 2, 6$	2
$H_*(\Omega^2 SO(8n+i); Z)$	$i = 2, 6$	2 or 2^2
$H_*(\Omega^2 SU(n); Z)$	$2^{r-1} < n \leq 2^r$	$\leq 2^r$
$H_*(\Omega^2 Sp(n); Z)$		2
$H_*(\Omega^2 G_2; Z)$		2
$H_*(\Omega^2 F_4; Z)$		2
$H_*(\Omega^2 E_6; Z)$		2 or 2^3
$H_*(\Omega^2 E_7; Z)$		2
$H_*(\Omega^2 E_8; Z)$		2

Table 1. 2–torsion

$H_*(\Omega^2 X; Z)$	condition	order of p–torsion
$H_*(\Omega^2 SO(2n); Z)$	$p^{r-1} < 2n-1 \leq p^r$	$\leq p^r$
$H_*(\Omega^2 SO(2n+1); Z)$	$p^{r-1} < 2n+1 \leq p^r$	$\leq p^r$
$H_*(\Omega^2 SU(n); Z)$	$p^{r-1} < n \leq p^r$	$\leq p^r$
$H_*(\Omega^2 Sp(n); Z)$	$p^{r-1} \leq 2n-1 < p^r$	$\leq p^r$
$H_*(\Omega^2 G_2; Z)$	$p \neq 5$	p
	$p = 5$	5^2
$H_*(\Omega^2 F_4; Z)$	$p = 3$	3
	$5 \leq p \leq 11$	p or p^2
	$p > 11$	p
$H_*(\Omega^2 E_6; Z)$	$p = 3$	3
	$5 \leq p \leq 11$	p or p^2
	$p > 11$	p
$H_*(\Omega^2 E_7; Z)$	$p = 3$	3 or 3^2
	$5 \leq p \leq 17$	p or p^2
	$p > 17$	p
$H_*(\Omega^2 E_8; Z)$	$p < 7$	p
	$7 \leq p \leq 29$	p or p^2
	$p > 29$	p

Table 2. p–torsion for odd primes

2. Preliminaries

Consider the short exact sequence of coefficients groups

$$Z \xrightarrow{\times p} Z \longrightarrow Z/(p)$$

which induces the exact couple

$$\begin{array}{ccc} H_*(X;Z) & \xrightarrow{\times p} & H_*(X;Z) \\ k \nwarrow & & \swarrow \rho \\ & H_*(X;\mathbb{F}_p) & \end{array}$$

where ρ is induced by the reduction homomorphism and k is the Bockstein homomorphism in the long exact sequence. From this exact couple we get the homology Bockstein spectral sequence $\{B^r, \beta^r\}$ [1] where β^r is the r-th differential in the Bockstein spectral sequence. Then $\beta^r(x) = y$ implies that there exists $v \in H_*(X;Z)$ such that $k(x) = p^{r-1}v$ where $y = \rho(v)$. Hence if $\beta^r(x) = y$ is nonzero differential and $|y| = s$, then we have p-torsion of order p^r in $H_s(X;Z)$.

By the universal coefficient theorem, the mod p homology has a following splitting:

$$H_q(X;\mathbb{F}_p) = H_q(X;Z) \otimes \mathbb{F}_p \oplus \mathrm{Ext}(H_{q-1}(X;Z), \mathbb{F}_p).$$

Hence a summand $Z/(p^n)$ in $H_q(X;Z)$ gives rise to summands $\mathbb{F}_p$ in $H_q(X;\mathbb{F}_p)$ and $H_{q+1}(X;\mathbb{F}_p)$. Conversely if there are summands $\mathbb{F}_p$ in $H_q(X;\mathbb{F}_p)$ and $H_{q+1}(X;\mathbb{F}_p)$, we can recover a summand in $H_q(X;Z)$ by determining the higher Bockstein actions in the Bockstein spectral sequence.

Now we mention the homology version of the Bockstein lemma [9, 11].

THEOREM 2.1. *Let $(E, p, B; F)$ be a fiber space. Let the element $u_{n+1} \in H_{n+1}(B;\mathbb{F}_p)$ be transgressive, and suppose that for some integer $i (i \geq 1)$ and some $v_{n+1} \in H_{n+1}(F;\mathbb{F}_p)$, u_{n+1} transgresses to $\beta^i v_{n+1}$. Then $\beta^{i+1} j_*(v_{n+1})$ is defined, and moreover*

$$p_*(\beta^{i+1} j_*(v_{n+1})) = -\beta^1 u_{n+1}$$

with the indeterminacy $\beta^1 p_*(H_{n+1}(E;\mathbb{F}_p))$ where j is the inclusion from F into E.

3. Torsions in the classical Lie groups

Let $E(x)$ be the exterior algebra on x and $\Gamma(x)$ the divided power algebra on x which is free over $\gamma_i(x)$ as a $\mathbb{F}_p$–module with product $\gamma_i(x)\gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x)$. We have homology operations, Dyer–Lashof operations, $Q_{i(p-1)}$ on the $(n+1)$-loop space $\Omega^{n+1}X$

$$Q_{i(p-1)} : H_q(\Omega^{n+1}X;\mathbb{F}_p) \to H_{pq+i(p-1)}(\Omega^{n+1}X;\mathbb{F}_p)$$

for $0 \le i \le n$ when $p = 2$, and for $0 \le i \le n$ and $i+q$ even when $p > 2$. They are natural with respect to $(n+1)$-loop maps [6]. In particular, we have $Q_0 x = x^p$. The iterated power Q_i^a denotes composition of Q_i's a times.

If G is a Lie group, G is homotopy equivalent to ΩBG. Hence $Q_{2(p-1)}$ is defined in $H_*(\Omega^2 G;\mathbb{F}_p)$. In this paper the subscript of the element always denotes the degree of that element unless stated otherwise. First we review the following basic fact.

LEMMA 3.1. *For any prime p, the order of p–torsions in* H_* $(\Omega^2 S^{2n+1};$ $Z)$ *is* p.

Proof. We have $H_*(\Omega^2 S^{2n+1};\mathbb{F}_2) = \mathbb{F}_2[Q_1^a x_{2n-1} : a \ge 0]$. Now we consider the Bockstein spectral sequence. Then $E_1 = H_*(\Omega^2 S^{2n+1};\mathbb{F}_2)$. By Nishida relation, we have $\beta Q_1^{a+1} x_{2n-1} = (Q_1^a x_{2n-1})^2$ for each $a \ge 0$. Since this Bockstein spectral sequence is a spectral sequence of an Hopf algebra, we have $E_2 = E(x_{2n-1})$. Hence there is no higher differential and $E_2 = E_\infty$. So the 2–torsions of $H_*(\Omega^2 S^{2n+1};\mathbb{F}_2)$ are all of order 2. Similarly we can show the same result for odd primes p. □

We recall the following facts for 2–torsions.

THEOREM 3.2. [3] *The order of 2–torsions in* $H_*(\Omega^2 Spin(4n+i);Z)$ *is 2 if* $i \ne 1$, *and 2 or* 2^2 *otherwise.*

Since $Spin(n)$ is a double covering space of $SO(n)$, we have Ω^2 $Spin(n) \simeq \Omega\ {}^2SO(n)$. So the 2–torsions of $H_*(\Omega^2 SO(n);Z)$ and H_* $(\Omega^2\ Spin(n);\ Z)$ are the same.

For the unitary case, we have the following.

THEOREM 3.3. [11] *Let r and n be such that $2^{r-1} < n \leq 2^r$. Then 2^r annihilates all 2–torsions in $H_*(\Omega^2 SU(n); Z)$, but 2^{r-1} does not.*

Consider the following fibration:

$$\Omega^2 Sp(n) \longrightarrow \Omega^2 Sp(n+1) \longrightarrow \Omega^2 S^{4n+3}.$$

Then from [2], the corresponding Serre spectral sequence collapses at the E_2–term and we have the following.

THEOREM 3.4. *The mod 2 homology of $\Omega^2 Sp(n+1)$ is*

$$H_*(\Omega^2 Sp(n+1); \mathbb{F}_2) = \bigotimes_{0 \leq i \leq n} H_*(\Omega^2 S^{4i+3}; \mathbb{F}_2)$$

and 2 annihilates all 2–torsions in $H_*(\Omega^2 Sp(n+1); Z)$.

Therefore 2 annihilates all 2–torsions in $H_*(\Omega^2 G; Z)$ where G is the following spaces: $Sp(n), Spin(4n), Spin(4n+1), Spin(4n+3)$

Now we turn to p–torsions for an odd prime p. Combining results from [10] and [11], we have the following theorem.

THEOREM 3.5. (a) *For an odd prime p, there are choices of generators x_i and y_i such that $H_*(\Omega^2 SU(n+1); \mathbb{F}_p)$ is isomorphic to*

$$E(Q^a_{(p-1)} x_{2i-1} : a \geq 0, 0 \leq i \leq n, i \neq 0 \ mod\ p)$$
$$\otimes \mathbb{F}_p[Q^a_{2(p-1)} y_{2i-2} : 0 \leq i \leq n, i \neq 0 \ mod\ p, p^a i > n]$$

and $\beta^t(Q^a_{(p-1)} x_{2i-1}) = Q^a_{2(p-1)} y_{2i-2}$ where t is the smallest integer such that $p^t i > n$.

(b) *Let p be an odd prime, and r and n be integers such that $p^{r-1} < n \leq p^r$. Then p^r annihilates all p–torsions in $H_*(\Omega^2 SU(n); Z)$, but p^{r-1} does not.*

We can derive the following results from the above mod p homology of the double loop space of $SU(n)$.

THEOREM 3.6. *Let p be an odd prime, and r and n be integers such that $p^{r-1} < 2n-1 \leq p^r$. Then p^r annihilates all p–torsions in $H_*(\Omega^2 SO(2n+1); Z)$, but p^{r-1} does not.*

Proof. For an odd prime p, we have the Harris splitting [7]

$$SU(2n+1) \simeq_p SU(2n+1)/SO(2n+1) \times SO(2n+1)$$

where $\simeq_p$ means homotopy equivalence localized at p. Hence by looping twice, we get

$$\Omega^2 SU(2n+1) \simeq_p \Omega^2(SU(2n+1)/SO(2n+1)) \times \Omega^2 SO(2n+1).$$

Therefore the mod p homology of $\Omega^2 SO(2n+1)$ for an odd prime p is one of direct summands of the mod p homology of $\Omega^2 SU(2n+1)$. Moreover, we have the corresponding mod p homologies:

$$\begin{aligned} &H_*(SU(2n+1); \mathbb{F}_p) = E(u_{2i+1} : 1 \leq i \leq 2n), \\ &H_*(SU(2n+1)/SO(2n+1); \mathbb{F}_p) = E(u_{4i+1} : 1 \leq i \leq n), \\ &H_*(SO(2n+1); \mathbb{F}_p) = E(u_{4i-1} : 1 \leq i \leq n). \end{aligned}$$

From the above Harris splitting, we can separate $H_*(\Omega^2 SU(2n+1))$ into two parts, so that $H_*(\Omega^2 SO(2n+1); \mathbb{F}_p)$ is isomorphic to

$$\begin{aligned} &E(Q^a_{(p-1)} x_{2i-1} : i \text{ odd }, 0 \leq i \leq 2n-1, i \neq 0 \bmod p, a \geq 0) \\ &\otimes \mathbb{F}_p[Q^a_{2(p-1)} y_{2i-2} : i \text{ odd }, 0 \leq i \leq 2n-1, i \neq 0 \bmod p, p^a i > 2n-1] \end{aligned}$$

and $\beta^t(Q^a_{(p-1)} x_{2i-1}) = Q^a_{2(p-1)} y_{2i-2}$ where t is the smallest integer such that $p^t i > 2n-1$. Therefore in the Bockstein spectral sequence, we have p–torsion of order p^t in $H_{2p^a i-2}(\Omega^2 SO(2n+1); Z)$. Then t becomes the largest number when $i = 1$. Hence if r is an integer such that $p^{r-1} \leq 2n-1 < p^r$, then p^r annihilates all p–torsions in $H_*(\Omega^2 SO(2n+1); Z)$, but p^{r-1} does not. □

THEOREM 3.7. *Let p be an odd prime, and r and n be integers such that $p^{r-1} < 2n-1 \leq p^r$. Then p^r annihilates all p–torsions in $H_*(\Omega^2 SO(2n+2); Z)$, but p^{r-1} does not.*

Proof. We have

$$H_*(SO(2n+1);Q) = E(u_{4i+3} : 0 \le i \le n-1),$$
$$H_*(SO(2n+2);Q) = E(u_{4i+1} : 0 \le i \le n-1) \otimes E(u_{2n+1}).$$

Hence $H_*(\Omega^2 SO(2n+2);Q) = E(u_{4i-1} : 0 \le i \le n-1) \otimes E(u_{2n-1})$. Consider the Serre spectral sequence converging to $H_*(\Omega^2 SO(2n);\mathbb{F}_p)$

$$\Omega^2 SO(2n+1) \xrightarrow{\ i\ } \Omega^2 SO(2n+2) \xrightarrow{\ p\ } \Omega^2 S^{2n+1}.$$

From the knowledge of the rational homology, $H_*(\Omega^2 SO(2n+2);Q)$, we can derive that the first differential from x_{2i-1} is trivial where

$$H_*(\Omega^2 S^{2n+1};\mathbb{F}_p) = E(Q^a_{p-1}x_{2i-1} : a \ge 0) \otimes \mathbb{F}_p[\beta Q^a_{p-1}x_{2i-1} : a > 0].$$

From commutativity between transgressions and homology operations, the Serre spectral sequence converging to $H_*(\Omega^2 SO(2n+2);\mathbb{F}_p)$ collapses at the E_2-term. Hence we have

$$H_*(\Omega^2 SO(2n+2);\mathbb{F}_p) = H_*(\Omega^2 SO(2n+1);\mathbb{F}_p) \otimes H_*(\Omega^2 S^{2n+1};\mathbb{F}_p)$$

and the conclusion follows. □

For an odd prime p, we have the following equivalence

$$SO(2n+1) \simeq_p Sp(n)$$

which was conjectured by Serre and proved by Harris [7]. Hence we have the following.

THEOREM 3.8. *Let p be an odd prime, and r and n be integers such that $p^{r-1} < 2n-1 \le p^r$. Then p^r annihilates all p-torsions in $H_*(\Omega^2 Sp(n);Z)$, but p^{r-1} does not.*

4. Torsions in the exceptional Lie groups

The exceptional Lie groups when localized at p split as followings [8]:

$$\begin{array}{lll}
G_2 & p=3 & B_2(3,11), \\
 & p=5 & B(3,11), \\
 & p>5 & S^3\times S^{11}, \\
 & & \\
F_4 & p=5 & B(3,11)\times B(15,23), \\
 & p=7 & B(3,15)\times B(11,23), \\
 & p=11 & B(3,23)\times S^{11}\times S^{15}, \\
 & p>11 & S^3\times S^{11}\times S^{15}\times S^{23}, \\
 & & \\
E_6 & p=5 & F_4\times B(9,17), \\
 & p>5 & F_4\times S^9\times S^{17}, \\
 & & \\
E_7 & p=5 & B(3,11,19,27,35)\times B(15,23), \\
 & p=7 & B(3,15,27)\times B(11,23,35)\times S^{19}, \\
 & p=11 & B(3,23)\times B(15,35)\times S^{11}\times S^{19}\times S^{27}, \\
 & p=13 & B(3,27)\times B(11,35)\times S^{15}\times S^{19}\times S^{23}, \\
 & p=17 & B(3,35)\times S^{11}\times S^{15}\times S^{19}\times S^{23}\times S^{27}, \\
 & p>17 & S^3\times S^{11}\times S^{15}\times S^{19}\times S^{23}\times S^{27}\times S^{35}, \\
 & & \\
E_8 & p=7 & B(3,15,27,39)\times B(23,35,47,59), \\
 & p=11 & B(3,23)\times B(15,35)\times B(27,47)\times B(39,59), \\
 & p=13 & B(3,27)\times B(15,39)\times B(23,47)\times B(35,39), \\
 & p=17 & B(3,35)\times B(15,47)\times B(27,59)\times S^{23}\times S^{39}, \\
 & p=19 & B(3,39)\times B(23,59)\times S^{15}\times S^{27}\times S^{35}\times S^{47}, \\
 & p=23 & B(3,47)\times B(15,59)\times S^{23}\times S^{27}\times S^{35}\times S^{39}, \\
 & p=29 & B(3,59)\times S^{15}\times S^{23}\times S^{27}\times S^{35}\times S^{39}\times S^{47}, \\
 & p>29 & S^3\times S^{15}\times S^{23}\times S^{27}\times S^{35}\times S^{39}\times S^{47}\times S^{59}.
\end{array}$$

The space $B(2n_1+1,\dots,2n_r+1)$ is built up from fibrations involving p–local spheres of the indicated dimensions and equivalent to a direct factor of the p–localization of $SU(n+p)/SU(n)$. The space $B(2n+1,2n+2p-1)$ is equivalent to a direct factor of the p–localization of $SU(n+p)/SU(n)$ and the cohomology of $B(2n+1,2n+2p-1)$ is

$$H^*(B(2n+1,2n+2p-1);\mathbb{F}_p)=E(x_{2n+1},x_{2n+2p-1})$$

with $\mathcal{P}^1 x_{2n+1} = x_{2n+2p-1}$.

From the above splitting, we get the following results immediately.

THEOREM 4.1. *p annihilates all p–torsions in* $H_*(\Omega^2 G; Z)$ *for*

$$\begin{array}{ll} p > 5 & G = G_2, \\ p > 11 & G = F_4, E_6, \\ p > 17 & G = E_7, \\ p > 29 & G = E_8. \end{array}$$

From now on we denote $H_*(\Omega^2 S^n; \mathbb{F}_p)$ by $\Omega_2(n)$ and $\otimes_{k=1}^r H_*(\Omega^2 S^{n_k}; \mathbb{F}_p)$ by $\Omega_2(n_1, \cdots, n_r)$.

The cases of $H_*(\Omega^2 G_2; \mathbb{F}_p)$ and $H_*(\Omega^2 F_4; \mathbb{F}_p)$ follow from [4] and the following homotopy equivalence,

$$\Omega^2 Spin(7) \simeq_2 \Omega^2 G_2 \times \Omega^2 S^7, \quad \Omega^2 Spin(7) \simeq \Omega^2 SO(7).$$

THEOREM 4.2. *The homology of* $\Omega^2 G_2$ *is*
(a) $H_*(\Omega^2 G_2; \mathbb{F}_2) = E(z_1) \otimes \mathbb{F}_2[\beta z_7] \otimes \mathbb{F}_2[Q_1^a z_7) : a \geq 0] \otimes \Omega_2(11)$.
(b) $H_*(\Omega^2 G_2; \mathbb{F}_3) = \Omega_2(3, 11)$.

THEOREM 4.3. *The homology of* $\Omega^2 F_4$ *is*
(a) $H_*(\Omega^2 F_4; \mathbb{F}_2) = E(z_1) \otimes \mathbb{F}_2[\beta z_7] \otimes \Omega_2(9, 11, 15, 23)$.
(b) $H_*(\Omega^2 F_4; \mathbb{F}_3) = E(z_1) \otimes \mathbb{F}_3[\beta z_{17}] \otimes \Omega_2(11, 15, 19, 23)$.

COROLLARY 4.4. *For primes* $p = 2, 3$, *p annihilates all p–torsions in* $H_*(\Omega^2 G; Z)$ *for* $G = G_2, F_4$.

Proof. We consider the Bockstein spectral sequence converging to $H_*(\Omega^2 G_2; Z)/\text{torsion} \otimes \mathbb{F}_2$ with $E_1 = H_*(\Omega^2 G_2; \mathbb{F}_2)$. With the first Bockstein differentials, we have $E_2 = E(z_1, z_9)$. Hence there is no higher differential, so that $E_2 = E_\infty$. So 2 annihilates all 2–torsions in $H_*(\Omega^2 G_2; Z)$. For p odd primes, we also have $E_2 = E(z_1, z_9)$ and $E_2 = E_\infty$. Note that $H_*(G_2; Q) = E(z_3, z_{11})$. Similar proof works for $G = F_4$. □

We recall the following theorem in [5].

THEOREM 4.5. *The homology of the double loop space of E_6, E_7 and E_8, are as follows:*

(a) $H_*(\Omega^2 E_6;\mathbb{F}_2) = E(z_1)\otimes\mathbb{F}_2\,[\beta^3 Q_1^2 z_7]\otimes(\bigotimes_{a\geq 0}(E(Q_1^a z_7)\otimes\mathbb{F}_2[Q_2^a z_{62}]))$
$\otimes\Omega_2(11,15,23)$, *where* $\beta^3 Q_1^{a+3} z_7 = Q_2^a z_{62}, a\geq 0,$
$H_*(\Omega^2 E_6;\mathbb{F}_3) = E(z_1)\otimes\mathbb{F}_3[z_{16}]\otimes(\bigotimes_{a\geq 0}(E(Q_2^a z_{17})\otimes\mathbb{F}_3[\beta Q_2^{a+1} z_{17}]))$
$\otimes\Omega_2(9,11,15,17,23)$.

(b) $H_*(\Omega^2\, E_7;\mathbb{F}_2) = E(z_1)\otimes\,\mathbb{F}_2\,[\beta z_{31}]\otimes\,\mathbb{F}_2[Q_1^a z_{31}: a\geq 0])$
$\otimes\Omega_2(11,15,19,23,27,35)$,
$H_*(\Omega^2 E_7;\mathbb{F}_3) = E(z_1)\otimes\,(\bigotimes_{a\geq 0}(E(Q_2^a z_{17})\otimes\,\mathbb{F}_3[\beta^2 Q_2^{a+1} z_{17}]))$
$\otimes\Omega_2(11,15,23,27,35)$.

(c) $H_*(\Omega^2 E_8;\mathbb{F}_2) = E(x_1)\otimes E(z_{13})\otimes\mathbb{F}_2\,[\beta z_{31},\beta z_{55}]$
$\otimes(\bigotimes_{a\geq 0}\mathbb{F}_2\,[Q_1^a z_{31}, Q_1^a z_{55}])$
$\otimes\Omega_2(23,27,35,39,47,59)$,
$H_*(\Omega^2 E_8;\mathbb{F}_3) = E(z_1)\otimes\mathbb{F}_3(\beta z_{53})\otimes(\bigotimes_{a\geq 0}(E(Q_2^a z_{53})\otimes\mathbb{F}_3[\beta Q_2^{a+1} z_{53}]))$
$\otimes\Omega_2(15,23,27,35,39,47,59)$,
$H_*(\Omega^2 E_8;\mathbb{F}_5) = E(z_1)\otimes\mathbb{F}_5[\beta z_{49}]\otimes(\bigotimes_{a\geq 0}(E(Q_4^a z_{49})\otimes\mathbb{F}_5[\beta Q_4^{a+1} z_{49}]))$
$\otimes\Omega_2(15,23,27,35,39,47,59)$.

From the Bockstein spectral sequence, we get the following corollaries.

COROLLARY 4.6. *The order of 2–torsions in $H_*(\Omega^2 E_6; Z)$ is 2 or 2^3 and the order of 3–torsions is 3.*

Proof. We consider the Bockstein spectral sequence with $E_1 = H_*(\Omega^2 E_6;\,\mathbb{F}_2)$. With the nontrivial first differentials, we have

$$E_2 = E(z_1)\otimes\mathbb{F}_2[\beta^3 Q_1^2 z_7]\otimes(\bigotimes_{a\geq 0}(E(Q_1^a z_7)\otimes\mathbb{F}_2[\beta^3 Q_1^{a+3} z_7])) \otimes E(z_9, z_{13}, z_{21}).$$

Since there is no nontrivial second Bockstein differential, we have $E_2 = E_3$. With the third Bockstein differentials, we have $E_4 = E(z_1,\, z_7,\, z_9,\, z_{13},\, z_{15},\, z_{21})$. Hence there is no higher differential and $E_4 = E_\infty$. So the order of 2–torsions in $H_*(\Omega^2 E_6; Z)$ is 2 or 2^3. Next we consider the Bockstein spectral sequence with $E_1 = H_*(\Omega^2 E_6;\mathbb{F}_3)$. Then after the first Bockstein differentials, we have $E_2 = E(z_1, z_7, z_9, z_{13}, z_{15}, z_{21})$. So 3–torsions of $H_*(\Omega^2 E_6; Z)$ are of order 3. Note that $H_*(E_6; Q) = E(z_3, z_9, z_{11}, z_{15}, z_{17}, z_{23})$. □

Similarly we get the following two corollaries from the Bockstein spectral sequence.

COROLLARY 4.7. *The order of* 2*–torsions in* $H_*(\Omega^2 E_7; Z)$ *is* 2 *and the order of* 3*–torsions is* 3 *or* 3^2.

COROLLARY 4.8. *The order of* p*–torsions in* $H_*(\Omega^2 E_8; Z)$ *is* p *for* $p = 2,\ 3,\ 5$.

Now the remaining cases are as follows:

$$\begin{array}{ll} p = 5, & G = G_2, \\ 5 \leq p \leq 11, & G = F_4, E_6, \\ 5 \leq p \leq 17, & G = E_7, \\ 7 \leq p \leq 29, & G = E_8. \end{array}$$

In order to get p–torsions of $H_*(\Omega^2 G; Z)$ for the above cases, we should study p–torsions in $H_*(\Omega^2 B(3, 2p+1); Z)$.

THEOREM 4.9. *For an odd prime* p,

$$H_*(\Omega^2 B(3, 2p+1); \mathbb{F}_p) = E(Q^a_{(p-1)} z_1 : a \geq 0) \otimes \mathbb{F}_p[\beta^2 Q^a_{(p-1)} z_1 : a \geq 2]).$$

Proof. First we compute $H^*(\Omega B(3, 2p+1); \mathbb{F}_p)$. In the Eilenberg Moore spectral sequence converging to $H^*(\Omega B(3, 2p+1); \mathbb{F}_p)$, we have

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(B(3,2p+1);\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \\ &= \Gamma(\sigma x_3, \sigma x_{2p+1}). \end{aligned}$$

Then it collapses at the E_2–term because E_2 is concentrated on even degrees. From the Steenrod operation on x_3 and the Cartan formula, we have

$$(\gamma_{p^i}(\sigma x_3))^p = \gamma_{p^i}(\sigma x_{2p+1}).$$

So the element $\gamma_{p^i}(\sigma x_3)$ generates a truncated polynomial algebra of height p^2 and $H^*(\Omega B(3, 2p+1); \mathbb{F}_p) = \otimes_{i \geq 0}(\mathbb{F}_p[\gamma_{p^i}(y_2)]/(\gamma_{p^i}(y_2)^{p^2}))$.

Now we consider the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^2 B(3, 2p+1); \mathbb{F}_p)$,

$$\begin{aligned} E^2 &= \mathrm{Ext}_{H^*(\Omega B(3,2p+1);\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \\ &= E(z_{2p^a-1} : a \geq 0) \otimes \mathbb{F}_p[z_{2p^{a+2}-2} : a \geq 0] \end{aligned}$$

and it collapses at the E^2-term by bidegree reason. If we consider the Serre spectral sequence corresponding to the fibration $\Omega^2S^3 \to \Omega^2B(3,2p+1) \to \Omega^2S^{2p+1}$, we have

$$\begin{aligned} E^2 = &\otimes_{a\geq 0}(E(Q^a_{(p-1)}\iota_{2p-1}) \otimes \mathbb{F}_p[\beta Q^{a+1}_{(p-1)}\iota_{2p-1}]) \\ &\otimes(\otimes_{a\geq 0}(E(Q^a_{(p-1)}\iota_1) \otimes \mathbb{F}_p[\beta Q^{a+1}_{(p-1)}\iota_1])) \end{aligned}$$

where $H_*(\Omega^2S^{2n+1};\mathbb{F}_p) = \otimes_{a\geq 0}(E(Q^a_{(p-1)}\iota_{2n-1}) \otimes \mathbb{F}_p[\beta Q^{a+1}_{(p-1)}\iota_{2n-1}])$. Then there are nontrivial differentials $d(Q^a_{(p-1)}\iota_{2p-1}) = \beta Q^{a+1}_{(p-1)}\iota_1$ for $a \geq 0$. Hence by the Bockstein lemma, $\beta^2 Q^{a+2}_{(p-1)}\iota_1 = \beta Q^{a+1}_{(p-1)}\iota_{2p-1}$. So

$$E^\infty = E(Q^a_{(p-1)}\iota_1 : a \geq 0) \otimes \mathbb{F}_p[\beta^2 Q^{a+2}_{(p-1)}\iota_1 : a \geq 0]\,.$$

Hence we have the conclusion. □

COROLLARY 4.10. *p^2 annihilates all p–torsions in $H_*(\Omega^2B(3,2p+1);Z)$ for an odd prime p.*

Proof. We consider the Bockstein spectral sequence for an odd prime p with $E_1 = H_*(\Omega^2B(3,2p+1);\mathbb{F}_p)$. Since there is no first Bockstein actions, we have $E_1 = E_2$. After the second Bockstein differentials, we have that $E_3 = E(\iota_1, Q_{(p-1)}\iota_1)$ and $E_3 = E_\infty$. So p–torsions of $H_*(\Omega^2B(3,2p+1);\mathbb{F}_p)$ are all of order p^2. □

From the above result, we can determine p–torsions of $H_*(\Omega^2G;Z)$ for the following cases.

COROLLARY 4.11. *p^2 annihilates all p–torsions in $H_*(\Omega^2G;Z)$ for the following cases:*

$$\begin{aligned} &p = 5, && G = G_2, \\ &5 \leq p \leq 11, && G = F_4, E_6, \\ &5 \leq p \leq 17, && G = E_7, \\ &7 \leq p \leq 29, && G = E_8. \end{aligned}$$

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Younggi Choi
Department of Mathematics Education
Seoul National University
Seoul 151-742, Korea
E-mail: yochoi@plaza.snu.ac.kr

Seonhee Yoon
Korean Information Security Agency
Seoul 137-070, Korea
E-mail: shyoon@kisa.or.kr